数理解析研究所講究録
第 1740 巻 2011 年 167-175

MULTIPLE SIGN-CHANGING SOLUTIONS FOR AN ASYMPTOTICALLY LINEAR ELLIPTIC PROBLEM

横浜国立大学大学院・工学研究院　塩路直樹 (NAOKI SHIOJI)
FACULTY OF ENGINEERING, YOKOHAMA NATIONAL UNIVERSITY

1. Introduction

We consider the problem

$$\begin{cases} -d^2\Delta u + u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $d > 0$, $f : \mathbb{R} \to \mathbb{R}$ is an appropriate function, and Ω is a bounded domain in $\mathbb{R}^N$. (For the sake of simplicity, we consider the case $N \geq 3$.) By using the Lusternik-Schnirelmann category theory, we will give a lower estimate of the number of sign-changing solutions of (1). Such a research was first studied for positive solutions by Benci and Cerami [4]. In [6], they obtained the following result; see also [5].

Theorem 1 (Benci-Cerami). *Assume*

(i) $f \in C^1(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, $f'(0) = 0$,
(ii) *there exist* $p \in (2, 2^*)$ *and* $C > 0$ *such that* $|f'(t)| \leq C(1 + |t|^{p-2})$ *for all* $t \in \mathbb{R}$, *where* $2^* = 2N/(N-2)$,
(iii) $f'(t) > f(t)/t$ *for all* $t \neq 0$,
(iv) *there exists* $\theta > 2$ *such that* $0 < \theta \int_0^t f(s)\,ds \leq tf(t)$ *for all* $t \neq 0$.

If $d > 0$ *is small enough, then problem* (1) *has at least* $\operatorname{cat}\Omega$ *positive solutions. Moreover if* Ω *is not contractible, then it has at least one other positive solution.*

Recently, Bartsch and Weth [2,3] studied sign-changing solutions of (1). For each $\rho > 0$ and $A \subset \mathbb{R}^N$, we set

$$\Omega_\rho = \{x \in \Omega : \operatorname{dist}_{\mathbb{R}^N}(x, \partial\Omega) \geq \rho\}, \qquad \Omega^\rho = \{x \in \mathbb{R}^N : \operatorname{dist}_{\mathbb{R}^N}(x, \Omega) \leq \rho\},$$
$$C_\rho A = \{(x, y) \in A \times A : |x - y| \geq \rho\}, \quad CA = \{(x, y) \in A \times A : x \neq y\}.$$

In [3], they showed the following result:

Theorem 2 (Bartsch-Weth). *Assume* (i)–(iv). *If* $d > 0$ *is small, then problem* (1) *has at least* $\operatorname{cat}(j_\rho) + 1$ *(with* $\rho > 0$ *small) sign-changing solutions and it is greater than or equal to* $\operatorname{cupl}(C\Omega) + 2$, *where* j_ρ *is the embedding*

$$j_\rho : (C_{2\rho}\Omega_\rho \times [-1, 1]^2, C_{2\rho}\Omega_\rho \times \partial[-1, 1]^2) \hookrightarrow (C\Omega^\rho \times \mathbb{R}^2, C\Omega^\rho \times (\mathbb{R}^2 \setminus \{(0, 0)\})).$$

In this paper, we study the case that f is asymptotically linear. We show a lower estimate of the number of the sign-changing solutions of (1), which is obtained in [12]. We note that we do not assume the differentiability of f, and that since we consider the case that f is asymptotically linear, f does not satisfy the so-called Ambrosetti-Rabinowitz superlinear condition (iv). Now, we show our result:

Theorem 3. *Assume*

(f1) $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$,
(f2) $t \mapsto f(t)/t$ *is strictly increasing on* $(0, \infty)$ *and strictly decreasing on* $(-\infty, 0)$,
(f3) $f'_+(0), f'_-(0) \in [0, 1)$, *where* $f'_\pm(0) = \lim_{t \to \pm 0} f(t)/t$,
(f4) $(\mathfrak{f}_+, \mathfrak{f}_-) \in (0, \infty) \times (0, \infty)$, *where* $\mathfrak{f}_\pm = \lim_{t \to \pm\infty} f(t)/t$.

Then there exists $d_0 > 0$ *such that for each* $d \in (0, d_0)$ *such that* $(\mathfrak{f}_+ - 1, \mathfrak{f}_- - 1)$ *is not a Fučík spectrum of* $-\Delta$ *on* $H_0^1(\Omega/d)$, *problem* (1) *has at least* $\operatorname{cat}(C\Omega \times [0,1]^2, C\Omega \times \partial[0,1]^2) + 1$ *sign-changing solutions.*

Remark 1. As in [3], we can give the following estimate:

$$\operatorname{cat}(C\Omega \times [0,1]^2, C\Omega \times \partial[0,1]^2) + 1 \geq \operatorname{cupl} C\Omega + 2 \geq \max\{\operatorname{cupl}\Omega + 2, 2\operatorname{cupl}\Omega\} + 1 \geq 3.$$

In the next section, we give some preliminaries. In Section 3, we give sketch of proofs for Theorems 1 and 3.

