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On the attainability for the best constant of the Sobolev-Hardy type inequality

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Abstract

We consider the existence of a minimizer for the best constant of the Hardy-Sobolev type inequality in arbitrary bounded smooth domain with $0\in\partial\Omega$. The Hardy-Sobolev inequality states that $\left(\int_{\Omega}\frac{|u|^{2^{*}}}{|x|^{s}}dx\right)^{\frac{2}{2^{*}}}\leqq C\int_{\Omega}|\nabla u|^{2}dx$ holds for all $u\in H_{0}^{1}(\Omega)$, where $n\geqq 3, 0<s<2$ and $2^{*}=2^{*}(s)=\frac{2(n-s)}{n-2}$. N.Ghoussoub and F.Robert[4] showed that the negativity of the mean curvature at 0 guarantees the attainability in the case $n\geqq 4$. In this paper, we treat the following minimizing problem, i.e.,

$$\mu_{s,p}^{\pm\lambda}(\Omega):=\inf\left\{\frac{\int_{\Omega}|\nabla u|^{2}dx\pm\lambda\left(\int_{\Omega}|u|^{p}dx\right)^{\frac{2}{p}}}{\left(\int_{\Omega}\frac{|u|^{2^{*}}}{|x|^{s}}dx\right)^{\frac{2}{2^{*}}}}; u\in H_{0}^{1}(\Omega)\setminus\{0\}\right\},$$

where $2\leqq p<\frac{2n}{n-2}$ and λ is a nonnegative constant. Our purpose is to make sure that the situation concerning the attainability is different between $\mu_{s,p}^{+\lambda}(\Omega)$ and $\mu_{s,p}^{-\lambda}(\Omega)$. In fact, the attainability of $\mu_{s,p}^{+\lambda}(\Omega)$ depends on the geometric assumption for Ω. On the other hand, $\mu_{s,p}^{-\lambda}(\Omega)$ can be achieved for any domain if $\frac{2n}{n-1}<p<\frac{2n}{n-2}$. These results are already generalized in the paper [6] by the same authors. In [6], we gave relatively a simple proof than the method by N.Ghoussoub and F.Robert[4]. However, in order to understand the detailed proof in [4], we followed their method in this article with the more general setting.

1 Introduction and main theorems

In this paper, we consider the attainability of the Sobolev-Hardy type inequalities. Let $n\geqq 3, s\in[0,2]$ and $2^{*}=2^{*}(s)=\frac{2(n-s)}{n-2}$. Then the Sobolev-Hardy inequality states that there exists a constant $C>0$ such that

$$\left(\int_{\mathbb{R}^{n}}\frac{|u|^{2^{*}}}{|x|^{s}}dx\right)^{\frac{2}{2^{*}}}\leqq C\int_{\mathbb{R}^{n}}|\nabla u|^{2}dx \tag{1.1}$$

holds for all $u \in H^1(\mathbb{R}^n)$. In what follows, let Ω be a domain in $\mathbb{R}^n$ and let $\mu_s(\Omega)$ be the sharp constant of (1.1), i.e.,

$$\mu_s(\Omega) := \inf \left\{ \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega \frac{|u|^{2^*}}{|x|^s} dx\right)^{\frac{2}{2^*}}} ; u \in H_0^1(\Omega) \setminus \{0\} \right\}.$$

Firstly, we mention the classical facts concerning $\mu_s(\mathbb{R}^n)$. E.H.Lieb[5] and G.Talenti[8] gave the exact values of $\mu_s(\mathbb{R}^n), 0 \leqq s < 2$ with minimizers of the form,

$$u(x) = (\kappa + |x|^{2-s})^{-\frac{n-2}{2-s}} \quad \text{for } x \in \mathbb{R}^n \text{ and } \kappa > 0.$$

Then the sharp constant of the Hardy inequality ($s = 2$) is obtained by $\mu_2(\mathbb{R}^n) = \lim_{s \uparrow 2} \mu_s(\mathbb{R}^n)$. However, H.Egnell[2] showed that $\mu_2(\mathbb{R}^n)$ is never attained. Next, it is well-known that in the non-singular case $s = 0$, $\mu_0(\Omega)$ is never attained provided $\Omega \neq \mathbb{R}^n$ (see for example M.Struwe[7]). The situation of the singular case $0 < s < 2$ is more complicated. H.Egnell[2] investigated the attainability of $\mu_s(\Omega)$ in the case that Ω is a cone Γ, which is defined by

$$\Gamma := \{x \in \mathbb{R}^n \ ; \ x = r\theta, \theta \in D, r > 0\},$$

where D is a domain in the unit sphere S^{n-1} in $\mathbb{R}^n$. Then it was proved that $\mu_s(\Gamma)$ can be achieved even if $\Gamma \neq \mathbb{R}^n$. The result of H.Egnell would make the motivation to consider $\mu_s(\Omega)$ with $0 \in \partial\Omega$ for general domains. In such a viewpoint, we refer to N.Ghoussoub and X.S.Kang[3]. Let Ω be a C^2-smooth domain in $\mathbb{R}^n, n \geqq 3$ with $0 \in \partial\Omega$. In [3], it was shown that $\mu_s(\Omega)$ is never attained provided Ω can be put into the half space $\mathbb{R}^n_-$ up to some rotation except for $\Omega = \mathbb{R}^n_-$. On the other hand, when $n \geqq 4$, the negativity of all principal curvatures of $\partial\Omega$ at 0 guarantees the attainability for $\mu_s(\Omega)$. Recently, the latter assertion was improved in N.Ghoussoub and F.Robert[4] so that the negativity of the mean curvature of $\partial\Omega$ at 0 implies the attainability under the slightly stronger assumption concerning the regularity for Ω.

Our purpose in this paper is to investigate the results in [3] and [4] with a lower perturbation, which means that we consider the following infimum,

$$\mu_{s,p}^{\pm\lambda}(\Omega) := \inf \left\{ \frac{\int_\Omega |\nabla u|^2 dx \pm \lambda \left(\int_\Omega |u|^p dx\right)^{\frac{2}{p}}}{\left(\int_\Omega \frac{|u|^{2^*}}{|x|^s} dx\right)^{\frac{2}{2^*}}} ; u \in H_0^1(\Omega) \setminus \{0\} \right\},$$

where $n \geqq 3, 2 \leqq p < \frac{2n}{n-2}$ and let Ω be a bounded (As for $\mu_{s,p}^{+\lambda}(\Omega)$, we necessarily need not assume the boundedness of Ω) domain with $0 \in \partial\Omega$. In addition, λ is a nonnegative constant such that

$$\begin{cases} \lambda \geqq 0 \quad \text{in} \quad \mu_{s,p}^{+\lambda}(\Omega), \\ 0 < \lambda < \Lambda_p \quad \text{in} \quad \mu_{s,p}^{-\lambda}(\Omega), \end{cases} \tag{1.2}$$

where Λ_p denotes the best constant of the Sobolev embedding, i.e.,

$$\Lambda_p := \inf \left\{ \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^p dx\right)^{\frac{2}{p}}} ; u \in H_0^1(\Omega) \setminus \{0\} \right\}.$$

We state our main results, which clarify the difference between $\mu_{s,p}^{+\lambda}(\Omega)$ and $\mu_{s,p}^{-\lambda}(\Omega)$ as for the minimizing problem. First, concerning $\mu_{s,p}^{+\lambda}(\Omega)$, we shall show the following.

Theorem 1.1. *(i) Let $n \geqq 3, s \in (0,2), 2 \leqq p < \frac{2n}{n-2}, \lambda \geqq 0$ and let Ω be a C^1-smooth domain with $0 \in \partial\Omega$. In addition, assume that Ω can be put into the half space $\mathbb{R}^n_-$. Then $\mu_{s,p}^{+\lambda}(\Omega) = \mu_s(\mathbb{R}^n_-)$ holds and $\mu_{s,p}^{+\lambda}(\Omega)$ is never attained provided $\Omega \neq \mathbb{R}^n_-$.*

(ii) Let $n \geqq 4, s \in (0,2), 2 \leqq p < \frac{2n}{n-1}, \lambda \geqq 0$ and let Ω be a smooth bounded domain with $0 \in \partial\Omega$. In addition, assume that the mean curvature of $\partial\Omega$ at 0 is negative. Then $\mu_{s,p}^{+\lambda}(\Omega)$ is attained.

Remark 1.2. *(i) With some technical reason, we cannot obtain the similar result for $n = 3$ and for the region $\frac{2n}{n-1} \leqq p < \frac{2n}{n-2}$ in Theorem 1.1 (ii). Theorem 1.1 implies that the attainability depends on the geometric assumption for the domain Ω at least for $n \geqq 4$ and for $2 \leqq p < \frac{2n}{n-1}$.*

(ii) The case $\lambda = 0$ in Theorem 1.1 (ii) coincides with the result in N.Ghoussoub and F.Robert[4] and our generalization is basically based on the strategy of them.

Next, we state the results concerning the attainability for $\mu_{s,p}^{-\lambda}(\Omega)$.

