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THE RAPIDLY DECREASING FUNCTIONS OF THE MICROSCOPICALLY-DESCRIPTIVE HYDRODYNAMIC EQUATIONS.
微視的記述流体力学方程式における急減少関数

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ABSTRACT. The "two-constant" theory introduced first by Laplace in 1805 still forms the basis of current theory describing isotropic, linear elasticity, describing the capillarity. By using "two-constant" theory, the Navier-Stokes equations are formulated. These equations with the two coefficients in the ratio 1 : 3 originated from Poisson [16] in 1831. Moreover, these equations contained both a linear and a nonlinear term developed earlier in Navier's equations [20] in 1827.

We show the process of formulation of *calculus of variations* using the two functions characterized from the attraction and repulsion, and his criticism to Laplace imaging the *Gaussian function* as the rapidly decreasing function by Gauss in 1830. And we introduce a contribution to the hydromechanics, partly because he was a comtenporary of the epock of formulation of the Navier-Stokes equations, which are our main theme in our paper.

Particularly, from the viewpoint of mathematics, several important topics such as integral theory in §4.3 which are his selling points. We show his unique *rapidly decreasing function* (we call it 'RDF' below) and reduction of integral from sextuplex to quadruplex, in the sections §4.1. In and after §4.2, we show his calculus of variations in the capillarity against the RDF and calculation of it by Laplace.

1. INTRODUCTION

[1] • At first, in section §2, we discuss the "two-constant" theory. In 1805, Laplace introduced the "two-constant" theory, so-called because of the prominence of two constants in his theory, in regard to capillary action with constants denoted by H and K. (cf. Table 1, 2). Thereafter, contributing investigators in formulating NS equations, i.e. equations describing equilibrium or capillary situations, have presented various pairs of constants. The original two-constant theory is commonly accepted as describing isotropic, linear elasticity. [3, p.121]. However, the persistence of just two constants in later developments is to be particularly noted.

• Next, another topic discussed in section §3 is the $RDFs$ which were kerneled in the "two-constant" and which provided the common, mathematical interpretation of fluid properties among the then progenitors, in particular by Gauss, a contemporary of the progenitors of the NS equations, who contributed to the formulation of fluid mechanics in the development of Laplace's capillarity.

• Then, we uncover reasons for the practice in naming these fundamental equations of fluid motion "NS equations". In Table 2, we present a chronology outlining this practice. The last entry from 1934 by Prandtl [27] grouped the equations containing three terms: (1) the nonlinear term, (2) the Laplacian term multiplied by ν, (3) the gradient term of divergence multiplied by $\frac{\nu}{3}$, which takes its rise in the fluid equation by Poisson, and used the nomenclature "the Navier-Stokes equations" for this set of equations. These equations with the two coefficients in the ratio 1 : 3 originated from Poisson [16] in 1831. Moreover, these equations contained both a linear and a nonlinear term developed earlier in Navier's equations [20] in 1827. Still earlier, the nonlinear term was introduced by Euler [7] in 1752-5. cf. Table 2.

• Finally, In section §4, we discuss Gauss' Latin paper[2] including the conceptions of microscopically-descriptive (we call it 'MD' below) formulation and RDF, which was published following the paper of the theory on curved surface [5].

2. A UNIVERSAL METHOD FOR THE "TWO-CONSTANT" THEORY

In this section, we propose a universal method to describe the kinetic equations that arise in isotropic, linear elasticity. This method is outlined as follows:

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[1] Throughout this paper, in citation of bibliographical sources, by surrounding our own paragraph or sentences of commentaries between (⇓) and (⇑) ((⇑) is used only when not following to next section,) and by $=^*$ or $\Rightarrow^*$, we detail the statement by Gauss, because we would like to discriminate and to avoid confusion from the descriptions by original authors. The mark : $\Rightarrow$ mean transformation of the statements in brevity by ours.

[2] (⇓) This free translation from Latin to English is of ours.

- The partial differential equations describing waves in elastic solids or flows in elastic fluids are expressed by using one constant or a pair of constants C_1 and C_2 such that: for elastic solids: $\frac{\partial^2 \mathbf{u}}{\partial t^2} - (C_1 T_1 + C_2 T_2) = \mathbf{f}$, for elastic fluids : $\frac{\partial \mathbf{u}}{\partial t} - (C_1 T_1 + C_2 T_2) + \cdots = \mathbf{f}$, where $T_1, T_2, \cdots$ are the terms depending on tensor quantities constituting our equations.
- The two coefficients C_1 and C_2 associated with the tensor terms are the two constants of the theory, definitions of which depend on the contributing author. For example, ε and E were introduced by Navier, R and G by Cauchy, k and K in elastic and $(K+k)\alpha$ and $\frac{(K+k)\alpha}{3}$ in fluid by Poisson, ε and $\frac{\varepsilon}{3}$ by Saint-Venant, and μ and $\frac{\mu}{3}$ by Stokes. Since Poisson, the ratio of two coefficient in fluid was fixed by 3. Moreover, C_1 and C_2 can be expressed in the following form:

$$\begin{cases} C_1 \equiv \mathcal{L} r_1 g_1 S_1. \\ C_2 \equiv \mathcal{L} r_2 g_2 S_2. \end{cases} \quad \begin{cases} S_1 = \iint g_3 \rightarrow C_3, \\ S_2 = \iint g_4 \rightarrow C_4, \end{cases} \quad \Rightarrow \quad \begin{cases} C_1 = C_3 \mathcal{L} r_1 g_1 = \frac{2\pi}{15} \mathcal{L} r_1 g_1. \\ C_2 = C_4 \mathcal{L} r_2 g_2 = \frac{2\pi}{3} \mathcal{L} r_2 g_2. \end{cases}$$

Here $\mathcal{L}$ corresponds to either $\sum_0^\infty$ as argued for by Poisson or $\int_0^\infty$ as argued for by Navier. A heated debate had developed between the two over this point. It is a matter of personnel preference as to how the two constants should be expressed.

3. THE *RDFs* KERNELED IN THE "TWO-CONSTANT"

In Table 1, we show the form of g_1 and g_2, which are kernel functions and with which the progenitors of the fluid equation developed their formulae. Here we refer to these functions as rapidly decreasing functions (*RDF*s). [3] While formulating the equilibrium equations, we obtain the competing theories of "two-constant" in capillary action between Laplace and Gauss.

In 1830, after Laplace's death, Gauss [6] started publishing his studies on capillarity following his famous paper on curved surfaces [5]. In the paper, Gauss criticized Laplace's calculations of 1805-7 in which the "two-constant" in his calculation of capillary action were introduced. At about this time, Gauss had studied what became to be called *Gaussian function* or *Gaussian curve* and using this as his *RDF* Gauss criticised Laplace's example function e^{-if} as the equivalent function of $\varphi(f)$. Here, $\varphi(f)$ is the *RDF*, which depends on distance f. In that paper, Gauss [6] pointed out various deficiencies: • 1. Laplace had mentioned only attractive action without considering the repulsive action; • 2. Laplace could not identify the correct example function as the equivalent function of the *RDF*; and • 3. Laplace lacked any proof from say a geometrical point of view. The following are Gauss' criticisms to Laplace in the preface of [6].

