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平面内の様々な勾配流 — モデリング，応用，そして多角形版[1)]
Various gradient flows in the plane — modeling, applications and polygonal analogues

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Contents.

1 Introduction

In this talk we study evolution of a family of closed smooth plane curves:

$$\boldsymbol{x}:\ [0,1]\times[0,T)\to\Gamma(t)=\{\boldsymbol{x}(u,t)\in\mathbb{R}^2;\ u\in[0,1]\subset\mathbb{R}/\mathbb{Z}\},$$

starting from a given initial curve $\Gamma(0)=\Gamma_0$, and driven by the evolution law:

$$\partial_t\boldsymbol{x}=\alpha\boldsymbol{t}+\beta\boldsymbol{n},$$

where $\boldsymbol{t}=\partial_u\boldsymbol{x}/|\partial_u\boldsymbol{x}|$ is the unit tangent vector, and $\boldsymbol{n}$ is the unit outward normal vector which satisfies $\det(\boldsymbol{n},\boldsymbol{t})=1$. Here and hereafter, we denote $\partial_\xi\mathsf{F}=\partial\mathsf{F}/\partial\xi$ and $|\boldsymbol{a}|=\sqrt{\boldsymbol{a}.\boldsymbol{a}}$ where $\boldsymbol{a}.\boldsymbol{b}$ is Euclidean inner product between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The solution curves are immersed or embedded such that $|\partial_u\boldsymbol{x}|>0$ holds.

We remark that the tangent velocity α has no effect of shape of solution curves and affect only parametrization. Therefore, the shape of solution curves are determined by the normal velocity β, and a nontrivial tangent velocity α will be chosen depending on the purpose.

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2 Gradient flows

A typical example of β is the **classical curvature flow** $\beta = -k$. Here k is the curvature in the direction $-\boldsymbol{n}$, which is defined from $\partial_s \boldsymbol{t} = -k\boldsymbol{n}$, $\partial_s \boldsymbol{n} = k\boldsymbol{t}$, and described as $k = \det(\partial_s \boldsymbol{x}, \partial_s^2 \boldsymbol{x})$, and $\partial_s \boldsymbol{x} = \boldsymbol{t}$ is the unit tangent vector. Here ∂_s is not partial differentiation. It means the operator $\partial_s \mathsf{F}(u,t) = g(u,t)^{-1}\partial_u \mathsf{F}(u,t)$, where $g(u,t) = |\partial_u \boldsymbol{x}(u,t)| > 0$ is called the local length and s is the arc length parameter determined from $ds = g(u,t)\,du$.

The classical curvature flow $\beta = -k$ is the gradient flow of the total length $L(t)$ of the curve $\Gamma(t)$ in the L^2 sense, since we have

$$L(t) = \int_{\Gamma(t)} ds = \int_0^1 g(u,t)\,du, \quad \partial_t L(t) = \int_{\Gamma(t)} k\beta\,ds$$

by means of the relation $\partial_t g = g(k\beta + \partial_s \alpha)$. Here $L(t)$ can be regarded as the functional $L(t) = L(\Gamma(t))$, and formally we have

$$\partial_t L(\Gamma(t)) = \langle \delta L(\Gamma), \beta \rangle$$

with the first variation δ and a prescribed inner product $\langle \cdot, \cdot \rangle$. In this talk, the L^2 inner product $\langle f, g \rangle = \displaystyle\int_{\Gamma(t)} fg\,ds$ will be chosen. Hence from $\delta L(\Gamma) = k$, we obtain the gradient flow $\beta = -\delta L(\Gamma) = -k$. Note that $\partial_t L(\Gamma(t)) = -\langle k, k \rangle \le 0$ holds, and in this sense the flow is often called the **curve-shortening flow** [15, 16].

This argument will be generalized as follows. Let $f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ be a smooth real-valued function defined on $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$, and let the general functional be

$$J(\Gamma(t)) = \int_{\Gamma(t)} f(\boldsymbol{x}, \partial_s \boldsymbol{x}, \partial_s^2 \boldsymbol{x})\,ds.$$

This includes several typical examples which will be mentioned below, since $f = f(\boldsymbol{x}, \boldsymbol{t}, -k\boldsymbol{n})$ holds. From elementary calculation using the relations

$$\partial_t \boldsymbol{n} = (\alpha k - \partial_s \beta)\boldsymbol{t}, \quad \partial_t \boldsymbol{t} = -(\alpha k - \partial_s \beta)\boldsymbol{n}, \quad \partial_t k = -\partial_s^2 \beta - k^2 \beta + \alpha \partial_s k, \tag{2.1}$$

we have $\partial_t J(\Gamma(t)) = \langle \delta J(\Gamma), \beta \rangle$, where

$$\delta J(\Gamma) = fk + \nabla_{\boldsymbol{x}} f.\boldsymbol{n} - \partial_s(\nabla_{\boldsymbol{y}} f.\boldsymbol{n}) + k^2(\nabla_{\boldsymbol{z}} f.\boldsymbol{n}) - k(\partial_s \nabla_{\boldsymbol{z}} f).\boldsymbol{t} + (\partial_s^2 \nabla_{\boldsymbol{z}} f).\boldsymbol{n}.$$

Hence we obtain the gradient flow $\beta = -\delta J(\Gamma)$ in the L^2 sense.