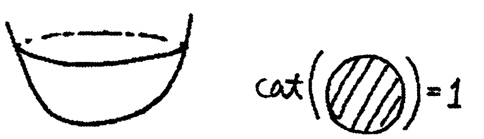
2. Preliminaries

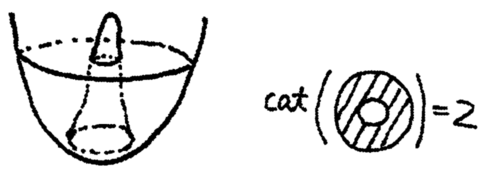
First, we recall the category in the sense of Lusternik-Schnirelmann. Let A be a topological space and let $B \subset A$. The category $\operatorname{cat}_A B$ is defined to be the least integer $n \in \mathbb{N} \cup \{0\}$ such that there exist open subsets $\{A_1, \dots, A_n\}$ (not $\{A_0, A_1, \dots, A_n\}$ as in other definitions below) of A such that $B \subset \bigcup_{i=1}^n A_i$ and each A_i is contractible in A. If there is no such open covering, we set $\operatorname{cat}_A B = \infty$. We set $\operatorname{cat} A = \operatorname{cat}_A A$ and we understand $\operatorname{cat}_A \emptyset = 0$.

By the proposition below, we can see that the category gives a lower estimate of the number of critical points in a level set of a functional which is bounded from below.

Proposition 1. *Let* H *be a Hilbert space and let* $I \in C^1(H, \mathbb{R})$ *be a functional such that it is bounded from below and it satisfies* (CPS), *i.e., each sequence* $\{u_n\} \subset H$ *satisfying* $\sup_n |I(u_n)| < \infty$ *and* $(1 + \|u_n\|)\|\nabla I(u_n)\| \to 0$ *has a convergent subsequence. Assume that* $c \in \mathbb{R}$ *is not a critical value for* I. *Then* $I^c \equiv \{u \in H : I(u) \leq c\}$ *has at least* $\operatorname{cat} I^c$ *critical points of* I.

We give simple examples for this proposition. In the figures below, we consider the case $H = \mathbb{R}^2$.





However, in general, it is difficult to find the value $\operatorname{cat} I^c$. So the conclusion of Proposition 1 may not make sense. In some cases, by the lemma below, we may give an estimate of $\operatorname{cat} I^c$.

Lemma 1. *Let Y be topological space. Assume that there exist $\alpha \in C(Y, I^c)$ and $\beta \in C(I^c, Y)$ such that $\beta \circ \alpha \simeq \mathrm{Id}_Y$, i.e., $\beta \circ \alpha$ is homotopic to Id_Y. Then $\operatorname{cat} I^c \geq \operatorname{cat} Y$.*

Now, we recall relative category for a pair of sets. Let A be a topological space and let $B \subset A$. The relative category $\operatorname{cat}(A, B)$ is defined to be the least integer $n \in \mathbb{N} \cup \{0\}$ such that there exists an open covering $\{A_0, A_1 \dots, A_n\}$ of A satisfying

- (i) $B \subset A_0$,
- (ii) there exists $h_0 \in C([0,1] \times A_0, A)$ such that $h_0(0,x) = x$ and $h_0(1,x) \in B$ for all $x \in A_0$, and $h_0(t,x) \in B$ for all $(t,x) \in [0,1] \times B$,
- (iii) each A_i is contractible in A for $i = 1, \dots, n$.

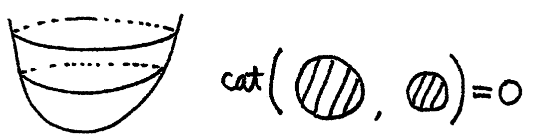
If there is no such open covering, we set $\operatorname{cat}(A, B) = \infty$.

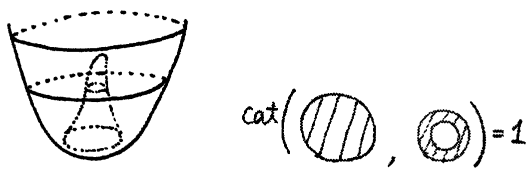
Remark 2. It holds that $\operatorname{cat} A = \operatorname{cat}(A, \emptyset)$.