> • Judging from the second dissertation: ≺ *Supplément à la théorie de l'action capillaire* ≻, Mr. Laplace investigated a little, not only the complete attraction, but also the partial one by $\varphi(f)$, and tacitly understood incompletely the general attraction; by the way, if we would refer the latter by him about our sensible modification, it is easy to see being conspicuous about it. [4]
>
> • He considers exponential e^{-if} as an example of equivalent function with $\varphi(f)$, denoting the large quantity by i, or $\frac{1}{i}$ becomes infinitesimal.
>
> But it is not at all necessary to limit the generality by such a large quantity, the things are more clear than words, we would see easiest, only to investigate if these integrations would be extended, not only infinite but also to an arbitrary sensible distance, or if anything, occurring wider in the finitely measurable distance in experiment. [6, p.33]

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[3] We show the then family of *RDF* by using our notation $f \in \mathcal{RFD}$, and f is a function kernelized in the two-constant belonging to the then rapidly decreasing function.

[4] (⇓) N.Bowditch, the editor of the complete works of Laplace, cites only the title of Gauss' paper : [6] but siding with Laplace with the following comments :

> This theory of capillary attraction was first published by La Place in 1806, and in 1807 he gave a supplement. In neither of these works is the repulsive force of the heat of fluid taken into consideration, because he supposed it to be unnecessary. But in 1819, he observed that this action could be taken into account, by supposing the force $\varphi(f)$ to represent the difference between the attractive force of the particles of the fluid $A(f)$, and the repulsive force of the heat $R(f)$ so that the combined action would be expressed by, $\varphi(f) = A(f) - R(f)$; $\cdots$ [9, p.685]

Maybe this was stated under the covering fire from Gauss' criticisms of Laplace. Gauss may not have read Laplace' works after 1819 in which he had changing his thoughts. As yet we have not been able to investigate this fact.

TABLE 1. The expression of the total momentum of molecular actions by Laplace, Gauss, Navier, Cauchy, Poisson, Saint-Venant & Stokes. (Remark. 6-8 : capillarity, except for euiliblium)

no	name	problem	C_1	C_2	C_3	C_4	$\mathcal{L}$	r_1	r_2	g_1	g_2	remark
1	Navier 1827 [12]	elastic solid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f\rho$		ρ : radius
2	Navier fluid 1827 [13]	motion of fluid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f(\rho)$		ρ : radius
				E		$\frac{2\pi}{3}$	$\int_0^\infty d\rho$		ρ^2		$F(\rho)$	
3	Cauchy 1828 [2]	system of particles	R		$\frac{2\pi}{15}$		$\int_0^\infty dr$	r^3		$f(r)$		$f(r) \equiv \pm[r\mathrm{f}'(r) - \mathrm{f}(r)]$
				G		$\frac{2\pi}{3}$	$\int_0^\infty dr$		r^3		f(r)	$f(r) \neq \mathrm{f}(r)$
4	Poisson 1829 [21]	elastic solid	k		$\frac{2\pi}{15}$		$\sum \frac{1}{\alpha^5}$	r^5		$\frac{d.\frac{1}{r}fr}{dr}$		
				K		$\frac{2\pi}{3}$	$\sum \frac{1}{\alpha^5}$		r^3		fr	
5	Poisson 1831 [22]	motion of fluid	k		$\frac{1}{30}$		$\sum \frac{1}{\varepsilon^3}$	r^3		$\frac{d.\frac{1}{r}fr}{dr}$		$C_3 = \frac{1}{4\pi}\frac{2\pi}{15} = \frac{1}{30}$
				K		$\frac{1}{6}$	$\sum \frac{1}{\varepsilon^3}$		r		fr	$C_4 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
6	Laplace 1806,7 [9]	capillary action	H		2π		$\int_0^\infty dz$	z		$\Psi(z)$		z : distance
				K		2π	$\int_0^\infty dz$				$\Psi(z)$	
6-2	Rewrited by Poisson 1831 [23]		H		$\frac{\pi}{4}\rho^2$		$\int_0^\infty dr$	r^4		φr		[23, pp.14-15]
				K		$\frac{2\pi}{3}\rho^2$	$\int_0^\infty dr$		r^3		φr	
7	Gauss 1830 [6]	capillary action										attraction : $-fx.dx = d\varphi x$, $\int fx.dx = -\varphi x$, repulsion : $-Fx.dx = d\Phi x$, $\int Fx.dx = -\Phi x$
8	Poisson 1831 [23]	capillary action	H		$\frac{\pi}{4}\rho^2$		$\int_0^\infty dr$	r^4		φr		[23, p.14]
				K		$\frac{2\pi}{3}\rho^2$	$\int_0^\infty dr$		r^3		φr	[23, p.12]
9	Saint-Venant 1843 [26]	fluid	ε	$\frac{\varepsilon}{3}$								
10	Stokes 1849 [27]	fluid	μ	$\frac{\mu}{3}$								
11	Stokes 1849 [27]	elastic solid	A	B								$A = 5B$

Here, we can consider these arguments on the *RDF*s as simple examples of today's distributions and hypergeometric functions of Schwarz in 1945, but which were popular in the 1830s, during the time the *NS* equations were being discussed in their microscopically-descriptive formulation.

In his historical descriptions about the study of capillariy action, we would like to recognize that there is no counterattack to Gauss, but the correct valuation. Gauss [7] stated his conclusion about Laplace's paper "his calulations in the pages, p.44 and the followings it,have non effect in vain."