Theorem 1.3. *Let $n \geqq 3, s \in (0,2), \frac{2n}{n-1} < p < \frac{2n}{n-2}, 0 < \lambda < \Lambda_p$ and let Ω be a C^2-smooth bounded domain with $0 \in \partial\Omega$. Then the infimum $\mu_{s,p}^{-\lambda}(\Omega)$ is achieved.*

Remark 1.4. *Theorem 1.3 implies that we no longer require the geometric assumption for the domain Ω provided p is big enough. Moreover, Theorem 1.1 implies that the condition $\lambda > 0$ cannot be removed in general. In the end, we note that the case $n = 3$ is also allowed in our statement.*

Theorem 1.5. *Let $s \in (0,2)$,*

$$\begin{cases} 2 < p < \frac{2n}{n-2} & \text{if} \quad n = 4, \\ 2 \leqq p < \frac{2n}{n-2} & \text{if} \quad n \geqq 5, \end{cases}$$

$0 < \lambda < \Lambda_p$ and let Ω be a C^2-smooth bounded domain with $0 \in \partial\Omega$. In addition, assume that Ω is flat near the origin. Then the infimum $\mu_{s,p}^{-\lambda}(\Omega)$ is achieved.

Remark 1.6. *The assumption that the domain Ω is flat near the origin allows us to obtain the attainability of $\mu_{s,p}^{-\lambda}(\Omega)$ for all $2 \leqq p < \frac{2n}{n-2}$, though $p = 2$ is excluded if $n = 4$. Unfortunately, we cannot obtain the corresponding fact in $n = 3$ because of the technical reason. Furthermore, as is mentioned in the previous remark, the case $\lambda = 0$ is still excluded under the situation in Theorem 1.5.*

For the proofs of main theorems, we first investigate the minimizing problem in the subcritical case, i.e.,

$$\mu_{s,p}^{\pm\lambda,\varepsilon}(\Omega) := \inf \left\{ \frac{\int_\Omega |\nabla u|^2 dx \pm \lambda \left(\int_\Omega |u|^p dx \right)^{\frac{2}{p}}}{\left(\int_\Omega \frac{|u|^{2^*-\varepsilon}}{|x|^s} dx \right)^{\frac{2}{2^*-\varepsilon}}} ; u \in H_0^1(\Omega) \setminus \{0\} \right\}, \tag{1.3}$$

where $\varepsilon \in (0, 2^* - 2)$. Then the compactness can be recovered and then the infimum $\mu_{s,p}^{\pm\lambda,\varepsilon}(\Omega)$ is achieved by a positive function $u_\varepsilon^\pm \in H_0^1(\Omega)$, see Proposition 2.1. The fact that $u_\varepsilon^\pm$ is a minimizer for $\mu_{s,p}^{\pm\lambda,\varepsilon}(\Omega)$ and the corresponding Euler-Lagrange equation satisfied by $u_\varepsilon^\pm$ tell us the boundedness of the norm $\|\nabla u_\varepsilon^\pm\|_{L^2(\Omega)}$ as $\varepsilon \to 0$. Then up to a subsequence, $u_\varepsilon^\pm$ converges to some function $u_0^\pm$ weakly in $H_0^1(\Omega)$ as $\varepsilon \to 0$. We shall show that $u_0^\pm$ is a minimizer for $\mu_{s,p}^{\pm\lambda}(\Omega)$ provided $u_0^\pm \neq 0$, respectively, see Proposition 2.2. On the other hand, section 3 is devoted to discuss the blow-up case $u_0^\pm = 0$. The goal of section 3 is to prove that the equality

$$\mu_{s,p}^{\pm\lambda}(\Omega) = \mu_s(\mathbb{R}_-^n)$$

holds if the blow-up case occurs, see Proposition 3.1. In section 4, we shall show main theorems. However, the proof of Theorem 1.1 and those of Theorems 1.3 and 1.5 are different. In the case of $\mu_{s,p}^{+\lambda}(\Omega)$, we prove that the blow-up case never occurs by using the negativity of the mean curvature at 0. On the other hand, we complete the proofs of of Theorems 1.3 and 1.5 by proving the strict inequality $\mu_{s,p}^{-\lambda}(\Omega) < \mu_s(\mathbb{R}_-^n)$.

2 Non blow-up case

We first note that a C^m-smooth domain $\Omega, m \in \mathbb{N}$ is expressed as the following which will be used throughout the paper. Let $x_0 \in \partial\Omega$. Then there exist an open interval $I \subset \mathbb{R}$, an open set $U' \subset \mathbb{R}^{n-1}$, an open set $V \subset \mathbb{R}^n$, a C^m-diffeomorphism $\varphi \in C^m(U, V), U = I \times U'$ and a function $\varphi_0 \in C^m(U')$ such that

$$\begin{aligned}
&\text{(i)}\, 0 \in U, x_0 \in V \text{ and } \varphi(0) = x_0\,;\\
&\text{(ii)}\, \varphi(U \cap \{x_1 < 0\}) = V \cap \Omega \text{ and } \varphi(U \cap \{x_1 = 0\}) = V \cap \partial\Omega\,;\\
&\text{(iii)}\, \varphi(x) = x_0 + (x_1 + \varphi_0(x'), x') \text{ for } x = (x_1, x') \in I \times U' = U\,;\\
&\text{(iv)}\, \varphi_0(0) = 0 \text{ and } \nabla'\varphi_0(0) = 0, \nabla' = (\partial_2, \cdots \partial_n).
\end{aligned}$$

Lemma 2.1. *Let $2 \leqq p < \frac{2n}{n-2}, 0 < s < 2, \lambda$ as in (1.2) and let Ω be a C^1-smooth domain with $0 \in \partial\Omega$ (As for $\mu_{s,p}^{-\lambda}(\Omega)$, we assume the boundedness for Ω). Then it follows*

$$\mu_{s,p}^{\pm\lambda}(\Omega) \leqq \mu_s(\mathbb{R}_-^n).$$

Proof. The proof of Lemma 2.1 will be done in a quite similar way as in Ghoussoub-Robert[4, Proposition 3.1] without any modification. Hence, we omit it here. □

Since the minimizing problem for $\mu_{s,p}^{\pm\lambda,\varepsilon}(\Omega)$ does not include any noncompact term. Thus by virtue of the compactness, the following proposition is elemental, and we give the statement without the proof.

Proposition 2.1. *Let $2 \leqq p < \frac{2n}{n-2}, 0 < s < 2, \lambda$ as in (1.2) and let Ω be a $C^{0,1}$-smooth bounded domain with $0 \in \overline{\Omega}$. In addition, for arbitrary $\varepsilon \in (0, 2^* - 2)$, define $\mu_{s,p}^{\pm\lambda,\varepsilon}(\Omega)$ as in (1.3). Then the infimum $\mu_{s,p}^{\pm\lambda,\varepsilon}(\Omega)$ is achieved by a nonnegative function $u_\varepsilon^\pm \in H_0^1(\Omega) \cap C(\overline{\Omega}) \cap C^2(\overline{\Omega} \setminus \{0\})$ satisfying the following equation,*

$$-\Delta u_\varepsilon^\pm = \mp\lambda \|u_\varepsilon^\pm\|_{L^p(\Omega)}^{-(p-2)} (u_\varepsilon^\pm)^{p-1} + \frac{(u_\varepsilon^\pm)^{2^*-1-\varepsilon}}{|x|^s} \quad \text{in } \Omega. \tag{2.1}$$

Furthermore, the strong maximum principle yields that $u_\varepsilon^- > 0$ *in* Ω.

Next, we prove that a minimizer of $\mu_{s,p}^{\pm\lambda}(\Omega)$ can be obtained as a limit-function of the minimizers $u_\varepsilon^\pm$ for $\mu_{s,p}^{\pm\lambda,\varepsilon}(\Omega)$ in the non blow-up case. It is easy to prove the continuity of $\mu_{s,p}^{\pm\lambda,\varepsilon}(\Omega)$ as $\varepsilon \to 0$, i.e., we have the following lemma. Its proof will be omitted here.

Lemma 2.2. *Let* $2 \leqq p < \frac{2n}{n-2}, 0 < s < 2, \lambda$ *as in* (1.2) *and let* Ω *be a bounded domain with* $0 \in \overline{\Omega}$. *Then it follows* $\lim_{\varepsilon\to 0} \mu_{s,p}^{\pm\lambda,\varepsilon}(\Omega) = \mu_{s,p}^{\pm\lambda}(\Omega)$, *respectively.*

Next, let u_ε^- be a minimizer of $\mu_{s,p}^{-\lambda,\varepsilon}(\Omega)$ given by Proposition 2.1. Taking u_ε^- as a test function in the equation (2.1), we have

$$\int_\Omega |\nabla u_\varepsilon^-|^2 dx - \lambda \left(\int_\Omega (u_\varepsilon^-)^p dx\right)^{\frac{2}{p}} = \int_\Omega \frac{(u_\varepsilon^-)^{2^*-\varepsilon}}{|x|^s} dx. \tag{2.2}$$

Then with (2.2) and the fact that u_ε^- is a minimizer, we see that

$$\mu_{s,p}^{-\lambda,\varepsilon}(\Omega) = \frac{\int_\Omega |\nabla u_\varepsilon^-|^2 dx - \lambda \left(\int_\Omega (u_\varepsilon^-)^p dx\right)^{\frac{2}{p}}}{\left(\int_\Omega \frac{(u_\varepsilon^-)^{2^*-\varepsilon}}{|x|^s} dx\right)^{\frac{2}{2^*-\varepsilon}}} = \left(\int_\Omega \frac{(u_\varepsilon^-)^{2^*-\varepsilon}}{|x|^s} dx\right)^{\frac{2^*-2-\varepsilon}{2^*-\varepsilon}}. \tag{2.3}$$