By the proposition below, we can understand that the relative category also gives a lower estimate of the number of critical points.

Proposition 2. *Let H be a Hilbert space and let $I \in C^1(H, \mathbb{R})$ satisfying* (CPS)*. Assume that $a, b \in \mathbb{R}$ $(a < b)$ are not critical values of I. Then $I^b \setminus I^a$ has at least $\operatorname{cat}(I^b, I^a)$ critical points of I.*

We also give some examples for this proposition. We consider the same functionals as before; see the examples just after Proposition 1. On the right hand side example, we can understand that it has at least two critical points from Propositions 1 and 2.





Next, (in order to understand Theorem 2,) we recall category for a map. Let (A, B) and (A', B') be pairs of topological spaces, i.e., A, A' are topological spaces and $B \subset A$, $B' \subset A'$. Then $\operatorname{cat}(g)$ is defined to be the least integer $n \in \mathbb{N} \cup \{0\}$ such that there exists an open covering $\{A_0, A_1, \dots, A_n\}$ of A satisfying

- (i) $B \subset A_0$,
- (ii) there exists $h_0 \in C([0,1] \times A_0, , A')$ such that $h_0(0,x) = g(x)$ and $h_0(1,x) \in B'$ for all $x \in A_0$, and $h_0(t,x) \in B'$ for all $(t,x) \in [0,1] \times B$,
- (iii) for each $i = 1, \dots, n$, there exists $h_i \in C([0,1] \times A_i, A')$ such that $h_i(0,x) = g(x)$ and $h_i(1,x) = h_i(1,y)$ for all $x, y \in A_i$.

If there is no such open covering, we set $\operatorname{cat}(g) = \infty$.

Remark 3. It holds that $\operatorname{cat}(A, B) = \operatorname{cat}(\mathrm{Id}_{(A,B)})$ and $\operatorname{cat} A = \operatorname{cat}(A, \emptyset) = \operatorname{cat}(\mathrm{Id}_{(A,\emptyset)})$.

Finally, we recall excisive category for a pair of topological spaces. Let (A, B) be a pair of topological spaces. The excisive category $\operatorname{ecat}(A, B)$ is defined to be a least integer $n \in \mathbb{N} \cup \{0\}$ such that there exists an open covering $\{A_0, \dots, A_n\}$ of A satisfying

(i) $B \subset A_0$,

(ii) there exists $h_0 \in C([0,1] \times A_0, A)$ such that $h_0(0, x) = x$ and $h_0(1, x) \in B$ for all $x \in A_0$, and for $(t, x) \in [0,1] \times A_0$ with $h_0(t, x) \in B$, $h_0(s, x) = h_0(t, x)$ for all $s \in [t, 1]$,

(iii) for each $i = 1, \dots, n$, $A_i \cap B = \emptyset$ and A_i is contractible in $A \setminus B$.

If there is no such open covering, we set $\operatorname{ecat}(A, B) = \infty$. From the definitions, we can easily see the following:

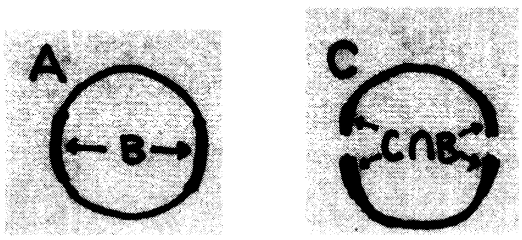
Lemma 2. $\operatorname{ecat}(A, B) \geq \operatorname{cat}(A, B)$.

The following is the property that ecat is named excisive category.

Lemma 3. *Let A be a topological space and let B, C be closed subset of A with $C \cup B = A$. Then* $\operatorname{ecat}(A, B) = \operatorname{ecat}(C, C \cap B)$.

We give simple examples for these lemmas. We can see that cat does not satisfy the lemma above. In the example below, we consider

$$A = \{(\cos\theta, \sin\theta) : \theta \in [-\pi/2, 3\pi/2)\},$$
$$B = \{(\cos\theta, \sin\theta) : \theta \in [-\pi/6, \pi/6] \cup [5\pi/6, 7\pi/6]\},$$
$$C = \{(\cos\theta, \sin\theta) : \theta \in [-\pi/2, -\pi/8] \cup [\pi/8, 7\pi/8] \cup [9\pi/8, 3\pi/2)\}.$$



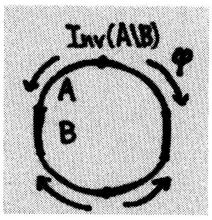
On the figures, we can easily see $\operatorname{ecat}(A, B) = 2 > 1 = \operatorname{cat}(A, B)$ and $\operatorname{ecat}(C, C \cap B) = 2 = \operatorname{cat}(C, C \cap B)$.