4. The *RDF* of Gauss in the capillary action

4.1. Three basic forces and two *RDFs* : f derived from φ and F derived from Φ.

We consider the force reducing to three basic forces. • I. Gravity. • II. The attractive force, which itself corresponds to the points $m, m', m'', \cdots$. The intensity of attraction of function is propotional with the

TABLE 2. The kinetic equations of the hydrodynamics until the "Navier-Stokes equations" was fixed. (Rem. HD : hydro-dynamics, N under entry-no : non-linear, gr.dv : grad.div, E : $\frac{\Delta}{gr.dv}$ of elastic, F : $\frac{\Delta}{gr.dv}$ and the group of entry 6-13 show $F = 3$ in fluid.)

no	name/prob	the kinetic equations	Δ	gr.dv	E	F
1 N	Euler (1752-55) [4, p.127] fluid	$\begin{cases} X - \frac{1}{h}\frac{dp}{dx} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ Y - \frac{1}{h}\frac{dp}{dy} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \qquad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \\ Z - \frac{1}{h}\frac{dp}{dz} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}, \end{cases}$				
2	Navier (1827)[12] elastic solid	$(6\text{-}1)_{N^{\varepsilon}} \quad \begin{cases} \frac{\Pi}{g}\frac{d^2x}{dt^2} = \varepsilon\left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dbda} + 2\frac{d^2z}{dcda}\right), \\ \frac{\Pi}{g}\frac{d^2y}{dt^2} = \varepsilon\left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dcdb}\right), \\ \frac{\Pi}{g}\frac{d^2z}{dt^2} = \varepsilon\left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right) \end{cases}$ where Π is density of the solid, g is acceleration of gravity.	ε	2ε	$\frac{1}{2}$	
3 N	Navier (1827)[13] fluid	$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\right) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\left(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\right) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\right) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w\ ; \end{cases}$	ε	2ε		$\frac{1}{2}$
4	Cauchy (1828)[2] system of particles in elastic and fluid	$\begin{cases} (L+G)\frac{\partial^2\xi}{\partial x^2} + (R+H)\frac{\partial^2\xi}{\partial y^2} + (Q+I)\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = \frac{\partial^2\xi}{\partial t^2}, \\ (R+G)\frac{\partial^2\eta}{\partial x^2} + (M+H)\frac{\partial^2\eta}{\partial y^2} + (P+I)\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = \frac{\partial^2\eta}{\partial t^2}, \\ (Q+G)\frac{\partial^2\zeta}{\partial x^2} + (P+H)\frac{\partial^2\zeta}{\partial y^2} + (N+I)\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = \frac{\partial^2\zeta}{\partial t^2}, \end{cases}$ $G = H = I, \quad L = M = N, \quad P = Q = R, \quad L = 3R$	$R+G$	$2R$	if $G = 0$ $\frac{1}{2}$	
5	Poisson (1831)[22] elastic solid in general equations	$\begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2u}{dx^2}, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2v}{dy^2}, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = \frac{\Pi}{\rho}\frac{d^2w}{dz^2}, \end{cases}$	$\frac{a^2}{3}$	$\frac{2a^2}{3}$	$\frac{1}{2}$	
6	Poisson (1831)[22] fluid in general equations	$\begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} + \alpha(K+k)\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} + \alpha(K+k)\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} + \alpha(K+k)\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \end{cases}$	β	$\frac{\beta}{3}$		3
7	Saint-Venant (1843)[26] fluid	His equations are not in his paper [26], however we are available for it by his tensor.	ε	$\frac{\varepsilon}{3}$		3
8	Stokes (1849)[27] fluid	$(12)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0. \end{cases}$	μ	$\frac{\mu}{3}$		3
9	Maxwell (1865-66) [11] HD	$\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{dp}{dx} - C_M\left[\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{dp}{dy} - C_M\left[\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho Y, \qquad \text{where, } C_M \equiv \frac{pM}{6k\rho\Theta_2} \\ \rho\frac{\partial w}{\partial t} + \frac{dp}{dz} - C_M\left[\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho Z \end{cases}$	C_M	$\frac{C}{3}$		3
10	Kirchhoff (1876)[?] HD	$\begin{cases} \mu\frac{du}{dt} + \frac{\partial}{\partial x} - C_K\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu X, \\ \mu\frac{dv}{dt} + \frac{\partial p}{\partial y} - C_K\left[\Delta v + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu Y, \\ \mu\frac{dw}{dt} + \frac{\partial p}{\partial z} - C_K\left[\Delta z + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu Z, \end{cases} \quad \begin{cases} \frac{1}{\mu}\frac{d\mu}{dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\ \text{where, } C_K \equiv \frac{1}{3\kappa}\frac{p}{\mu} \end{cases}$	C_K	$\frac{\Delta}{3}$		3
11 N	Rayleigh (1883)[25] HD	$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = -\frac{du}{dt} + \nu\nabla^2 u - u\frac{du}{dx} - v\frac{du}{dy}, \\ \frac{1}{\rho}\frac{dp}{dy} = -\frac{dv}{dt} + \nu\nabla^2 v - u\frac{dv}{dx} - v\frac{dv}{dy} \end{cases} , \quad \frac{du}{dx} + \frac{dv}{dy} = 0$	ν			
12	Boltzmann (1895)[1] HD	$(221)_B \quad \begin{cases} \rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \mathcal{R}\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} - \mathcal{R}\left[\Delta v + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho Y, \\ \rho\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} - \mathcal{R}\left[\Delta w + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho Z \end{cases}$	$\mathcal{R}$	$\frac{\mathcal{R}}{3}$		3
13 N	Prandtl (1934)[24] HD	$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$ FOR INCOMPRESSIBLE, IT IS SIMPLIFIED DIV $\mathbf{w} = 0$, $\quad \frac{D\mathbf{w}}{dt} = g - \frac{1}{\rho}\text{GRAD } p + \nu\Delta\mathbf{w}$	ν	$\frac{\nu}{3}$		3

TABLE 3. Cross-indexed differences on the *RDFs* $f \in \mathcal{RFD}$ (Remark. 1,5,6 : on capillarity)

		1	2	3	4	5
no	Name/Problem/ Bibl. (Year read) - Year published/	Laplace	Poisson	Navie	$f(r)$ at $r=0$	$f(r)$ at $r=\infty$
1	Laplace capillary action : [9] 1806-07	$L_1 : K, H$ L_2 :force attractive only and $f \simeq c^{-if}$, $f \in \mathcal{RDF}$			0	0
2	Poisson elastic : [18],(1828)-28; [21],1829;[22],(1829)-31 fluid : [22],(1829)-31 disputing origin: [18],1828 (with Navier : [19],1828;[20],1828)	Refer to Laplace's $f \in \mathcal{RFD}$	k, K	$P_1 \rightarrow N_1$: $f \simeq ab^{-(\frac{r}{n\alpha})^m}$ $P_2 \rightarrow N_2$: not by integral but by sum because $k = -K = 0$ at once. $P_3 \rightarrow N_3 : k = \varepsilon$ of Navier $P_4 \rightarrow N_4 : f \in \mathcal{RFD}$	0	0
3	Navier elastic:[12],(1821)-27 fluid:[13],(1822)-27 (with Poisson : [14],1828; [15],1829; [16],1829; [17],1829 with Arago[17],1829)	Refer to Laplace's integral	$N_1 \rightarrow P_1 : f \simeq e^{-k\rho}$ $N_2 \rightarrow P_2$: not by sum but by integral as Laplace does $N_3 \rightarrow P_3 : [r^4 f(r)]_0^\infty \neq 0$, $\varepsilon \neq k$ $N_4 \rightarrow P_4 : r^4 f(r)$ for $r = 0$, $f \in \mathcal{RFD}$ but only in $r = \infty$, $f(r) \not\to 0$ as $r \to 0$	ε in elastic ε, E in fluid	$\neq 0$	0
4	Cauchy elastic & fluid :[2]				0	0
5	Gauss capillary action : [6] (to Laplace [6],1830 to Bessel[7],1830)	$G_1 \rightarrow L_1$:Laplace's deduction is conspicuous. $G_2 \rightarrow L_2$:no necessary to limit i of c^{-if} to be very large.			-	
6	Poisson capillary action : [23],1831, (to Gauss[23])	Same K and H with Laplace			0	0