Hence, from (2.2), (2.3) and Lemma 2.2 it follows that

$$\int_\Omega |\nabla u_\varepsilon^-|^2 dx \leqq \frac{\Lambda_p}{\Lambda_p - \lambda} \left(\int_\Omega |\nabla u_\varepsilon^-|^2 dx - \lambda \left(\int_\Omega (u_\varepsilon^-)^p dx\right)^{\frac{2}{p}}\right) = \frac{\Lambda_p}{\Lambda_p - \lambda} \int_\Omega \frac{(u_\varepsilon^-)^{2^*-\varepsilon}}{|x|^s} dx$$
$$= \frac{\Lambda_p}{\Lambda_p - \lambda} \mu_{s,p}^{-\lambda,\varepsilon}(\Omega)^{\frac{2^*-\varepsilon}{2^*-2-\varepsilon}} \to \frac{\Lambda_p}{\Lambda_p - \lambda} \mu_{s,p}^{-\lambda}(\Omega)^{\frac{2^*}{2^*-2}}$$

as $\varepsilon \to 0$. Therefore, we see that there exist $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0, 2^*-2)$ with $\varepsilon_j \to 0$ as $j \to \infty$ and $u_0^- \in H_0^1(\Omega)$ such that

$$\begin{cases} u_{\varepsilon_j}^- \to u_0^- & \text{weakly in } H_0^1(\Omega), \\ u_{\varepsilon_j}^- \to u_0^- & \text{strongly in } L^p(\Omega), \\ u_{\varepsilon_j}^- \to u_0^- & \text{a.e. in } \Omega \end{cases} \tag{2.4}$$

as $j \to \infty$. The following proposition shows that $\mu_{s,p}^{-\lambda}(\Omega)$ is achieved in the non blow-up case. Obviously, the same manner as above works for $\mu_{s,p}^{+\lambda}(\Omega)$ and we see that

$$\begin{cases} u_{\varepsilon_j}^+ \to u_0^+ & \text{weakly in } H_0^1(\Omega), \\ u_{\varepsilon_j}^+ \to u_0^+ & \text{strongly in } L^p(\Omega), \\ u_{\varepsilon_j}^+ \to u_0^+ & \text{a.e. in } \Omega. \end{cases}$$

Proposition 2.2. *Let* $u_0^\pm$ *be a function in* $H_0^1(\Omega)$ *constructed in the previous way. Then* $u_0^\pm$ *is a minimizer for* $\mu_{s,p}^{\pm\lambda}(\Omega)$ *provided* $u_0^\pm \neq 0$, *respectively.*

Proof. We shall show Proposition 2.2 only for $\mu_{s,p}^{-\lambda}(\Omega)$ since the proof is quite similar. The equation (2.1) satisfied by $u_{\varepsilon_j}^-$ with u_0^- as a test function yields that

$$\int_\Omega \nabla u_{\varepsilon_j}^- \cdot \nabla u_0^- dx - \lambda \|u_{\varepsilon_j}^-\|_{L^p(\Omega)}^{-(p-2)} \int_\Omega (u_{\varepsilon_j}^-)^{p-1} u_0^- dx = \int_\Omega \frac{(u_{\varepsilon_j}^-)^{2^*-1-\varepsilon_j} u_0^-}{|x|^s} dx. \tag{2.5}$$

By using weak convergences, we have as $j \to \infty$,

$$\begin{cases} \int_\Omega \frac{(u_{\varepsilon_j}^-)^{2^*-1-\varepsilon_j} u_0^-}{|x|^s} dx \to \int_\Omega \frac{(u_0^-)^{2^*}}{|x|^s} dx, \\ \int_\Omega \nabla u_{\varepsilon_j}^- \cdot \nabla u_0^- dx \to \int_\Omega |\nabla u_0^-|^2 dx, \\ \int_\Omega (u_{\varepsilon_j}^-)^{p-1} u_0^- dx \to \int_\Omega (u_0^-)^p dx. \end{cases} \tag{2.6}$$

Thus recalling $u_0^- \neq 0$ and letting $j \to \infty$ in (2.5),

$$\int_\Omega |\nabla u_0^-|^2 dx - \lambda \left(\int_\Omega (u_0^-)^p dx \right)^{\frac{2}{p}} = \int_\Omega \frac{(u_0^-)^{2^*}}{|x|^s} dx.$$

Then we see that

$$\mu_{s,p}^{-\lambda}(\Omega) \leqq \frac{\int_\Omega |\nabla u_0^-|^2 dx - \lambda \left(\int_\Omega (u_0^-)^p dx\right)^{\frac{2}{p}}}{\left(\int_\Omega \frac{(u_0^-)^{2^*}}{|x|^s} dx\right)^{\frac{2}{2^*}}} = \left(\int_\Omega \frac{(u_0^-)^{2^*}}{|x|^s} dx \right)^{\frac{2^*-2}{2^*}},$$

and we have

$$\mu_{s,p}^{-\lambda}(\Omega)^{\frac{2^*}{2^*-2}} \leqq \int_\Omega \frac{(u_0^-)^{2^*}}{|x|^s} dx. \tag{2.7}$$

Therefore, from (2.3), (2.7), Lemma 2.2 and Fatou's lemma, we obtain that

$$\mu_{s,p}^{-\lambda}(\Omega)^{\frac{2^*}{2^*-2}} \leqq \int_\Omega \frac{(u_0^-)^{2^*}}{|x|^s} dx \leqq \liminf_{j\to\infty} \int_\Omega \frac{(u_{\varepsilon_j}^-)^{2^*-\varepsilon_j}}{|x|^s} dx = \liminf_{j\to\infty} \mu_{s,p}^{-\lambda,\varepsilon}(\Omega)^{\frac{2^*-\varepsilon_j}{2^*-2-\varepsilon_j}} = \mu_{s,p}^{-\lambda}(\Omega)^{\frac{2^*}{2^*-2}}.$$

Consequently, we have

$$\int_\Omega |\nabla u_0^-|^2 dx - \lambda \left(\int_\Omega (u_0^-)^p dx \right)^{\frac{2}{p}} = \int_\Omega \frac{(u_0^-)^{2^*}}{|x|^s} dx = \mu_{s,p}^{-\lambda}(\Omega)^{\frac{2^*}{2^*-2}}. \tag{2.8}$$

In the end, we see that

$$\begin{aligned} \int_\Omega |\nabla u_{\varepsilon_j}^- - \nabla u_0^-|^2 dx = &\int_\Omega |\nabla u_{\varepsilon_j}^-|^2 dx - \lambda \left(\int_\Omega (u_{\varepsilon_j}^-)^p dx \right)^{\frac{2}{p}} + \lambda \left(\int_\Omega (u_{\varepsilon_j}^-)^p dx \right)^{\frac{2}{p}} \\ &- 2\int_\Omega \nabla u_{\varepsilon_j}^- \cdot \nabla u_0^- dx + \int_\Omega |\nabla u_0^-|^2 dx. \end{aligned}$$

Then by (2.2), (2.3), (2.4), (2.6), (2.8) and Lemma 2.2, we have

$$\int_\Omega |\nabla u_{\varepsilon_j}^- - \nabla u_0^-|^2 dx \to 0$$

as $j \to \infty$. As easily checked, u_0^- is a minimizer of $\mu_{s,p}^{-\lambda}(\Omega)$. □

3 Blow-up case

In this section, we investigate the blow-up case where the minimizers $\{u_{\varepsilon_j}^{\pm}\}_{j\in\mathbb{N}}$ given by Proposition 2.1 converges to 0 weakly in $H_0^1(\Omega)$ as $j\to\infty$.