The following is an important property that shows why excisive category appears; see [3]. In the following, we understand that B is an "exit" set for A.

Lemma 4. *Let A be a metric space and let $\varphi : [0, \infty) \times A \to A$, a semiflow. Let B be a closed subset of A such that B is strictly positively invariant by φ, i.e., $\varphi(t, u) \in \operatorname{Int}(B)$ for each $u \in B$ and $t > 0$. Then $\operatorname{cat}_{A \setminus B}(\operatorname{Inv}(A \setminus B)) \geq \operatorname{ecat}(A, B)$, where*

$$\operatorname{Inv}(A \setminus B) = \{u \in A \setminus B : \varphi(t, u) \in A \setminus B \text{ for all } t \geq 0\}.$$

We also give a simple example for this lemma. On the figure below, we consider that A and B as in the example above and $\varphi : [0, \infty) \times A \to A$ is a semiflow as in the figure whose fixed points are $(0, -1)$, $(1, 0)$, $(0, 1)$ and $(-1, 0)$. Then we can see $\operatorname{Inv}(A \setminus B) = \{(0, -1), (0, 1)\}$ and the following:



$$\operatorname{cat}_{A\setminus B}(\operatorname{Inv}(A\setminus B)) = 2 = \operatorname{ecat}(A,B).$$

3. Sketch of Proofs of Theorems 1 and 3

In this section, we give sketch of proofs of Theorems 1 and 3. For $a \in \mathbb{R}$, we set $a^+ = \max\{a, 0\}$ and $a^- = \min\{a, 0\}$. We note that $a = a^+ + a^-$. For each domain $G \subset \mathbb{R}^N$, we consider $H_0^1(G) \subset H^1(\mathbb{R}^N)$. For $u \in H_0^1(G)$, we set $(u,v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)\, dx$ and $\|u\|^2 = (u,u)$. We recall a generalized barycenter map β on $L^2(\mathbb{R}^N)$ defined in [2]. For $u \in L^2(\mathbb{R}^N) \setminus \{0\}$, we define a bounded continuous function $\check{u}$ on $\mathbb{R}^N$ by $\check{u}(x) = \int_{B_1(x)} |u(y)|^2\, dy$ for $x \in \mathbb{R}^N$, and we set $\Lambda(u) = \{x \in \mathbb{R}^N : \check{u}(x) \geq |\check{u}|_\infty/2\}$. We define $\beta : L^2(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ by

$$\beta(u) = \frac{\int_{\Lambda(u)} x\, (\check{u}(x) - |\check{u}|_\infty/2) dx}{\int_{\Lambda(u)} (\check{u}(x) - |\check{u}|_\infty/2) dx} \quad \text{for } u \in L^2(\mathbb{R}^N) \setminus \{0\}.$$

We note that β is continuous from $L^2(\mathbb{R}^N) \setminus \{0\}$ into $\mathbb{R}^N$.

We note that problem (1) is equivalent to

$$\begin{cases} -\Delta u + u = f(u) & \text{in } \Omega/d, \\ u = 0 & \text{on } \partial\Omega/d. \end{cases} \tag{2}$$

We set

$$F_+(t) = \int_0^t f^+(s)\, ds \quad \text{for } t \in \mathbb{R}.$$

For each domain G (we consider $G = \mathbb{R}^N$ or $G = \Omega/d$), we set

$$\begin{aligned} \Phi_{G,+}(u) &= \int_G \left(\frac{1}{2}(|\nabla u|^2 + |u|^2) - F_+(u) \right) dx, \quad u \in H_0^1(G), \\ \mathcal{N}_{G,+} &= \{u \in H_0^1(G) : u^+ \neq 0, (\nabla \Phi_{G,+}(u), u) = 0\}, \\ c_{G,+} &= \inf\{\Phi_{G,+}(u) : u \in \mathcal{N}_{G,\pm}\}, \\ \mathcal{K}^+ &= \{u \in \mathcal{N}_{\mathbb{R}^N,+} : \Phi_{\mathbb{R}^N,+}(u) = c_{\mathbb{R}^N,+},\ u(0) = \max_{x \in \mathbb{R}^N} u(x)\}. \end{aligned}$$

By the concentration compactness principle, we have the following:

Proposition 3. *Let $d_n \to 0$ and $\{u_n\} \subset \mathcal{N}_{\Omega_{d_n},+}$ such that $\Phi_{\Omega/d_n,+}(u_n) \to c_{\mathbb{R}^N,+}$. Then there exist a subsequence $\{u_{n_m}\}$ of $\{u_n\}$, $\{y_m\} \subset \mathbb{R}^N$ and $v \in \mathcal{K}^+$ such that*