distance if this function, the $\prec$ *characteristic* $\succ$ denoted by f in mass and supposed that the attraction is uniformly concentrated in the point. • III. The forces, $m, m', m'', \cdots$ are attractive to the infinitesimal fixed points. For these forces, with the similar way, we will designate the $\prec$ *characteristic* F $\succ$ such that the inverse-directional distance is used, and with $M, M', M'', \cdots$, which are treated as a fixed point in one case, or a mass in the other case, which are supposed in these concentrate. For brevity, we express :

$$\Omega = -gc\int zds + \frac{1}{2}c^2 \iint ds.ds'.\varphi(ds, ds') + cC \iint ds.dS.\Phi(ds, dS) \tag{1}$$

where, s, s' are specially denoted spaces (satisfied with the mobile material), however we must integrate twice with the element to resolve it, because φ and Φ are defined as the functions such that : $-fx.dx = d\varphi x$, $\int fx.dx = -\varphi x$, and $-Fx.dx = d\Phi x$, $\int Fx.dx \equiv -\Phi x$. Then the integral (1) contains sextuplex integral. ($\Downarrow$) Here the integral (1) contains sextuplex integral.($\Uparrow$)

We would like to show that the spacial elements, depending on the three variables, which imply that *the sextuplex integral are to be reduced to the quadruplex integral.* [5] Our integral (I) neglecting the insensible factors : $= -\int \pi\theta'\rho.d\tau + \int \pi\theta'\rho.d\tau'$. Clearly this is not important, either the parts τ and τ' or to the surface T to t is rather important. The value of the sextuplex integral in the left hand-side of the following expression becomes

$$\iint ds.dS.\varphi(ds, dS) = 4\pi\sigma\psi 0 - \pi T\theta 0 + \pi T'\theta 0 - \pi\int d\tau.\theta'\rho + \pi\int d\tau'.\theta'\rho \tag{2}$$

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[5] ($\Downarrow$) Poisson recognizes this Gauss' achievement in [23].

4.2. **Variation problem to be solved by geometric method.**

In the application of previous survey to the evolution the second term of the expression Ω in the art. 3,in the art. 6 denote by S in the art.16 σ, T, T' will be use as $s, t, 0$, if t is the total surface of the space s, in which the fluid is filled. Therefore whenever this space extensional sensible part however insensible concentration is kept, this sort of gap (crevice), the part of the second part of the expression Ω of (1) becomes $= \frac{1}{2}\pi c^2(s\phi 0 - t\theta 0)$. In static equilibrium it is due to the maximum value, this turns into $-gc\int zds + \frac{1}{2}c^2 s\psi_0 - \frac{1}{2}\pi c^2 t\theta_0 + \pi cCT\Theta_0$. In an arbitrary fluid, of which the figure is yield oneself to the space s meaning invariant, of which the expression becomes as follows : $\int zds + \frac{\pi c\theta_0}{2g}.t - \frac{\pi CT\Theta_0}{g}.T$, and in an equilibrium state which is due to *minimum*. Here, we denote $\frac{\pi c\theta_0}{2g} \equiv \alpha^2$, $\frac{\pi CT\Theta_0}{2g} \equiv \beta^2$, $t \equiv T + U$, and by W, then

$$W \equiv \int zds + (\alpha^2 - 2\beta^2)T + \alpha^2 U \qquad (3)$$

Here, we consider : the surface, denoted by s, a part U, on which all the points is determined by the coordinate x, y, z, these three values are the distances to an arbitrary horizontal plane. It is capable to recognize z is, for example, as the indeterminated function by x, y, for these secondary partial differential with our conventional method, by omitting a bracket, we show it by $\frac{dz}{dx}.dx$, $\frac{dz}{dy}.dy$. The structure we are considering is as follows :

(1) We define the points consisted of an arbitrary and every points on the surface, denoting s with respect to the rectanglar surface, normal to the exterior direction of s, and in addition, we set an angle by cosine between this normal direction to the axis of rectanglar coordinate x, y and z with parallel, which we denote by ξ, η and ζ. Thereby it will be :

$$\xi^2 + \eta^2 + \zeta^2 = 1, \quad \frac{dz}{dx} = -\frac{\xi}{\zeta}, \quad \frac{dz}{dy} = -\frac{\eta}{\zeta}. \quad \Rightarrow^* \quad 1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 = \frac{1}{\zeta^2} \qquad (4)$$

(2) The boundary of surface U become linear in itself, as the same as denoted by P, and while the motion is supposed necessarily, this element dP (as the same way of dU as the surface) is treated as positive only.

(3) The angle by cosine, that directions of the element dP are expressed with the axis of coordinate of x, y, z, denoted by X, Y, Z : since we would avoid giving ambiguous sense about the direction, we define these angles as follows :

• at first, we assume that the normal direction in the element dP to the surface U, and draw a tangent • next, looking this line innerward, we draw the second side. • finally, in the normal direction with respect to the surface, we put the third side in the space s to the exterior, and constituting similarly the next system of three rectangles and the coordinate axis x, y, z.

Thus, we see easily the following expressions (cf. *Disquisitiones generales circa superficies curvas*), using the angle by cosine with the direction to the axis of the coordinates x, y, z are respectively

$$\eta^0 Z - \zeta^0 Y, \quad \zeta^0 X - \xi^0 Z, \quad \xi^0 Y - \zeta^0 X. \qquad (5)$$

Here, we suppose that ξ^0, η^0, ζ^0 are the values of ξ, η, ζ for the points of the element dP. (cf. (20))

Now, we assume a triangle consisted of three points : P_1, P_2, P_3.[6] We put the element of U by a triangle dU consisted of these points, of which the coordinates are : $P_1 : (x,\ y,\ z)$, $P_2 : (x + dx,\ y + dy,\ z + \frac{dz}{dx}.dx + \frac{dz}{dy}.dy)$. $P_3 : (x + d'x,\ y + d'y,\ z + \frac{dz}{dx}.d'x + \frac{dz}{dy}.d'y)$.