Let Ω be a C^2-smooth bounded domain with $0\in\partial\Omega$. Recall that the minimizers $u_\varepsilon^{\pm}\in H_0^1(\Omega)\setminus\{0\}$ are solutions to

$$\begin{cases} -\Delta u_\varepsilon^{\pm} = \mp\lambda\|u_\varepsilon^{\pm}\|_{L^p(\Omega)}^{-(p-2)}(u_\varepsilon^{\pm})^{p-1} + \frac{(u_\varepsilon^{\pm})^{2^*-1-\varepsilon}}{|x|^s} & \text{in } \Omega, \\ u_\varepsilon^{\pm}>0 \quad \text{in } \Omega, \end{cases} \tag{3.1}$$

where $2\leqq p<\frac{2n}{n-2}, 0<s<2, \lambda$ as in (1.2) and $\varepsilon\in(0,2^*-2)$. For the regularity of the solutions u_ε, we can prove $u_\varepsilon^{\pm}\in C^\alpha(\overline{\Omega})$ for some $\alpha\in(0,1)$ depending only on s by the iteration method, see N.Ghoussoub and F.Robert[4, Proposition 8.1] for instance. Thus from the standard elliptic theory and the strong maximum principle, we obtain $u_\varepsilon^{\pm}\in C^2(\overline{\Omega}\setminus\{0\})\cap C^1(\overline{\Omega})$ and $u_\varepsilon^- >0$ in Ω. Furthermore, $u_\varepsilon^{\pm}$ satisfies

$$\int_\Omega \frac{(u_\varepsilon^{\pm})^{2^*-\varepsilon}}{|x|^s}dx = \mu_{s,p}^{\lambda}(\Omega)^{\frac{2^*}{2^*-2}} + o(1)$$

as $\varepsilon\to 0$. Then in the quite same argument in section 2, we have that there exist $\{\varepsilon_j\}_{j\in\mathbb{N}}\subset(0,2^*-2)$ with $\varepsilon_j\to 0$ as $j\to\infty$ and $u_0^{\pm}\in H_0^1(\Omega)$ such that

$$\begin{cases} u_{\varepsilon_j}^{\pm}\to u_0^{\pm} & \text{weakly in } H_0^1(\Omega), \\ u_{\varepsilon_j}^{\pm}\to u_0^{\pm} & \text{strongly in } L^p(\Omega), \\ u_{\varepsilon_j}^{\pm}\to u_0^{\pm} & \text{a.e. in } \Omega \end{cases}$$

as $j\to\infty$. In addition, we assume that the blow-up occurs, i.e., the limit-function $u_0^{\pm}=0$. Our goal in this section is to prove the following proposition.

Proposition 3.1. *Assume that the blow-up case occurs as above. Then we have the equality*

$$\mu_{s,p}^{\pm\lambda}(\Omega)=\mu_s(\mathbb{R}_-^n).$$

In the rest of this section, we treat only the case $\mu_{s,p}^{-\lambda}(\Omega)$ since the proof of Proposition 3.1 is quite same as in the case of $\mu_{s,p}^{+\lambda}(\Omega)$. We mainly follow the strategy developed by N.Ghoussoub and F.Robert[4] who treated the case $\lambda=0$ or the case $p=2$. However, note that the term $\|u_\varepsilon^{\pm}\|_{L^p(\Omega)}^{-(p-2)}(u_\varepsilon^{\pm})^{p-1}$ in the equation (3.1) is no longer linear in the case $p>2$ and the coefficient depends on ε which make some difficulty to show the attainability. We prepare several lemmas.

Let $x_{\varepsilon_j}\in\Omega$ be a maximum point of $u_{\varepsilon_j}^-$, that is, $0<\max_\Omega u_{\varepsilon_j}^- = u_{\varepsilon_j}^-(x_{\varepsilon_j})$ holds, and we define positive constants $\nu_{\varepsilon_j}>0$ and $\kappa_{\varepsilon_j}>0$ by

$$\nu_{\varepsilon_j} := u_{\varepsilon_j}^-(x_{\varepsilon_j})^{-\frac{2}{n-2}} \quad\text{and}\quad \kappa_{\varepsilon_j} := \nu_{\varepsilon_j}^{\frac{2^*-2-\varepsilon_j}{2^*-2}}. \tag{3.2}$$

Lemmas 3.1-3.4 below will be proved in the quite same way as in N.Ghoussoub and F.Robert[4]. Hence, we will omit them here.

Lemma 3.1. *Up to a subsequence, it follows* $\lim_{j\to\infty} \nu_{\varepsilon_j} = 0$.

Lemma 3.2. *It follows that* $|x_{\varepsilon_j}| = O(\kappa_{\varepsilon_j})$ *as* $j \to \infty$.

Let φ be a local chart at $0 \in \partial\Omega$ introduced in section 2 and define

$$v_{\varepsilon_j}(x) := \frac{(u^-_{\varepsilon_j} \circ \varphi)(\kappa_{\varepsilon_j} x)}{u^-_{\varepsilon_j}(x_{\varepsilon_j})}$$

for $x \in \frac{U}{\kappa_{\varepsilon_j}} \cap \{x_1 \leqq 0\}$. Since $\kappa_{\varepsilon_j} \to 0$ as $j \to \infty$, for any $\eta \in C_c^\infty(\mathbb{R}^n)$, we see that $\operatorname{supp}\eta \subset \frac{U}{\kappa_{\varepsilon_j}}$ for all $j \in \mathbb{N}$ large enough, and then it follows $\eta v_{\varepsilon_j} \in \dot{H}^1_0(\mathbb{R}^n_-)$, where $\dot{H}^1_0(\mathbb{R}^n_-)$ denotes the closure of $C_c^\infty(\mathbb{R}^n_-)$ in the Sobolev space endowed with the norm $\|\nabla \cdot \|_{L^2(\mathbb{R}^n_-)} + \| \cdot \|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_-)}$.

Lemma 3.3. *There exists* $v \in \dot{H}^1_0(\mathbb{R}^n_-)\backslash\{0\}$ *such that for any* $\eta \in C_c^\infty(\mathbb{R}^n)$*, up to a subsequence,* ηv_{ε_j} *converges to* ηv *weakly in* $\dot{H}^1_0(\mathbb{R}^n_-)$ *as* $j \to \infty$*. In addition, there exists* $\alpha \in (0,1)$ *such that* $v \in C^\alpha_{loc}(\overline{\mathbb{R}^n_-})$ *and for any* $K > 0$*, up to a subsequence,* v_{ε_j} *converges to* v *in* $C^\alpha_{loc}(\overline{B_K(0)} \cap \{x_1 \leqq 0\})$ *as* $j \to \infty$.

Lemma 3.4. $v \in \dot{H}^1_0(\mathbb{R}^n_-)$ *constructed in Lemma 3.3 satisfies*

$$-\Delta v = \frac{v^{2^*-1}}{|x|^s} \quad in \quad \mathbb{R}^n_-.$$

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. Lemma 3.4 says that $v \in \dot{H}^1_0(\mathbb{R}^n_-)$ satisfies

$$-\Delta v = \frac{v^{2^*-1}}{|x|^s} \quad \text{in} \quad \mathbb{R}^n_-.$$

Taking v as a test function,

$$\int_{\mathbb{R}^n_-} |\nabla v|^2 dx = \int_{\mathbb{R}^n_-} \frac{v^{2^*}}{|x|^s} dx.$$

From the definition of $\mu_s(\mathbb{R}^n_-)$, we obtain

$$\mu_s(\mathbb{R}^n_-) \leqq \frac{\int_{\mathbb{R}^n_-} |\nabla v|^2 dx}{\left(\int_{\mathbb{R}^n_-} \frac{v^{2^*}}{|x|^s} dx\right)^{\frac{2}{2^*}}} = \left(\int_{\mathbb{R}^n_-} |\nabla v|^2 dx\right)^{\frac{2^*-2}{2^*}},$$

and then we have

$$\mu_s(\mathbb{R}^n_-)^{\frac{2^*}{2^*-2}} \leqq \int_{\mathbb{R}^n_-} |\nabla v|^2 dx. \tag{3.3}$$

The direct computation yields that

$$\int_{\mathbb{R}^n_-} |\nabla(\eta_R v_{\varepsilon_j})|^2 dx \leqq C_\delta \|\nabla \eta_R\|^2_{L^n(\mathbb{R}^n)} \|v_{\varepsilon_j}\|^2_{L^{\frac{2n}{n-2}}(\operatorname{supp}|\nabla\eta_R| \cap \{x_1<0\})}$$

$$+ (1+\delta)\, \nu_{\varepsilon_j}^{\frac{(n-2)\varepsilon_j}{2^*-2}} (1+O(\kappa_{\varepsilon_j}))\|\eta_R\|^2_{L^\infty(\mathbb{R}^n)}\|\nabla u^-_{\varepsilon_j}\|^2_{L^2(\Omega)}$$

$$= C_\delta \|\nabla \eta_1\|^2_{L^n(\mathbb{R}^n)}\|v_{\varepsilon_j}\|^2_{L^{\frac{2n}{n-2}}(\mathrm{supp}\,|\nabla\eta_R|\cap\{x_1<0\})} + (1+\delta)\, \nu_{\varepsilon_j}^{\frac{(n-2)\varepsilon_j}{2^*-2}} (1+O(\kappa_{\varepsilon_j}))\|\nabla u^-_{\varepsilon_j}\|^2_{L^2(\Omega)}. \tag{3.4}$$