(i) *$y_m \in \Omega/d_{n_m}$ for all $m \in \mathbb{N}$, and $\operatorname{dist}(y_m, \partial\Omega/d_{n_m}) \to \infty$,*
(ii) *$\|u_{n_m} - v(\cdot - y_m)\| \to 0$.*

Remark 4. In the proposition above, we note that $d_n \to 0$ and $\{u_n\} \subset \mathcal{N}_{\Omega_{d_n},+}$ imply $\|\Phi_{\Omega/d_n,+}(u_n)\| \to 0$. Indeed, in Theorem 1, we treat the case that f is superlinear. We note that in the case of f is asymptotically linear, the proposition above may not hold.

We also know the following.

Lemma 5. *For each $d > 0$, $\Phi_{\Omega/d,+}$ satisfies Palais-Smale condition on $\mathcal{N}_{\Omega/d,+}$.*

Now, we give the sketch of proof of Theorem 1. Let $0 < \delta_0 \ll 1$, $0 < d \ll 1$ and set $X = \{u \in \mathcal{N}_{\Omega/d,+} : \Phi_{\Omega/d,+}(u) \le c_{\Omega/d,+} + \delta_0\}$. By the following steps, we can give the proof of Theorem 1.

(I) Since $\Phi_{\Omega/d,+}$ satisfies Palais-Smale condition on $\mathcal{N}_{\Omega/d,+}$, by a similar proof of that of Proposition 1, we can show that X has at least $\operatorname{cat} X$ critical points of $\Phi_{\Omega/d,+}$.

(II) By Proposition 3, for each $u \in X$, there exist $v \in \mathcal{K}^+$ and $y \in \Omega/d$ such that $\operatorname{dist}(y, \partial\Omega/d) \approx \infty$ and $\|u - v(\cdot - y)\| \approx 0$. Hence we have $\beta(u) \approx dy$.

(III) Fix $w \in \mathcal{K}^+$ and define $\alpha \in C(\Omega, X)$ by $\alpha(z) \approx w(\cdot - z/d)$. Then by (II), we can infer that $\beta \circ \alpha \simeq \mathrm{Id}_\Omega$. Hence by Lemma 1, we have $\operatorname{cat} X \ge \operatorname{cat} \Omega$. (Technically, α should be a continuous function from Ω_ρ (for the definition of Ω_ρ, see Section 1) into X with small $\rho > 0$ and we need a precise argument.)

(IV) In the case when Ω is not contractible, we can find a critical point u of $\Phi_{\Omega/d,+}$ such that $\Phi_{\Omega/d,+}(u) > c_{\Omega/d,+} + \delta_0$.

From (I)–(III), we can see that problem (1) has at least $\operatorname{cat}\Omega$ positive solutions, and from (IV), we can see that if Ω is not contractible, problem (1) has at least one other positive solution.

Next, we go to Theorem 3. Although f has changed to a function satisfying (f1)–(f4), we also set

$$F(t) = \int_0^t f(s)\,ds \quad \text{and} \quad F_\pm(t) = \int_0^t f^\pm(s)\,ds \quad \text{for } t \in \mathbb{R},$$

and for each domain G (we consider $G = \mathbb{R}^N$ or $G = \Omega/d$), we set

$$\begin{aligned}
\Phi_G(u) &= \int_G \left(\frac{1}{2}(|\nabla u|^2 + |u|^2) - F(u)\right) dx, \quad u \in H_0^1(G),\\
\Phi_{G,\pm}(u) &= \int_G \left(\frac{1}{2}(|\nabla u|^2 + |u|^2) - F_\pm(u)\right) dx, \quad u \in H_0^1(G),\\
\mathcal{N}_G &= \{u \in H_0^1(G) \setminus \{0\} : (\nabla\Phi_G(u), u) = 0\},\\
\mathcal{N}_{G,\pm} &= \{u \in H_0^1(G) : u^\pm \ne 0, (\nabla\Phi_{G,\pm}(u), u) = 0\},\\
c_G &= \inf\{\Phi_G(u) : u \in \mathcal{N}_G\},\\
c_{G,\pm} &= \inf\{\Phi_{G,\pm}(u) : u \in \mathcal{N}_{G,\pm}\},\\
\mathcal{K}^+ &= \{u \in \mathcal{N}_{\mathbb{R}^N,+} : \Phi_{\mathbb{R}^N,+}(u) = c_{\mathbb{R}^N,+},\ u(0) = \max_{x\in\mathbb{R}^N} u(x)\},\\
\mathcal{K}^- &= \{u \in \mathcal{N}_{\mathbb{R}^N,-} : \Phi_{\mathbb{R}^N,-}(u) = c_{\mathbb{R}^N,-},\ u(0) = \min_{x\in\mathbb{R}^N} u(x)\}.
\end{aligned}$$