If we assume $dx.d'y - dy.d'x > 0$, then the twice area of this triangle is gained by our principle as follows :

$$(dx.d'y - dy.d'x)\sqrt{\left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]} \qquad (6)$$

$\Downarrow$(6) becomes $\frac{(dx.d'y - dy.d'x)}{\zeta}$ from (4). ($\Uparrow$)

- location value by perturbation of P_1 : $(x + \delta x,\ y + \delta y,\ z + \delta z)$.

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[6] ($\Downarrow$) The symbols : P_1, P_2, P_3 are of ours insted of "the first point", etc.

- Location value by perturbation of P_2 :

$$\begin{bmatrix} x+dx \\ y+dy \\ z+\frac{dz}{dx}.dx+\frac{dz}{dy}.dy \end{bmatrix} . \begin{bmatrix} \delta x+\frac{d\delta x}{dx}.dx+\frac{d\delta x}{dy}.dy \\ \delta y+\frac{d\delta y}{dx}.dx+\frac{d\delta y}{dy}.dy \\ \delta z+\frac{d\delta z}{dx}.dx+\frac{d\delta z}{dy}.dy \end{bmatrix} , \begin{bmatrix} (x+\delta x)+(1+\frac{d\delta x}{dx}).dx+\frac{d\delta x}{dy}.dy \\ (y+\delta y)+\frac{d\delta y}{dx}.dx+(1+\frac{d\delta y}{dy}).dy \\ (z+\delta z)+(\frac{dz}{dx}+\frac{d\delta z}{dx}).dx+(\frac{dz}{dy}+\frac{d\delta z}{dy}).dy \end{bmatrix}$$

- Location value by perturbation of P_3, by the same way : (omitted.)

(⇓) We can also show the matrix with only variation as follows :

$$\begin{bmatrix} \delta x & \delta y & \delta z \\ (1+\frac{d\delta x}{dx}).dx+\frac{d\delta x}{dy}.dy & \frac{d\delta y}{dx}.dx+(1+\frac{d\delta y}{dy}).dy & E.dx+D.dy \\ (1+\frac{d\delta x}{dx}).d'x+\frac{d\delta x}{dy}.d'y & \frac{d\delta y}{dx}.d'x+(1+\frac{d\delta y}{dy}).d'y & E.d'x+D.d'y \end{bmatrix} \text{ where, } \begin{cases} E\equiv\frac{dz}{dx}+\frac{d\delta z}{dx}, \\ D\equiv\frac{dz}{dy}+\frac{d\delta z}{dy} \end{cases} \tag{7}$$

By the way, these principle comes from Lagrange [8, pp.189-236], [7] in which Lagrange states his *méthode des variations* [8] in hydrostatics. (⇑) *The duplex triangles* [9] including these points, by the same method, for brevity, by denoting the sum by N, (6) is expressed as follows : $(dx.d'y - dy.d'x)\sqrt{N}$.

(⇓) These values : $dxd'y - dyd'x$. $dzd'x - dxd'z$ and $dyd'z - dzd'y$ are calculated in permutation by *Jacobian* $|J|$ of the three determinants extracted from (7) :

$$(x,y) : \begin{vmatrix} 1+\frac{d\delta x}{dx} & \frac{d\delta x}{dy} \\ \frac{d\delta y}{dx} & 1+\frac{d\delta y}{dy} \end{vmatrix} , \quad (x,z) : \begin{vmatrix} 1+\frac{d\delta x}{dx} & \frac{d\delta x}{dy} \\ E & D \end{vmatrix} , \quad (y,z) : \begin{vmatrix} 1+\frac{d\delta y}{dy} & \frac{d\delta y}{dx} \\ D & E \end{vmatrix} \qquad (⇑)$$

We denote temporarily the following sum by N, then

$$\begin{aligned} N \quad = \quad & \Big[\Big(1+\frac{d\delta x}{dx}\Big)\Big(1+\frac{d\delta y}{dy}\Big)-\frac{d\delta x}{dy}.\frac{d\delta y}{dx}\Big]^2+\Big[\Big(1+\frac{d\delta x}{dx}\Big)\Big(\frac{dz}{dy}+\frac{d\delta z}{dy}\Big)-\frac{d\delta x}{dy}\Big(\frac{dz}{dx}+\frac{d\delta z}{dx}\Big)\Big]^2 \\ + \quad & \Big[\Big(1+\frac{d\delta y}{dy}\Big)\Big(\frac{dz}{dx}+\frac{d\delta z}{dx}\Big)-\frac{d\delta y}{dx}\Big(\frac{dz}{dy}+\frac{d\delta z}{dy}\Big)\Big]^2 =^* C^2+\Big[D_1^2+D_2^2\Big]D^2+\Big[E_1^2+E_2^2\Big]E^2-2\Big[D_1E_2+E_1D_2\Big], \end{aligned}$$

$$\text{where,} \quad C\equiv\Big(1+\frac{d\delta x}{dx}\Big)\Big(1+\frac{d\delta y}{dy}\Big)-\frac{d\delta x}{dy}.\frac{d\delta y}{dx}=1+\frac{d\delta x}{dx}+\frac{d\delta y}{dy},\ D\equiv\frac{dz}{dy}+\frac{d\delta z}{dy},\ E\equiv\frac{dz}{dx}+\frac{d\delta z}{dx} \tag{8}$$

and D_1, D_2, E_1, E_2 are the two terms consisting of D and E respectively, and these coefficients are correspond to the variables of the equation on the theory of curved surface by Gauss [5]. Extending (8) with neglecting the second order of δ, for example, $\frac{d\delta x}{dy}.\frac{d\delta y}{dx}$ or $(\frac{d\delta y}{dy})^2$, etc., and for brevity, denoting the sum by L, then

$$\sqrt{N}=\Big(\Big[1+(\frac{dz}{dx})^2+(\frac{dz}{dy})^2\Big].\Big[1+\frac{L}{1+(\frac{dz}{dx})^2+(\frac{dz}{dy})^2}\Big]\Big)^{\frac{1}{2}} =^* \Big(L+1+(\frac{dz}{dx})^2+(\frac{dz}{dy})^2\Big)^{\frac{1}{2}}$$

where, L is gained by extracting only one order terms in the expanded terms from (8) :

(⇓) Here, we see the coefficient 2 included in L in (9) come from two triangles, mentioned in the footnote (9). [10]

$$\begin{aligned} N \quad & =^* \quad C^2+(\bullet)D^2+(\bullet)E^2+(\bullet)DE \\ & =^* \quad 2\Big[\frac{d\delta x}{dx}\Big\{[1+\Big(\frac{dz}{dy}\Big)^2\Big\}-\frac{dz}{dx}\frac{dz}{dy}\Big(\frac{d\delta x}{dy}+\frac{d\delta y}{dx}\Big)+\frac{d\delta y}{dy}\Big\{[1+\Big(\frac{dz}{dx}\Big)^2\Big\}+\Big(\frac{dz}{dy}\frac{d\delta z}{dy}+\frac{dz}{dx}\frac{d\delta z}{dx}\Big)\Big] \\ & + \quad \Big[1+\Big(\frac{dz}{dx}\Big)^2+\Big(\frac{dz}{dy}\Big)^2\Big] =^* 2L+\Big[1+\Big(\frac{dz}{dx}\Big)^2+\Big(\frac{dz}{dy}\Big)^2\Big] \end{aligned} \tag{9}$$

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[7] (⇓) Section 7. *De l'equilibre des fluids incompressibles*, §2. *Oú l'on déduit les dois générales de l'équibre des fluides incompressibles de la nature des particules qui les composent.* [8, pp.204-236]

[8] (⇓) Lagrange[8, p.201]. Today's mathematical nomenclature is *calculus of variations* or *calcul des variations* by *The mathematical dictionary* (4th edition in 2007) edited by MSJ, 1954, p.432, (Japanese).