Here, we give several remarks. Taking $u^-_{\varepsilon_j}$ as a test function in (3.1), we have

$$\int_\Omega |\nabla u^-_{\varepsilon_j}|^2 dx - \lambda \left(\int_\Omega (u^-_{\varepsilon_j})^p dx\right)^{\frac{2}{p}} = \int_\Omega \frac{(u^-_{\varepsilon_j})^{2^*-\varepsilon_j}}{|x|^s} dx = \mu^\lambda_{s,p}(\Omega)^{\frac{2^*}{2^*-2}} + o(1)$$

as $j \to \infty$. Since $\lim_{j\to\infty} \|u^-_{\varepsilon_j}\|_{L^p(\Omega)} = 0$, we then get

$$\int_\Omega |\nabla u^-_{\varepsilon_j}|^2 dx \to \mu^\lambda_{s,p}(\Omega)^{\frac{2^*}{2^*-2}}$$

as $j \to \infty$. Moreover, from Lemma 3.3, we obtain

$$\|v_{\varepsilon_j}\|_{L^{\frac{2n}{n-2}}(\mathrm{supp}\,|\nabla\eta_R|\cap\{x_1<0\})} = \|v_{\varepsilon_j}\|_{L^{\frac{2n}{n-2}}((B_{2R}(0)\setminus B_R(0))\cap\{x_1<0\})} \to \|v\|_{L^{\frac{2n}{n-2}}((B_{2R}(0)\setminus B_R(0))\cap\{x_1<0\})}$$

as $j \to \infty$. In addition, since $\eta_R v_{\varepsilon_j}$ converges to v_R weakly in $\dot{H}^1_0(\mathbb{R}^n_-)$, taking the weak-limit yields $\|\nabla v_R\|_{L^2(\mathbb{R}^n_-)} \leqq \liminf_{j\to\infty} \|\nabla(\eta_R v_{\varepsilon_j})\|_{L^2(\mathbb{R}^n_-)}$. After all, letting $j \to \infty$ in (3.4) shows that

$$\|\nabla v_R\|^2_{L^2(\mathbb{R}^n_-)} \leqq C_\delta \|\nabla\eta_1\|^2_{L^n(\mathbb{R}^n)}\|v\|^2_{L^{\frac{2n}{n-2}}((B_{2R}(0)\setminus B_R(0))\cap\{x_1<0\})}$$
$$+ (1+\delta)\left(\liminf_{j\to\infty} \nu_{\varepsilon_j}^{\varepsilon_j}\right)^{\frac{n-2}{2^*-2}} \mu^\lambda_{s,p}(\Omega)^{\frac{2^*}{2^*-2}}.$$

Here, $v \in L^{\frac{2n}{n-2}}(\mathbb{R}^n_-)$ guarantees that $\|v\|_{L^{\frac{2n}{n-2}}((B_{2R}(0)\setminus B_R(0))\cap\{x_1<0\})} \to 0$ as $R \to \infty$. Since v_{R_j} converges v weakly in $\dot{H}^1_0(\mathbb{R}^n_-)$ as $j \to \infty$ and δ is arbitrary, we get

$$\|\nabla v\|^2_{L^2(\mathbb{R}^n_-)} \leqq \left(\liminf_{j\to\infty} \nu_{\varepsilon_j}^{\varepsilon_j}\right)^{\frac{n-2}{2^*-2}} \mu^\lambda_{s,p}(\Omega)^{\frac{2^*}{2^*-2}}. \tag{3.5}$$

As a consequence, since $\nu^{\varepsilon_j} \leqq 1$ for $j \in \mathbb{N}$ large enough, from Lemma 2.1, (3.3) and (3.5), we have

$$\|\nabla v\|^2_{L^2(\mathbb{R}^n_-)} \leqq \left(\liminf_{j\to\infty} \nu_{\varepsilon_j}^{\varepsilon_j}\right)^{\frac{n-2}{2^*-2}} \mu^\lambda_{s,p}(\Omega)^{\frac{2^*}{2^*-2}} \leqq \mu^\lambda_{s,p}(\Omega)^{\frac{2^*}{2^*-2}} \leqq \mu_s(\mathbb{R}^n_-)^{\frac{2^*}{2^*-2}} \leqq \|\nabla v\|^2_{L^2(\mathbb{R}^n_-)},$$

and then it follows that $\mu^\lambda_{s,p}(\Omega) = \mu_s(\mathbb{R}^n_-)$. □

4 Proof of theorems

This section is devoted to prove main theorems. We shall show the blow-up case argued in section 3 never occurs under the assumption in theorems. First, we shall give the proofs of Theorem 1.3 and Theorem 1.5. By virtue of Lemma 2.1, Proposition 2.2 and Proposition 3.1, it suffices to prove the following.

Proposition 4.1. *Let $n \geqq 3, s \in (0,2), \frac{2n}{n-1} < p < \frac{2n}{n-2}, 0 < \lambda < \Lambda_p$ and let Ω be a C^1-smooth bounded domain. Then it follows*

$$\mu_{s,p}^{-\lambda}(\Omega) < \mu_s(\mathbb{R}^n_-).$$

Proposition 4.2. *Let $s \in (0,2)$,*

$$\begin{cases} 2 < p < \frac{2n}{n-2} & \text{if} \quad n = 4, \\ 2 \leqq p < \frac{2n}{n-2} & \text{if} \quad n \geqq 5, \end{cases} \tag{4.1}$$

$0 < \lambda < \Lambda_p$, and let Ω be a bounded domain. Furthermore, assume that Ω is flat near the origin. Then it follows

$$\mu_{s,p}^{-\lambda}(\Omega) < \mu_s(\mathbb{R}^n_-).$$

Remark 4.1. *Obviously, Proposition 4.1 and Proposition 4.2 show Theorem 1.3, Theorem 1.5, respectively.*

First, we prove Proposition 4.1.

Proof of Proposition 4.1. We make use of the minimizer $v \in H^1_0(\mathbb{R}^n_-) \setminus \{0\}$ for $\mu_s(\mathbb{R}^n_-)$ constructed by H.Egnell[2] satisfying the following properties. First, the minimizer v enjoys

$$\begin{cases} -\Delta v = \frac{v^{2^*-1}}{|x|^s} \quad \text{in} \quad \mathbb{R}^n_-, \\ v > 0 \quad \text{in} \quad \mathbb{R}^n_-. \end{cases} \tag{4.2}$$

In addition, the following pointwise estimates hold,

$$|v(x)| \leqq \frac{C}{|x|^{n-2}} \quad \text{and} \quad |\nabla v(x)| \leqq \frac{C}{|x|^{n-1}} \tag{4.3}$$

for all $x \in \mathbb{R}^n_-$. Furthermore, K.S.Chou and C.W.Chu[1, Proposition 4.4] showed that $v \in L^\infty_{loc}(\mathbb{R}^n_-)$. They considered this regularity problem in the whole space $\mathbb{R}^n$. However, by imitating the argument in [1], we get the regularity of v on the half space. Then the standard elliptic theory yields $v \in C^1(\overline{\mathbb{R}^n_-}) \cap C^2(\overline{\mathbb{R}^n_-} \setminus \{0\})$. Hence, with (4.3), we obtain

$$|v(x)| \leqq \frac{C}{(1+|x|)^{n-2}} \quad \text{and} \quad |\nabla v(x)| \leqq \frac{C}{(1+|x|)^{n-1}} \tag{4.4}$$

for all $x \in \mathbb{R}^n_-$. Next, we claim that the decay estimate for v is slightly improved, i.e.,

$$|v(x)| \leqq \frac{C}{(1+|x|)^{n-1}} \tag{4.5}$$

holds for all $x \in \mathbb{R}^n_-$. Indeed, let $\tilde{v}$ be the Kelvin transform of v as follows,

$$\tilde{v}(x) := \frac{1}{|x|^{n-2}} v\left(\frac{x}{|x|^2}\right)$$

for $x \in \overline{\mathbb{R}^n_-} \setminus \{0\}$ and $\tilde{v}(0) := 0$. We easily see that $\tilde{v} \in C^2(\overline{\mathbb{R}^n_-} \setminus \{0\})$. Moreover, by using (4.2) and (4.4), we get

$$\begin{cases} -\Delta \tilde{v} = \frac{\tilde{v}^{2^*-1}}{|x|^s} & \text{in} \quad \mathbb{R}^n_-, \\ \tilde{v}(x) \leqq \frac{C}{(1+|x|)^{n-2}} & \text{for} \quad x \in \mathbb{R}^n_-. \end{cases}$$

Since $\tilde{v}$ vanishes on $\partial\mathbb{R}^n_-$, the standard elliptic theory yields $\tilde{v} \in C^1(\overline{\mathbb{R}^n_-})$, and then it follows that

$$\tilde{v}(x) \leqq \|\nabla \tilde{v}\|_{L^\infty(B_1(0)\cap\{x_1<0\})}|x|$$

for all $x \in B_1(0) \cap \{x_1 < 0\}$, which implies (4.5).