For a domain $G \subset \mathbb{R}^N$, we say $(a, b) \in \mathbb{R}^2$ is a Fučík spectrum of $-\Delta$ on $H_0^1(G)$ if there exists $u \in H_0^1(G) \setminus \{0\}$ such that

$$\begin{cases} -\Delta u = au^+ + bu^- & \text{in } G,\\ u = 0 & \text{on } \partial G. \end{cases}$$

In the case that f is asymptotically linear in Theorem 3, we know that Fučík spectrum plays an important role to show the existence of solutions; for example see [1,7–11]. First, we show the following which is obtained in [12].

Theorem 4. *Either $-\Delta$ on $H^1(\mathbb{R}^N)$ or $-\Delta$ on $H_0^1(\mathbb{R}_+^N)$ does not have a Fučík spectrum, where $\mathbb{R}_+^N = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_N > 0\}$.*

From the theorem above, we can show the following:

Lemma 6. *Let $d_n \to 0$ and $\{u_n\} \subset H^1(\mathbb{R}^N)$ such that $u_n \in H_0^1(\Omega/d_n)$ for all $n \in \mathbb{N}$, $\{(1 + \|u_n\|)\|\nabla\Phi_{\Omega/d_n}(u_n)\|\}$ is bounded and $\{\Phi_{\Omega/d_n}(u_n)\}$ is bounded from above. Then $\{\|u_n\|\}$ is bounded.*

For each $\varepsilon, d > 0$, we set

$$\mathcal{M}_{\varepsilon,d} = \{u \in \Phi_{\Omega/d}^{3c_{\mathbb{R}^N}} : u^+, u^- \in \mathcal{N}_{\Omega/d}, \|\nabla\Phi_{\Omega/d}(u)\| \leq \varepsilon\}, \tag{3}$$

where

$$\Phi_{\Omega/d}^c = \{u \in H_0^1(\Omega/d) : \Phi_{\Omega/d}(u) \leq c\}, \quad c \in \mathbb{R}.$$

Using Lemma 6, we can show the following property.

Proposition 4. *Let $\varepsilon_n \to 0$, $d_n \to 0$ and $\{u_n\} \subset H^1(\mathbb{R}^N)$ such that $u_n \in H_0^1(\Omega/d_n)$ for all $n \in \mathbb{N}$, $\operatorname{dist}(u_n, \mathcal{M}_{\varepsilon_n,d_n}) \to 0$ and $\overline{\lim}_{n\to\infty} \Phi_{\Omega/d_n}(u_n) \leq c_{\mathbb{R}^N}$. Then there exist a subsequence $\{u_{n_m}\}$ of $\{u_n\}$, $\{y_m^1\}, \{y_m^2\} \subset \mathbb{R}^N$, and $v^1 \in \mathcal{K}^+, v^2 \in \mathcal{K}^-$ such that*

(i) $|y_m^1 - y_m^2| \to \infty$,
(ii) $y_m^i \in \Omega/d_{n_m}$ *for all* $m \in \mathbb{N}$, *and* $\operatorname{dist}(y_m^i, \partial\Omega/d_{n_m}) \to \infty$ $(i = 1, 2)$,
(iii) $\|u_{n_m} - v^1(\cdot - y_m^1) - v^2(\cdot - y_m^2)\| \to 0$,
(iv) $|u_{n_m}^+ - v^1(\cdot - y_m^1)|_{L^2} \to 0$ *and* $|u_{n_m}^- - v^2(\cdot - y_m^2)|_{L^2} \to 0$.

Remark 5. Since we define $\mathcal{M}_{\varepsilon,d}$ by (3), we can show the proposition above. See Remark 4.

Fix $0 < \varepsilon_0 \ll 1$.

Lemma 7. *There exist $\delta_0 \in (0, \varepsilon_0)$ and $d_0 > 0$ such that*

$$\|\nabla\Phi_d(u)\| \geq \frac{24\delta}{\varepsilon_0}$$

for each $\delta \in (0, \delta_0)$, $d \in (0, d_0)$ and $u \in \Phi_d^{c_{\Omega/d}+2\delta}$ with $\varepsilon_0/2 < \operatorname{dist}(u, \mathcal{M}_{\varepsilon_0,d}) \leq \varepsilon_0$.

Fix $0 < d \ll 1$ such that $(\mathfrak{f}_+ - 1, \mathfrak{f}_- - 1)$ is not a Fučík spectrum of $-\Delta$ on $H_0^1(\Omega/d)$. Then we can show the following:

Lemma 8. *$\Phi_{\Omega/d}$ satisfies* (CPS).