[9] (⇓) The duplex triangles mean a rectangle made of two adjoining triangles.

[10] (⇓) We show the four terms in N (9) as follows :

- $C^2=\Big(1+\frac{d\delta x}{dx}+\frac{d\delta y}{dy}\Big)^2 \cong 1+2\Big(\frac{d\delta x}{dx}+\frac{d\delta y}{dy}\Big)$, $\bullet\Big[\Big(1+\frac{d\delta x}{dx}\Big)^2+\Big(\frac{d\delta y}{dx}\Big)^2\Big]D^2 \cong \Big(\frac{dz}{dy}\Big)^2+2\frac{d\delta x}{dx}\Big(\frac{dz}{dy}\Big)^2+2\frac{dz}{dy}\frac{d\delta z}{dy}$,
- $\Big[\Big(\frac{d\delta x}{dy}\Big)^2+\Big(1+\frac{d\delta y}{dy}\Big)^2\Big]E^2 \cong \Big(\frac{dz}{dx}\Big)^2+2\frac{d\delta y}{dy}\Big(\frac{dz}{dx}\Big)^2+2\frac{dz}{dx}\frac{d\delta z}{dx}$,
- $-2\Big[\Big(1+\frac{d\delta x}{dx}\Big)\frac{d\delta x}{dy}+\Big(1+\frac{d\delta y}{dy}\Big)\frac{d\delta y}{dx}\Big]DE \cong -2\frac{dz}{dx}\frac{dz}{dy}\Big(\frac{d\delta x}{dy}+\frac{d\delta y}{dx}\Big)$

(⇑) From (9)

$$\begin{aligned} L &= \Big[\frac{d\delta x}{dx}\Big\{[1+\Big(\frac{dz}{dy}\Big)^2\Big\} - \frac{dz}{dx}\frac{dz}{dy}\Big(\frac{d\delta x}{dy}+\frac{d\delta y}{dx}\Big) + \frac{d\delta y}{dy}\Big\{[1+\Big(\frac{dz}{dx}\Big)^2\Big\} + \Big(\frac{dz}{dy}\frac{d\delta z}{dy}+\frac{dz}{dx}\frac{d\delta z}{dx}\Big)\Big] \\ &=^* \frac{1}{2}\Big[N - \Big\{1+\Big(\frac{dz}{dx}\Big)^2+\Big(\frac{dz}{dy}\Big)^2\Big\}\Big] \end{aligned} \tag{10}$$

[11] Here we may recall (4), then the followings hold : the ratio of the first triangle to the second and plus 1 becomes, $1+\frac{L}{1+(\frac{dz}{dx})^2+(\frac{dz}{dy})^2} =^* 1+\frac{\text{1st triangle}}{\text{2nd triangle}} =^* 1+\zeta^2 L$. [12] Moreover, this is independent of the figure of a triangle dU, then, it turns out,

$$\delta dU = \frac{LdU}{1+(\frac{dz}{dx})^2+(\frac{dz}{dy})^2} =^* \zeta^2 LdU \tag{11}$$

Expanding L in (11) using (4) and (10), then

$$\delta dU = dU\Big[\frac{d\delta x}{dx}\Big(\eta^2+\zeta^2\Big) - \Big(\frac{d\delta x}{dy}+\frac{d\delta y}{dx}\Big)\xi\eta + \frac{d\delta y}{dy}\Big(\xi^2+\zeta^2\Big) - \frac{d\delta z}{dx}\xi\zeta - \frac{d\delta z}{dy}\eta\zeta\Big], \tag{12}$$

4.3. Integral expression by decomposing dU into dQ and dU.

From (12), all variation of the surface U is obtained by the following two integrals :

$$\int dU\Big[\Big(\eta^2+\zeta^2\Big)\frac{d\delta x}{dx} - \xi\eta\Big(\frac{d\delta y}{dx}\Big) - \xi\zeta\frac{d\delta z}{dx}\Big] \equiv A, \int dU\Big[-\xi\eta\frac{d\delta x}{dy} + \Big(\xi^2+\zeta^2\Big)\frac{d\delta y}{dy} - \eta\zeta\frac{d\delta z}{dy}\Big] \equiv B, \tag{13}$$

and these are separately treated. We consider as follows : • at first, we take the plane, rectangle to the coordinate axis y, and such as, the value determinated by itself, suitable it, it is between peripheral, the last value, which y has in the surface U. • next, for this plane, on the peripheral P, we cut in two part, or four, or six, etc., the points, of which the first coordinate will be followed by $x^0, x', x'', \cdots$; • then, as if the other quantities, we put suitablly the indicies for these points ; by the same way, we cut the surface with other plane, this infinite neighbourhood and parallel, which encounters with the second coordinate at the point of $y+dy$; • finally, between these planes, we could get the elements of peripheral $dP^0, dP', dP'', \cdots$, then we could see easily the left-hand side being expressed as follows :

$$dy = -Y^0 dP^0 = +Y'dP' = -Y''dP'' = +Y'''dP''' \text{ etc.} \tag{14}$$

If, in addition to, we consider the infinitely many planes, rectangles to the coordinate axis x, of which the element dx between x^0 and x', or between x'' and x''', or etc., it corresponds to the element : [13]

$$dU = \frac{dx.dy}{\zeta}, \tag{15}$$

$$\begin{aligned} \int \delta dU = & \int \Big[dU\Big(\eta^2+\zeta^2\Big)\frac{d\delta x}{dx} - \frac{d\delta y}{dx}\xi\eta - \frac{d\delta z}{dx}\xi\zeta\Big] + \int dU\Big[\Big(\xi^2+\zeta^2\Big)\frac{d\delta y}{dy} - \frac{d\delta x}{dy}\xi\eta - \frac{d\delta z}{dy}\eta\zeta\Big] \\ = & \; dy\int dx\frac{1}{\zeta}\Big[\Big(\eta^2+\zeta^2\Big).\frac{d\delta x}{dx} - \frac{d\delta y}{dx}\xi\eta - \frac{d\delta z}{dx}\xi\zeta\Big] + dx\int dy\frac{1}{\zeta}\Big[\Big(\xi^2+\zeta^2\Big).\frac{d\delta y}{dy} - \frac{d\delta x}{dy}\xi\eta - \frac{d\delta z}{dy}\eta\zeta\Big] \end{aligned}$$