Let φ be a local chart at $0 \in \partial\Omega$ introduced in section 2. Take a ball $B_{R_0}(0)$ with $\overline{B_{R_0}(0)} \subset V$ and $\zeta \in C^\infty_c(V)$ such that $\zeta \equiv 1$ in $B_{R_0}(0)$. For any $\delta > 0$, define

$$w_\delta(x) := v\left(\frac{\varphi^{-1}(x)}{\delta}\right)$$

for $x \in \Omega \cap V$. Then we easily see that $\zeta w_\delta \in H^1_0(\Omega) \setminus \{0\}$ for all δ small enough. From the definition of $\mu^{-\lambda}_{s,p}(\Omega)$, we obtain that

$$\mu^{-\lambda}_{s,p}(\Omega) \leqq \frac{\int_\Omega |\nabla(\zeta w_\delta)|^2 dx - \lambda\left(\int_\Omega |\zeta w_\delta|^p dx\right)^{\frac{2}{p}}}{\left(\int_\Omega \frac{|\zeta w_\delta|^{2^*}}{|x|^s} dx\right)^{\frac{2}{2^*}}} \leqq \frac{\int_{\Omega\cap V} |\nabla(\zeta w_\delta)|^2 dx - \lambda\left(\int_{\Omega\cap B_{R_0}(0)} w_\delta^p dx\right)^{\frac{2}{p}}}{\left(\int_{\Omega\cap B_{R_0}(0)} \frac{w_\delta^{2^*}}{|x|^s} dx\right)^{\frac{2}{2^*}}}. \tag{4.6}$$

for all $\delta > 0$. We estimate the integrals in the right-hand side in (4.6). The direct calculation yields that

$$\begin{aligned} \int_{\Omega\cap V} |\nabla(\zeta w_\delta)|^2 dx &= \int_{\Omega\cap V} |w_\delta \nabla\zeta|^2 dx + 2\int_{\Omega\cap V} w_\delta\, \zeta\, \nabla w_\delta \cdot \nabla\zeta dx + \int_{\Omega\cap V} |\zeta\nabla w_\delta|^2 dx \\ &= \int_{(\Omega\cap V)\setminus B_{R_0}(0)} |w_\delta\nabla\zeta|^2 dx + 2\int_{(\Omega\cap V)\setminus B_{R_0}(0)} w_\delta\,\zeta\,\nabla w_\delta\cdot\nabla\zeta dx + \int_{(\Omega\cap V)\setminus B_{R_0}(0)} |\zeta\nabla w_\delta|^2 dx + \int_{\Omega\cap B_{R_0}(0)} |\nabla w_\delta|^2 dx \\ &\leqq 2\int_{(\Omega\cap V)\setminus B_{R_0}(0)} |w_\delta\nabla\zeta|^2 dx + 2\int_{(\Omega\cap V)\setminus B_{R_0}(0)} |\zeta\nabla w_\delta|^2 dx + \int_{\Omega\cap B_{R_0}(0)} |\nabla w_\delta|^2 dx =: 2I_1 + 2I_2 + I_3. \end{aligned}$$

First, we estimate I_1. By a change of the variable and (4.5), we have

$$\begin{aligned} I_1 &\leqq \delta^n \|\nabla\zeta\|^2_{L^\infty(V)} \int_{\left\{x\in\frac{U\cap\{x_1<0\}}{\delta}\,;\,|\varphi(\delta x)|\geqq R_0\right\}} v^2 dx \leqq \delta^n\|\nabla\zeta\|^2_{L^\infty(V)} \int_{\{x\in\mathbb{R}^n_-\,;\,|\delta x|\geqq C>0\}} v^2 dx \\ &\leqq \delta^n \|\nabla\zeta\|^2_{L^\infty(V)} \int_{\{x\in\mathbb{R}^n\,;\,|\delta x|\geqq C\}} |x|^{-2(n-1)} dx = C\delta^{2(n-1)}. \end{aligned}$$

Therefore, we get $I_1 = O(\delta^{2(n-1)})$ as $\delta \to 0$. Next, note that $|\nabla w_\delta(x)| \leqq C\delta^{-1}\left|(\nabla v)\left(\frac{\varphi^{-1}(x)}{\delta}\right)\right|$ holds for all $x \in \Omega \cap V$, and then with (4.4), I_2 is estimated as follows,

$$I_2 \leqq C\delta^{-2}\|\zeta\|^2_{L^\infty(V)} \int_{(\Omega\cap V)\setminus B_{R_0}(0)} \left|(\nabla v)\left(\frac{\varphi^{-1}(x)}{\delta}\right)\right|^2 dx = C\delta^{n-2}\int_{\left\{x\in\frac{U\cap\{x_1<0\}}{\delta}\,;\,|\varphi(\delta x)|\geqq R_0\right\}} |\nabla v|^2 dx$$
$$\leqq C\delta^{n-2}\int_{\{x\in\mathbb{R}^n_-\,;\,|\delta x|\geqq C>0\}} |\nabla v|^2 dx \leqq C\delta^{n-2}\int_{\{x\in\mathbb{R}^n\,;\,|\delta x|\geqq C\}} |x|^{-2(n-1)}dx = C\delta^{2(n-2)}.$$

Hence, we get $I_2 = O(\delta^{2(n-2)})$ as $\delta \to 0$. Thirdly, it follows that

$$\begin{aligned} I_3 = {} & \delta^{-2}\int_{\Omega\cap B_{R_0}(0)} \left|(\nabla v)\left(\frac{\varphi^{-1}(x)}{\delta}\right)\right|^2 dx \\ & - 2\delta^{-2}\int_{\Omega\cap B_{R_0}(0)} (\partial_1 v)\left(\frac{\varphi^{-1}(x)}{\delta}\right)(\nabla' v)\left(\frac{\varphi^{-1}(x)}{\delta}\right)\cdot\nabla'\varphi_0(x')dx \\ & + \delta^{-2}\int_{\Omega\cap B_{R_0}(0)} (\partial_1 v)\left(\frac{\varphi^{-1}(x)}{\delta}\right)^2 |\nabla'\varphi_0(x')|^2 dx. \end{aligned} \tag{4.7}$$

Here, since $\frac{2n}{n-1} < p$, there exists $\alpha_0 \in (0,1)$ such that $\frac{2n}{n-1} < \frac{2n}{n-2+\alpha_0} < p$. With the fact $\nabla'\varphi_0(0) = 0$, we have that

$$|(\nabla'\varphi_0)((\varphi(\delta x))')| \leqq C|(\varphi(\delta x))'|^{\alpha_0} \leqq C\delta^{\alpha_0}|x|^{\alpha_0} \tag{4.8}$$

for all $x \in \frac{U\cap\{x_1<0\}}{\delta}$. From (4.4) and (4.8), we obtain that

$$\begin{aligned} & \delta^{-2}\int_{\Omega\cap B_{R_0}(0)} \left|(\nabla v)\left(\frac{\varphi^{-1}(x)}{\delta}\right)\right|^2 |\nabla'\varphi_0(x')|^2 dx \\ & \leqq \delta^{-2}\||\nabla'\varphi_0|\|_{L^\infty(U')}\int_{\Omega\cap B_{R_0}(0)} \left|(\nabla v)\left(\frac{\varphi^{-1}(x)}{\delta}\right)\right|^2 |\nabla'\varphi_0(x')|dx \\ & \leqq C\delta^{n-2}\int_{\frac{U\cap\{x_1<0\}}{\delta}} |\nabla v(x)|^2|(\nabla'\varphi_0)((\varphi(\delta x))')|dx \leqq C\delta^{n-2+\alpha_0}\int_{\mathbb{R}^n_-}|\nabla v|^2|x|^{\alpha_0}dx = C\delta^{n-2+\alpha_0}. \end{aligned} \tag{4.9}$$

Note that the last integral in the above estimate is finite by virtue of (4.4). Combining (4.7) with (4.9), we get

$$I_3 = \delta^{-2}\int_{\Omega\cap B_{R_0}(0)} \left|(\nabla v)\left(\frac{\varphi^{-1}(x)}{\delta}\right)\right|^2 dx + O(\delta^{n-2+\alpha_0}) = \delta^{n-2}\int_{\frac{\tilde{U}}{\delta}} |\nabla v|^2 dx + O(\delta^{n-2+\alpha_0})$$

as $\delta \to 0$, where $\tilde{U} := \{\varphi^{-1}(x)\,;\, x \in \Omega \cap B_{R_0}(0)\}$. As a consequence, it follows that

$$\int_{\Omega\cap V} |\nabla(\zeta w_\delta)|^2 dx \leqq \delta^{n-2}\int_{\frac{\tilde{U}}{\delta}} |\nabla v|^2 dx + O(\delta^{n-2+\alpha_0}) \tag{4.10}$$

as $\delta \to 0$. Furthermore, by changing the variable, we have

$$\left(\int_{\Omega\cap B_{R_0}(0)} w_\delta^p dx\right)^{\frac{2}{p}} = \delta^{\frac{2n}{p}}\left(\int_{\frac{\tilde{U}}{\delta}} v^p dx\right)^{\frac{2}{p}} \quad \text{and} \quad \int_{\Omega\cap B_{R_0}(0)} \frac{w_\delta^{2^*}}{|x|^s}dx = \delta^{n-s}\int_{\frac{\tilde{U}}{\delta}} \frac{v^{2^*}}{\left|\frac{\varphi(\delta x)}{\delta}\right|^s}dx. \tag{4.11}$$

After all, (4.6), (4.10) and (4.11) show that

$$\begin{aligned}\mu_{s,p}^{-\lambda}(\Omega) &\leqq \frac{\delta^{n-2}\int_{\frac{\tilde{U}}{\delta}}|\nabla v|^2dx + O(\delta^{n-2+\alpha_0}) - \lambda\delta^{\frac{2n}{p}}\left(\int_{\frac{\tilde{U}}{\delta}} v^p dx\right)^{\frac{2}{p}}}{\left(\delta^{n-s}\int_{\frac{\tilde{U}}{\delta}}\frac{v^{2^*}}{\left|\frac{\varphi(\delta x)}{\delta}\right|^s}dx\right)^{\frac{2}{2^*}}} \\ &= \frac{\int_{\frac{\tilde{U}}{\delta}}|\nabla v|^2dx + O(\delta^{\alpha_0}) - \lambda\delta^{\frac{2n}{p}-(n-2)}\left(\int_{\frac{\tilde{U}}{\delta}} v^p dx\right)^{\frac{2}{p}}}{\left(\int_{\frac{\tilde{U}}{\delta}}\frac{v^{2^*}}{\left|\frac{\varphi(\delta x)}{\delta}\right|^s}dx\right)^{\frac{2}{2^*}}}.\end{aligned}$$