Let $\varphi : [0, \infty) \times H_0^1(\Omega/d) \to H_0^1(\Omega/d)$ defined by

$$\begin{cases} \varphi(0, u) = u, \\ \dfrac{\partial\varphi}{\partial t}(t, u) = -\dfrac{(1 + \|\varphi(t, u)\|)^2\nabla\Phi_d(\varphi(t, u))}{(1 + \|\varphi(t, u)\|)^2\|\nabla\Phi_d(\varphi(t, u))\|^2 + 1}. \end{cases}$$

Here, for the sake of simplicity, we assume that $\nabla\Phi_d$ is locally Lipschitz. Technically, we need to approximate $\nabla\Phi_d$ by a pseudo-gradient vector field.

Fix $0 < a_0 \ll 1$, and we set

$$\mathcal{P} = \{u \in H_0^1(\Omega/d) : u \geq 0\},$$
$$\mathcal{D}_{a_0} = \{u \in H_0^1(\Omega/d) : \mathrm{dist}(u, \mathcal{P} \cup -P) \leq a_0\}.$$

Since $0 < a_0 \ll 1$, we can show the following:

Lemma 9. *$\mathcal{D}_{a_0}$ is strictly positively invariant by φ.*

We fix $T_0 \gg 1$ and we set

$$A_0 = \{u \in \Phi_{\Omega/d}^{c_{\Omega/d}+\delta_0} : \mathrm{dist}(u, \mathcal{M}_{\varepsilon_0,d}) \leq \varepsilon_0\},$$
$$\mathcal{E}_{T_0} = \{u \in H_0^1(\Omega/d) : \varphi(T_0, u) \in \mathcal{D}_{a_0} \cup \Phi_d^{c_d-\delta_0}\}.$$

Since $T_0 \gg 1$, we can show the following by using Lemma 7:

Lemma 10. $\Phi_{\Omega/d}^{c_d+\delta_0} \cup \mathcal{E}_{T_0} = A_0 \cup \mathcal{E}_{T_0}$.

Since for each $v \in H^1(\mathbb{R}^N)\setminus\{0\}$, $t \mapsto \int_{\mathbb{R}^N} f(tv)v/t\,dx : [0,\infty) \to \mathbb{R}$ is strictly increasing, for each $u \in H^1(\mathbb{R}^N)$, we can define $\tau(u) \in (0,\infty]$ by

$$\tau(u) = \begin{cases} t \in (0,\infty) \text{ satisfying } \|u\|^2 = \displaystyle\int_{\mathbb{R}^N} \frac{f(tu)u}{t}\,dx & \text{if } \|u\|^2 < \displaystyle\lim_{t\to\infty}\int_{\mathbb{R}^N} \frac{f(tu)}{t}u\,dx, \\ \infty & \text{otherwise.} \end{cases}$$

Now, we give the sketch of proof of Theorem 3.

(I) Since $\Phi_{\Omega/d}$ satisfies (CPS) and $\mathcal{E}_{T_0}$ is a closed subset of $\Phi_{\Omega/d}^{c_{\Omega/d}+\delta_0} \cup \mathcal{E}_{T_0}$ which is strictly positively invariant by φ, by a similar proof of that of Proposition 2, we can show that there exist at least $\mathrm{cat}_{\Phi_{\Omega/d}^{c_{\Omega/d}+\delta_0}\setminus\mathcal{E}_{T_0}}(\mathrm{Inv}(\Phi_{\Omega/d}^{c_{\Omega/d}+\delta_0}\setminus\mathcal{E}_{T_0}))$ critical points of $\Phi_{\Omega/d}$.

(II) For each $u \in A_0$, there exist $v^1 \in \mathcal{K}^+$, $v^2 \in \mathcal{K}^-$, $y^1, y^2 \in \Omega/d$ such that $|y^1 - y^2| \approx \infty$, $\mathrm{dist}(y^i, \partial\Omega/d) \approx \infty$, $\|u^+ - v^1(\cdot - y^1)\| \approx 0$ and $\|u^- - v^2(\cdot - y^2)\| \approx 0$ by Proposition 4. Hence, we have $(\beta(u^+), \beta(u^-)) \approx (dy^1, dy^2)$ for all $u \in A_0$.