(⇑)

Therefore, from here, it is clear for a part of integration by parts : A, that corresponds to the part of the surface depending on between the interval : y, $y+dy$, to have by the following integral, i.e., substituting the right hand-side of (15) into A of (13), then $A = dy\int dx\Big(\frac{\eta^2+\zeta^2}{\zeta}.\frac{d\delta x}{dx} - \frac{\xi\eta}{\zeta}.\frac{d\delta y}{dx} - \xi d\delta z\Big)$, by extending from $x = x^0$ to $x = x'$, next, from $x = x''$ to $x = x'''$ etc. In fact, considering the limit of this integration by parts, we express A and B by (14) and (15), as follows :

$$A = \int\Big(\frac{\eta^2+\zeta^2}{\zeta}\delta x - \frac{\xi\eta}{\zeta}\delta y - \xi\delta z\Big)YdP - \int\zeta dU\Big(\delta x\frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx} - \delta y\frac{d\frac{\xi\eta}{\zeta}}{dx} - \delta z\frac{d\xi}{dx}\Big) \tag{16}$$

$$B = \int\Big(\frac{\xi\eta}{\zeta}\delta x - \frac{\xi^2+\zeta^2}{\zeta}\delta y - \eta\delta z\Big)XdP + \int\zeta dU\Big(\delta x\frac{\frac{\xi\eta}{\zeta}}{dy} - \delta y\frac{d\frac{\xi^2+\zeta^2}{\zeta}}{dy} + \delta z\frac{d\eta}{dy}\Big) \tag{17}$$

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[11] (⇓) According to Gauss' notation, L denotes a first triangle, of which N is consisted.

[12] (⇓) The two triangles of first and second are contiguous and construct a quadrilateral by two dU.

[13] (⇓) In fact, comparerig the two expressions : (13) with (16) and (13) with (17) respectively, then this correspondence is deduced.

Here we determine for all the circumference P, we get ζQ from the first terms of both (16) and (17), $\left[X\xi\eta + Y\left(\eta^2+\zeta^2\right)\right]\delta x - \left[X(\xi^2+\zeta^2) + Y\xi\eta\right]\delta y + (X\eta\zeta - Y\xi\zeta)\delta z = \zeta Q$. Moreover, for every point of the surface U, we get V from the second terms of both (16) and (17),

$$\left(\frac{d\frac{\xi\eta}{\zeta}}{dy} - \frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx}\right)\zeta\delta x + \left(\frac{d\frac{\xi\eta}{\zeta}}{dx} - \frac{d\frac{\xi^2+\zeta^2}{\zeta}}{dy}\right)\zeta\delta y + \left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)\zeta\delta z \equiv V \tag{18}$$

That is, we can put

$$\delta U = \int QdP + \int VdU \tag{19}$$

The first integral is to be extended along all the circumference P, and the second is on all surface U.[14] Formulae for Q and V notably contradict $X\xi + Y\eta + Z\zeta = 0$,[15] Q has always the symmetric form as follows :

$$Q = (Y\zeta - Z\eta)\delta x + (Z\xi - X\zeta)\delta y + (X\eta - Y\xi)\delta z \quad \Rightarrow^{*} \quad Q = \begin{vmatrix} \delta x & \delta y & \delta z \\ X & Y & Z \\ \xi & \eta & \zeta \end{vmatrix} \tag{20}$$

When we see the form of V, we can reduce from the formulae (4), and moreover, from $\xi^2 + \eta^2 + \zeta^2 = 1$, we can deduce $\xi\frac{d\xi}{dx} + \eta\frac{d\eta}{dx} + \zeta\frac{d\zeta}{dx} = 0$, then by dividing this expression with ζ from the both side of hand, then

$$\Rightarrow \quad \frac{\xi}{\zeta}\frac{d\xi}{dx} = -\left(\frac{\eta}{\zeta}\frac{d\eta}{dx} + \frac{d\zeta}{dx}\right) \quad \Rightarrow \quad \frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx} = \eta\frac{d\frac{\eta}{\zeta}}{dx} + \left(\frac{\eta}{\zeta}.\frac{d\eta}{dx} + \frac{d\zeta}{dx}\right) = \eta\frac{d\frac{\eta}{\zeta}}{dx} - \frac{\xi}{\zeta}.\frac{d\xi}{dx} \tag{21}$$

We may replace the coefficient of $\zeta\delta x$ in V of (18), using (4) and (21),

$$\frac{d\frac{\xi\eta}{\zeta}}{dy} - \frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx} = \frac{d\frac{\xi\eta}{\zeta}}{dy} - \eta\frac{d\frac{\eta}{\zeta}}{dx} + \frac{\xi}{\zeta}.\frac{d\xi}{dx} = \left(\frac{\xi}{\zeta}\frac{d\eta}{dy} + \eta\cancel{\frac{d\frac{\xi}{\zeta}}{dy}}\right) - \eta\cancel{\frac{d\frac{\xi}{\zeta}}{dy}} + \frac{\xi}{\zeta}.\frac{d\xi}{dx} = \frac{\xi}{\zeta}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)$$

Similarly for $\zeta\delta y$, $\frac{d\frac{\xi\eta}{\zeta}}{dx} - \frac{d\frac{\xi^2+\zeta^2}{\zeta}}{dy} = \frac{\eta}{\zeta}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)$. Then V of (18) is reduced as follows : $V = (\xi\delta x + \eta\delta y + \zeta\delta z)\left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)$. Before going forward, we must illustrate conveniently the important geometrical expression. Here we restrict the various direction, we would like to present the following its intuitionally facile method, which we introduced in *Disquisitiones generales circa superficies curvas*. We consider the following geometric structure. • At first, we put the sphere, of which the radius = 1 at the center of an arbitrary surface, we denote the axis of the coordinates x, y and z by the points (1), (2) and (3), • next, taking exterior domain denoted by s, we number a point denoting by the point (4) toward the normal direction on surface ; • then, at an arbitrary point on surface, drawing various rectangle direction toward point of itself, which we denote by the point (5), • finally, for the variation of itself, we suppose that the quantity $\sqrt{\delta x^2 + \delta y^2 + \delta z^2}$ is always positive, and we denote the quantity by δe for brevity, then [16] $\delta x = \delta e.\cos(1,5)$. $\delta y = \delta e.\cos(2,5)$, $\delta z = \delta e.\cos(3,5)$.