Hence, since v in a minimizer for $\mu_s(\mathbb{R}^n_-)$, we see that

$$\begin{aligned}&\mu_{s,p}^{-\lambda}(\Omega) - \mu_s(\mathbb{R}^n_-) \\ &\leqq \frac{\left(\int_{\frac{\tilde{U}}{\delta}}|\nabla v|^2dx + O(\delta^{\alpha_0}) - \lambda\delta^{\frac{2n}{p}-(n-2)}\left(\int_{\frac{\tilde{U}}{\delta}} v^p dx\right)^{\frac{2}{p}}\right)\left(\int_{\mathbb{R}^n_-}\frac{v^{2^*}}{|x|^s}dx\right)^{\frac{2}{2^*}} - \int_{\mathbb{R}^n_-}|\nabla v|^2dx\left(\int_{\frac{\tilde{U}}{\delta}}\frac{v^{2^*}}{\left|\frac{\varphi(\delta x)}{\delta}\right|^s}dx\right)^{\frac{2}{2^*}}}{\left(\int_{\frac{\tilde{U}}{\delta}}\frac{v^{2^*}}{\left|\frac{\varphi(\delta x)}{\delta}\right|^s}dx\right)^{\frac{2}{2^*}}\left(\int_{\mathbb{R}^n_-}\frac{v^{2^*}}{|x|^s}dx\right)^{\frac{2}{2^*}}}.\end{aligned} \tag{4.12}$$

Moreover, by virtue of (4.4) and (4.5), it follows that

$$\int_{\frac{\tilde{U}}{\delta}}|\nabla v|^2dx = \int_{\mathbb{R}^n_-}|\nabla v|^2dx + O(\delta^{n-2}) \quad \text{and} \quad \int_{\frac{\tilde{U}}{\delta}}|v|^p dx = \int_{\mathbb{R}^n_-}|v|^p dx + O(\delta^{(n-1)p-n}) \tag{4.13}$$

as $\delta \to 0$, respectively. In order to investigate the integral $\int_{\frac{\tilde{U}}{\delta}}\frac{v^{2^*}}{\left|\frac{\varphi(\delta x)}{\delta}\right|^s}dx$, we use the elementary inequality as follows. Let $0 < t_1 \leqq t_2 \leqq 1$. Then there exists a constant C such that

$$|a^{t_1} - b^{t_1}| \leqq C\, a^{-(t_2-t_1)}\, |a-b|^{t_2} \tag{4.14}$$

holds for all $a \geqq 0$ and $b \geqq 0$. Now we set

$$\int_{\frac{\tilde{U}}{\delta}}\frac{v^{2^*}}{\left|\frac{\varphi(\delta x)}{\delta}\right|^s}dx = \int_{\mathbb{R}^n_-}\frac{v^{2^*}}{|x|^s}dx + J_1 - J_2,$$

where
$$J_1 := \int_{\frac{\tilde U}{\delta}} \frac{v^{2^*}}{\left|\frac{\varphi(\delta x)}{\delta}\right|^s}dx - \int_{\frac{\tilde U}{\delta}} \frac{v^{2^*}}{|x|^s}dx \quad \text{and} \quad J_2 := \int_{\mathbb{R}^n_-\setminus\frac{\tilde U}{\delta}} \frac{v^{2^*}}{|x|^s}dx.$$

We distinguish two cases.

Case 1. Let $0 < s \leqq 1$. For any $x \in \frac{\tilde U}{\delta}$, there exists $\theta \in (0,1)$ such that $\varphi_0(\delta x') = (\nabla'\varphi_0)(\theta\delta x')\cdot \delta x'$, and then with (4.14), we get

$$\left||x|^s - \left|\frac{\varphi(\delta x)}{\delta}\right|^s\right| \leqq C|x|^{-(1-s)}\left||x| - \left|\frac{\varphi(\delta x)}{\delta}\right|\right| \leqq C|x|^{-(1-s)}\left|x - \frac{\varphi(\delta x)}{\delta}\right| = C|x|^{-(1-s)}\frac{|\varphi_0(\delta x')|}{\delta}$$
$$\leqq C|x|^{-(1-s)}|x'||(\nabla'\varphi_0)(\theta\delta x')| \leqq C|x|^{-(1-s)}|x'|\left(\sum_{i=2}^n \|\nabla[\partial_i\varphi_0]\|^2_{L^\infty(U')}|\theta\delta x'|^2\right)^{\frac{1}{2}} \leqq C\delta|x|^{-(1-s)}|x'|^2. \tag{4.15}$$

In addition, since the inequality $|\varphi(\delta x)| \geqq \delta|x'|$ holds for all $x \in \frac{\tilde U}{\delta}$, J_1 can be estimated as follows,

$$|J_1| \leqq \int_{\frac{\tilde U}{\delta}} \frac{\left||x|^s - \left|\frac{\varphi(\delta x)}{\delta}\right|^s\right|}{\left|\frac{\varphi(\delta x)}{\delta}\right|^s|x|^s} v^{2^*}dx \leqq C\delta\int_{\frac{\tilde U}{\delta}} \frac{|x'|^{2-s}}{|x|}v^{2^*}dx \leqq C\delta\int_{\mathbb{R}^n_-}|x|^{1-s}v^{2^*}dx = C\delta,$$

where (4.5) guarantees the boundedness of the last integral in the above estimate.

Case 2. Let $1 < s < 2$. In this case, from (4.15) with $s = 1$, we see that

$$\left||x|^s - \left|\frac{\varphi(\delta x)}{\delta}\right|^s\right| \leqq C\left(|x|^{s-1} + \left|\frac{\varphi(\delta x)}{\delta}\right|^{s-1}\right)\left||x| - \left|\frac{\varphi(\delta x)}{\delta}\right|\right| \leqq C\delta|x|^{s-1}|x'|^2.$$

Then in the quite same manner as in Case 1, we get $J_1 = O(\delta)$ as $\delta \to 0$.

In both cases, we have $J_1 = O(\delta)$ as $\delta \to 0$. Furthermore, by (4.5), we easily see that $J_2 = O(\delta^{\frac{n(n-s)}{n-2}})$ as $\delta \to 0$. Since $1 < \frac{n(n-s)}{n-2}$, it follows that

$$\int_{\frac{\tilde U}{\delta}} \frac{v^{2^*}}{\left|\frac{\varphi(\delta x)}{\delta}\right|^s}dx = \int_{\mathbb{R}^n_-}\frac{v^{2^*}}{|x|^s}dx + O(\delta), \tag{4.16}$$

as $\delta \to 0$. After all, from (4.12), (4.13) and (4.16), we obtain that

$$\left(\int_{\frac{\tilde U}{\delta}}|\nabla v|^2dx + O(\delta^{\alpha_0}) - \lambda\delta^{\frac{2n}{p}-(n-2)}\left(\int_{\frac{\tilde U}{\delta}}v^p dx\right)^{\frac{2}{p}}\right)\left(\int_{\mathbb{R}^n_-}\frac{v^{2^*}}{|x|^s}dx\right)^{\frac{2}{2^*}} - \int_{\mathbb{R}^n_-}|\nabla v|^2dx\left(\int_{\frac{\tilde U}{\delta}}\frac{v^{2^*}}{\left|\frac{\varphi(\delta x)}{\delta}\right|^s}dx\right)^{\frac{2}{2^*}}$$
$$= \left(\int_{\mathbb{R}^n_-}|\nabla v|^2dx + O(\delta^{\alpha_0}) - \lambda\delta^{\frac{2n}{p}-(n-2)}\left(\int_{\mathbb{R}^n_-}|v|^pdx + O(\delta^{(n-1)p-n})\right)^{\frac{2}{p}}\right)\left(\int_{\mathbb{R}^n_-}\frac{v^{2^*}}{|x|^s}dx\right)^{\frac{2}{2^*}}$$

$$-\int_{\mathbb{R}^n_-}|\nabla v|^2dx\left(\int_{\mathbb{R}^n_-}\frac{v^{2^*}}{|x|^s}dx+O(\delta)\right)^{\frac{2}{2^*}}$$

$$=\left(\int_{\mathbb{R}^n_-}|\nabla v|^2dx+O(\delta^{\alpha_0})-\lambda\delta^{\frac{2n}{p}-(n-2)}\left(\int_{\mathbb{R}^n_-}|v|^pdx\right)^{\frac{2}{p}}+O(\delta^{\frac{2n}{p}-(n-2)+(n-1)p-n})\right)\left(\int_{\mathbb{R}^n_-}\frac{v^{2^*}}{|x|^s}dx\right)^{\frac{2}{2^*}}$$

$$-\int_{\mathbb{R}^n_-}|\nabla v|^2dx\left(\int_{\mathbb{R}^n_-}\frac{v^{2^*}}{|x|^s}dx\right)^{\frac{2}{2^*}}+O(\delta)$$

$$=-\lambda\delta^{\frac{2n}{p}-(n-2)}\left(\int_{\mathbb{R}^n_-}|v|^pdx\right)^{\frac{2}{p}}\left(\int_{\mathbb{R}^n_-}\frac{v^{2^*}}{|x|^s}dx\right)^{\frac{2}{2^*}}+O(\delta^{\alpha_0})<0$$

for all $\delta>0$ small enough since we have $\frac{2n}{n-2+\alpha_0}<p$, which ends the proof. □

Next, we shall show Proposition 4.2 in which the basic strategy is the same as the proof of Proposition 4.1.