(III) By Lemmas 9, 4, 3 and 2, we have

$$\begin{aligned}\mathrm{cat}_{\Phi_{\Omega/d}^{c_{\Omega/d}+\delta_0}\setminus\mathcal{E}_{T_0}}(\mathrm{Inv}(\Phi_{\Omega/d}^{c_{\Omega/d}+\delta_0} \setminus \mathcal{E}_{T_0})) &\geq \mathrm{ecat}\,(\Phi_{\Omega/d}^{c_{\Omega/d}+\delta_0} \cup \mathcal{E}_{T_0}, \mathcal{E}_{T_0}) \\ &= \mathrm{ecat}\,(A_0, A_0 \cap \mathcal{E}_{T_0}) \geq \mathrm{cat}\,(A_0, A_0 \cap \mathcal{E}_{T_0}).\end{aligned}$$

We fix $w_+ \in \mathcal{K}^+$, $w_- \in \mathcal{K}^-$, and we define $h : \{u \in H_0^1(\Omega/d) : u^\pm \neq 0\} \to C\Omega \times [0,1]^2$ and $\alpha : C\Omega \times [0,1]^2 \to \{u \in H_0^1(\Omega/d) : u^\pm \neq 0\}$ by

$$h(u) = (d\beta(u^+), d\beta(u^-), \chi(\tau(\varphi(T_0,u)^+)), \chi(\tau(\varphi(T_0,u)^-))),$$
$$\alpha(x_1, x_2, s_1, s_2) \approx s_1 w_+(\cdot - x_1/d) + s_2 w_-(\cdot - x_2/d),$$

where $\chi : (0,\infty) \to [0,1]$ is an appropriate strictly decreasing function with $\chi(1) = 1/2$. Using (II) and some properties of τ, we can show $\beta \circ \alpha$ is homotopic to the identity mapping on $(C\Omega \times [0,1]^2, C\Omega \times \partial[0,1]^2)$. Then by a similar proof of that of Lemma 1, we can also show

$$\text{cat}\,(A_0, A_0 \cap \mathcal{E}_{T_0}) \geq \text{cat}(C\Omega \times [0,1]^2, C\Omega \times \partial[0,1]^2).$$

(For the precise argument, we need to define h and α in a little bit different way.)

(IV) There exists at least one other critical point $u \in H_0^1(\Omega/d)$ such that it is sign-changing and $\Phi_{\Omega/d}(u) > c_{\Omega/d} + \delta_0$.

From these steps, we can find that problem 1 has at least $\text{cat}(C\Omega \times [0,1]^2, C\Omega \times \partial[0,1]^2) + 1$ sign-changing solutions.

References

[1] T. Bartsch, Z.-Q. Wang, and Z. Zhang, *On the Fučik point spectrum for Schrödinger operators on* $\mathbb{R}^N$, J. Fixed Point Theory Appl. **5** (2009), no. 2, 305–317.

[2] T. Bartsch and T. Weth, *Three nodal solutions of singularly perturbed elliptic equations on domains without topology*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), no. 3, 259–281.

[3] ———, *The effect of the domain's configuration space on the number of nodal solutions of singularly perturbed elliptic equations*, Topol. Methods Nonlinear Anal. **26** (2005), no. 1, 109–133.

[4] V. Benci and G. Cerami, *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems*, Arch. Rational Mech. Anal. **114** (1991), no. 1, 79–93.

[5] V. Benci, G. Cerami, and D. Passaseo, *On the number of the positive solutions of some nonlinear elliptic problems*, Nonlinear analysis, Sc. Norm. Super. di Pisa Quaderni, Scuola Norm. Sup., Pisa, 1991, pp. 93–107.

[6] V. Benci and G. Cerami, *Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology*, Calc. Var. Partial Differential Equations **2** (1994), no. 1, 29–48.

[7] S. Carl and K. Perera, *Sign-changing and multiple solutions for the p-Laplacian*, Abstr. Appl. Anal. **7** (2002), no. 12, 613–625.

[8] E. N. Dancer and Z. Zhang, *Fucik spectrum, sign-changing, and multiple solutions for semilinear elliptic boundary value problems with resonance at infinity*, J. Math. Anal. Appl. **250** (2000), no. 2, 449–464.

[9] S. Li and Z. Zhang, *Fucik spectrum, sign-changing and multiple solutions for semilinear elliptic boundary value problems with jumping nonlinearities at zero and infinity*, Sci. China Ser. A **44** (2001), no. 7, 856–866.

[10] M. Tanaka, *Existence of a non-trivial solution for the p-Laplacian equation with Fučík-type resonance at infinity. III*, Nonlinear Anal. **72** (2010), no. 1, 507–526.

[11] Z. Zhang and S. Li, *On sign-changing and multiple solutions of the p-Laplacian*, J. Funct. Anal. **197** (2003), no. 2, 447–468.

[12] N. Shioji, *Existence of multiple sign-changing solutions for an asymptotically linear elliptic problem and the topology of the configuration space of the domain*, preprint.