Here, we consider the every point on the surface. In this boundary, if we call the periphery P, we can consider the two directions. (⇓) (Remark. About the expression of cos, when (•) is a unique point naming, (•, •) means the angle between two points taking an intermidiate of the origin.) (⇑) • At first, we denote the corresponding point to dP by the point (6), • next, we draw the rectangle direction to the surface, which is the inner normally-directed tangential to the surface, then we denote the point by (7), • then, by the hypothesis, these points (6), (7) and (4) look toward the same direction , [17] • finally, using above-mentionhed (1), (2) and (3) then (4, 6), (4, 7) and (6, 7) make a cube, [18] when we consider the angles as the rectangles. Thus, the above-mentioned equations (5) are transformed into

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[14] (⇓) This is what is called the *Gaussian integral formula* in two dimensions.

[15] (⇓) This means $X\xi + Y\eta + Z\zeta \neq 0$.

[16] (⇓)By the way, for understanding Gauss' method of description of angle, we can see the same method by Lagrange in 1788.

[17] (⇓) This image is considered that there are three directions emitting from a common point and making a certain angle with two directions (i.e. points.)

[18] (⇓) (4, 6), (4, 7) and (6, 7) make a plane consisting of a cube respectively.

TABLE 4. Comparison of Q and V in $\delta U = \int QdP + \int VdU$ between two methods

no	value	analytic method	geometric method
1	Q	$Q = \left(\frac{\xi\eta}{\zeta}\delta x - \frac{\xi^2+\zeta^2}{\zeta}\delta y - \eta\delta z\right)X + \left(\frac{\eta^2+\zeta^2}{\zeta}\delta x - \frac{\xi\eta}{\zeta}\delta y - \xi\delta z\right)Y$	$Q = -\delta e.\cos(5,7)$
2	V	$V = \left(\frac{d\frac{\xi\eta}{\zeta}}{dy} - \frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx}\right)\zeta\delta x + \left(\frac{d\frac{\xi\eta}{\zeta}}{dx} - \frac{d\frac{\xi^2+\zeta^2}{\zeta}}{dy}\right)\zeta\delta y + \left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)\zeta\delta z$	$V = \delta e.\cos(4,5).\left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)$ $= \delta e.\cos(4,5).\left(\frac{1}{R} + \frac{1}{R'}\right)$

$\eta Z - \zeta Y = \cos(1,7)$, $\zeta X - \xi Z = \cos(2,7)$, $\xi Y - \eta X = \cos(3,7)$. In the previous article, these forms are as follows :

$$Q = -\delta e.\cos(5,7), \quad V = \delta e.\cos(4,5).\left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right) \tag{22}$$

cos(4, 5) clearly indicates, the translation of Finally, we get the value of the right-hand side in V. [19]

$$\begin{aligned} \frac{d\xi}{dx} + \frac{d\eta}{dy} &= \frac{1}{R} + \frac{1}{R'} = -\zeta^3\Big[\frac{d^2z}{dx^2}\Big\{1 + \Big(\frac{dz}{dy}\Big)^2\Big\} - \frac{2d^2z}{dx.dy}.\frac{dz}{dx}.\frac{dz}{dy} + \frac{d^2z}{dy^2}\Big\{1 + \Big(\frac{dz}{dx}\Big)^2\Big\}\Big], \\ \text{where,}\quad \zeta^3 &= \Big[1 + \big(\frac{dz}{dx}\big)^2 + \big(\frac{dz}{dy}\big)^2\Big]^{-\frac{3}{2}}, \end{aligned} \tag{23}$$

where, R and R' are the radii of curvature respectively. From (19), (22) and (23), we get the five expressions. (I) $\delta U = \int QdP + \int VdU = -\int \delta e.\cos(5,7).dP + \int \delta e.\cos(4,5).(\frac{1}{R} + \frac{1}{R'})dU$. Evolving further the variation, for the expression W is explained by the variation of figure of the space s, we would like to start to argue at first, from the variation of the space s. Recalling that we consider that the prism with the equal sides and oriented to the solid body, then, on this point, we can see that this prism has the following relations : (II) $\delta s = \int dU.\delta e.\cos(4,5)$. (III) $\delta \int zds = \int zdU.\delta e.\cos(4,5)$. (IV) $\delta T = \int dP.\delta e.\cos(5,8)$, If we introduce here the angle $(7,8) \equiv i$ as the boundary angle, we can formulate (V) as follows : (V) $\cos(5,7) = \cos(5,8).\cos i$, where $\delta e = \sqrt{\delta x^2 + \delta y^2 + \delta z^2}$.

By the combination of above formulae I, $\cdots$, IV, we get the variational expression of W, where, W is the value of (3).

$$\delta W = \int dU.\delta e.\cos(4,5).\Big[z + \alpha^2\Big(\frac{1}{R} + \frac{1}{R'}\Big)\Big] - \int dP.\delta e.\cos(5,8).(\alpha^2\cos\ i - \alpha^2 + 2\beta^2), \tag{24}$$

where, $z + \alpha^2\Big(\frac{1}{R} + \frac{1}{R'}\Big) = \text{Const}$. If we set Const $= 0$, then $z = -\alpha^2\Big(\frac{1}{R} + \frac{1}{R'}\Big)$, and, z is the height of capillary action, α and β are the values defined in (3). From (24)

$$\delta W = -\int dP.\delta e.\cos(5,8).(\alpha^2\cos i - \alpha^2 + 2\beta^2) = \alpha^2\int dP.\delta e.\cos(5,8).\Big(1 - 2\big(\frac{\beta}{\alpha}\big)^2 - \cos\ i\Big)$$

Here, we assume A such that $\cos A = 1 - 2\sin^2(\frac{A}{2}) = 1 - 2\frac{\beta^2}{\alpha^2}$. If $\sin\frac{A}{2} = \frac{\beta}{\alpha}$, then , $\delta W = \alpha^2\int dP.\delta e.\cos(5.8).(\cos A - \cos i)$, where, the integral is to be extended along the total line P.

5. Conclusions

The "two-constant" were defined in terms of kernel functions of $RDFs$, describing the characteristics of dissipation or diffusion within isotropic and homogeneous fluids that were necessary for the interpretation of the nature of fluid or the formulation of the equations of the fluid mechanics including kinetics, equilibrium and capillarity. With their origin perhaps arising in the work of Laplace in 1805, these sorts of functions are simple examples of today's distributions and hypergeometric function of Schwarz proposed in 1945. Gauss [6] also contributed to develop fundamental conception of RDF or $MDNS$ equations for fluid mechanics including capillary action, because he formulated the equations with two-function instead of two-constant and these were the the superior method from other contemporaries with the progenitors of NS equations.

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[19](⇓) cf. Laplace [9, 10] had deduced his same expression with Gauss' (23). cf. Poisson [22], p.105.

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