Proof of Proposition 4.2. First, the condition that the domain Ω is flat near the origin allows us to assume there exist an open interval $I_0\subset\mathbb{R}$ and a ball $B(0)\subset\mathbb{R}^{n-1}$ such that $0\in I_0$, $B(0)\subset\partial\Omega$ and $U\cap\{x_1<0\}\subset\Omega$, where $U:=I_0\times B(0)$. We again use the minimizer $v\in H^1_0(\mathbb{R}^n_-)$ for $\mu_s(\mathbb{R}^n_-)$ in the proof of Proposition 4.1. Take a ball $\tilde{B}(0)\subset\mathbb{R}^n$ with $\overline{\tilde{B}(0)}\subset U$ and $\zeta\in C^\infty_c(U)$ such that $\zeta\equiv 1$ in $\tilde{B}(0)$. Define $w_\delta(x):=v\left(\frac{x}{\delta}\right)$ for $\delta>0$ and $x\in U\cap\{x_1\leqq 0\}$. Then we see that $\zeta w_\delta\in H^1_0(\Omega)\setminus\{0\}$ for all $\delta>0$ small enough since $v\neq 0$. Hence, it follows that

$$\mu^{-\lambda}_{s,p}(\Omega)\leqq\frac{\int_\Omega|\nabla(\zeta w_\delta)|^2dx-\lambda\left(\int_\Omega|\zeta w_\delta|^pdx\right)^{\frac{2}{p}}}{\left(\int_\Omega\frac{|\zeta w_\delta|^{2^*}}{|x|^s}dx\right)^{\frac{2}{2^*}}}\leqq\frac{\int_{U\cap\Omega}|\nabla(\zeta w_\delta)|^2dx-\lambda\left(\int_{\tilde{B}(0)\cap\Omega}w_\delta^pdx\right)^{\frac{2}{p}}}{\left(\int_{\tilde{B}(0)\cap\Omega}\frac{w_\delta^{2^*}}{|x|^s}dx\right)^{\frac{2}{2^*}}}.$$

In the quite same way as in the proof of Proposition 4.1, we obtain that

$$\left(\int_{\frac{\tilde{B}(0)\cap\Omega}{\delta}}\frac{v^{2^*}}{|x|^s}dx\right)^{\frac{2}{2^*}}\left(\int_{\mathbb{R}^n_-}\frac{v^{2^*}}{|x|^s}dx\right)^{\frac{2}{2^*}}\left(\mu^{-\lambda}_{s,p}(\Omega)-\mu_s(\mathbb{R}^n_-)\right)$$

$$\leqq\left(\int_{\frac{\tilde{B}(0)\cap\Omega}{\delta}}|\nabla v|^2dx+O(\delta^{n-2})-\lambda\delta^{\frac{2n}{p}-(n-2)}\left(\int_{\frac{\tilde{B}(0)\cap\Omega}{\delta}}v^pdx\right)^{\frac{2}{p}}\right)\left(\int_{\mathbb{R}^n_-}\frac{v^{2^*}}{|x|^s}dx\right)^{\frac{2}{2^*}}$$

$$-\int_{\mathbb{R}^n_-}|\nabla v|^2dx\left(\int_{\frac{\tilde{B}(0)\cap\Omega}{\delta}}\frac{v^{2^*}}{|x|^s}dx\right)^{\frac{2}{2^*}}$$

$$=\left(\int_{\mathbb{R}^n_-}|\nabla v|^2dx+O(\delta^{n-2})-\lambda\delta^{\frac{2n}{p}-(n-2)}\left(\int_{\mathbb{R}^n_-}|v|^pdx+O(\delta^{(n-1)p-n})\right)^{\frac{2}{p}}\right)\left(\int_{\mathbb{R}^n_-}\frac{v^{2^*}}{|x|^s}dx\right)^{\frac{2}{2^*}}$$

$$-\int_{\mathbb{R}^n_-}|\nabla v|^2dx\left(\int_{\mathbb{R}^n_-}\frac{v^{2^*}}{|x|^s}dx+O(\delta^{\frac{n(n-s)}{n-2}})\right)^{\frac{2}{2^*}}$$

$$= \left(\int_{\mathbb{R}^n_-} |\nabla v|^2 dx + O(\delta^{n-2}) - \lambda \delta^{\frac{2n}{p}-(n-2)} \left(\int_{\mathbb{R}^n_-} |v|^p dx \right)^{\frac{2}{p}} + O(\delta^{\frac{2n}{p}-(n-2)+(n-1)p-n}) \right) \left(\int_{\mathbb{R}^n_-} \frac{v^{2^*}}{|x|^s} dx \right)^{\frac{2}{2^*}}$$

$$- \int_{\mathbb{R}^n_-} |\nabla v|^2 dx \left(\int_{\mathbb{R}^n_-} \frac{v^{2^*}}{|x|^s} dx \right)^{\frac{2}{2^*}} + O(\delta^{\frac{n(n-s)}{n-2}})$$

$$= -\lambda \delta^{\frac{2n}{p}-(n-2)} \left(\int_{\mathbb{R}^n_-} |v|^p dx \right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^n_-} \frac{v^{2^*}}{|x|^s} dx \right)^{\frac{2}{2^*}} + O(\delta^{n-2}) + O(\delta^{\frac{2n}{p}-(n-2)+(n-1)p-n})$$

as $\delta \to 0$. In the last equality, we used the fact $n-2 < \frac{n(n-s)}{n-2}$. Under the assumption (4.1), we get $\frac{2n}{p} - (n-2) < n-2$, and then taking $\delta > 0$ small enough shows that $\mu^{-\lambda}_{s,p}(\Omega) - \mu_s(\mathbb{R}^n_-) < 0$. □

In what follows, we shall prove Theorem 1.1.

Proof of Theorem 1.1. First we give the proof of (i) which is a corollary of Lemma 2.1. Indeed, since the infimum $\mu^{+\lambda}_{s,p}(\Omega)$ is invariant for the rotation, we have

$$\mu^{+\lambda}_{s,p}(\Omega) = \mu^{+\lambda}_{s,p}(T(\Omega)) \geqq \mu^{+0}_{s,p}(T(\Omega)) = \mu_s(T(\Omega)) \geqq \mu_s(\mathbb{R}^n_-), \tag{4.17}$$

where the last inequality in the above estimates is obtained by the facts that $T(\Omega) \subset \mathbb{R}^n_-$ and $H^1_0(T(\Omega)) \subset H^1_0(\mathbb{R}^n_-)$. Then combining Lemma 2.1 with (4.17) implies that

$$\mu^{+\lambda}_{s,p}(\Omega) = \mu_s(\mathbb{R}^n_-). \tag{4.18}$$

Furthermore, we proceed to the contradiction argument, and assume that $\mu^{+\lambda}_{s,p}(\Omega)$ is achieved by some nonnegative function $u_0 \in H^1_0(\Omega) \setminus \{0\}$. However, the equality (4.18) says that u_0 is a minimizer for $\mu_s(\mathbb{R}^n_-)$ satisfying

$$-\Delta u_0 = \frac{u_0^{2^*-1}}{|x|^s} \quad \text{in} \quad \mathbb{R}^n_-.$$

Then by the standard elliptic theory and the strong maximum principle, we get $u_0 \in C^1(\overline{\mathbb{R}^n_-}) \cap C^2(\overline{\mathbb{R}^n_-} \setminus 0)$ and $u > 0$ in $\mathbb{R}^n_-$, which is a contradiction.

Next, we shall show Theorem 1.1(ii). However, in the case $2 \leqq p < \frac{2n}{n-1}$, the quite same strategy as in the case $p = 2$ shown by N.Ghoussoub and F.Robert[4] works. That is, if the blow-up case occurs, then up to a subsequence, we eventually obtain the following equality,

$$\lim_{j\to\infty} \frac{\varepsilon_j}{\nu_{\varepsilon_j}} = \frac{(n-s)H(0)}{(n-2)^2 \mu_s(\mathbb{R}^n_-)^{\frac{n-s}{2-s}}} \int_{\mathbb{R}^n_-} |x'|^2 |(\nabla v)(0,x')|^2 dx', \tag{4.19}$$

where $H(0)$ denotes the mean curvature of $\partial\Omega$ at 0, ν_{ε_j} is defined as in (3.2) and $v \in \dot{H}^1_0(\mathbb{R}^n_-)$ is a function constructed in Lemma 3.3. The equality (4.19) is a contradiction to $H(0) < 0$, which implies that the blow-up case cannot happen, and then we have a minimizer for $\mu^{+\lambda}_{s,p}(\Omega)$. In the end, we mention that the condition $p < \frac{2n}{n-1}$ is necessary to get the regularity for $v \in C^1(\overline{\mathbb{R}^n_-})$.□

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