3 Several examples

We will show some typical examples of gradient flows.

Anisotropic curvature flow $\beta = -w(\theta)k$ is an extension of classical curvature flow, where θ is the angle of $\boldsymbol{n}$, i.e., $\boldsymbol{n} = (\cos\theta, \sin\theta)$ and $\boldsymbol{t} = (-\sin\theta, \cos\theta)$, and w is a positive weight function. Then it is often called **weighted curvature flow**. This is not a gradient flow in general, but a typical case will be shown as follows.

Let us consider the gradient flow of an energy functional defined on $\Gamma(t)$:

$$E_\gamma(\Gamma(t)) = \int_{\Gamma(t)} \gamma(\boldsymbol{n})\, ds,$$

where $\gamma(\boldsymbol{n}) > 0$ is an interfacial energy density. If γ is a homogeneous of degree 1 function in $C^2(\mathbb{R}^2\backslash\{\boldsymbol{0}\})$, then we have $\partial_t E_\gamma(\Gamma(t)) = \langle \delta E_\gamma(\Gamma), \beta\rangle$ with

$$\delta E_\gamma(\Gamma) = w_\gamma(\boldsymbol{n})k, \quad w_\gamma(\boldsymbol{n}) = \mathrm{Hess}\,\gamma(\boldsymbol{n})\boldsymbol{n}^\perp.\boldsymbol{n}^\perp, \quad \boldsymbol{n}^\perp = \boldsymbol{t}.$$

We put $\sigma(\theta) = \gamma(\boldsymbol{n}(\theta))$. Then meaning of the weight $w_\gamma(\boldsymbol{n})$ will be more clearly:

$$w_\gamma(\boldsymbol{n}(\theta)) = \sigma(\theta) + \sigma''(\theta).$$

Thus we obtain the anisotropic curvature flow $\beta = -(\sigma + \sigma'')k$.

Forced curvature flow $\beta = -w(\boldsymbol{x}, \theta)k + F(\boldsymbol{x}, \theta)$ is the extension of anisotropic curvature flow with an external force F. A special case is the gradient flow of

$$E_F(\Gamma(t)) = \int_{\Gamma(t)} \gamma(\boldsymbol{x})\, ds,$$

where $\gamma(\boldsymbol{x}) > 0$ is an inhomogeneous energy density. If γ is differentiable, then we have $\partial_t E_F(\Gamma(t)) = \langle \delta E_F(\Gamma), \beta\rangle$, where

$$\delta E_F(\Gamma) = \gamma(\boldsymbol{x})k + \nabla\gamma(\boldsymbol{x}).\boldsymbol{n},$$

and we obtain the gradient flow $\beta = -\delta E_F(\Gamma)$. For some γ, an initially convex curve may lose its convexity in finite time [32]. This phenomena is a contrast to the curve-shortening flow, in which any initial nonconvex curve becomes convex in finite time [16]. The image segmentation is another application of this flow, which will be mentioned later.

The Willmore flow is the gradient flow of the Willmore functional or bending energy

$$W(\Gamma(t)) = \frac{1}{2}\int_{\Gamma(t)} k^2\, ds.$$

Hence we have the Willmore flow $\beta = -\delta W(\Gamma)$, where

$$\delta W(\Gamma) = -\partial_s^2 k - \frac{1}{2}k^3.$$

See, e.g. [6] and references therein.

Area-preserving flows are described as the form $\beta = \langle \beta_0 \rangle - \beta_0$ formally, where

$$\langle \mathsf{F} \rangle = \frac{1}{L(\Gamma(t))}\int_{\Gamma(t)} \mathsf{F}\, ds$$

is the average of F, since the enclosed area and its time derivative are given as

$$A(\Gamma(t)) = \frac{1}{2}\int_{\Gamma(t)} \boldsymbol{x}.\boldsymbol{n}\, ds, \quad \partial_t A(\Gamma(t)) = \int_{\Gamma(t)} \beta\, ds,$$

respectively. In particular, using Lagrange multiplier λ, the gradient flow of $J(\Gamma(t)) + \lambda A(\Gamma(t))$ is $\beta = -(\delta J(\Gamma) + \lambda)$, and from $\partial_t A(\Gamma(t)) = 0$ we have $\lambda = -\langle \delta J(\Gamma)\rangle$. Hence $\beta = \langle \delta J(\Gamma)\rangle - \delta J(\Gamma)$ is the area-preserving gradient flows of J. In the particular case $J(\Gamma(t)) = L(\Gamma(t))$, we have the classical area-preserving curvature flow $\beta = 2\pi/L - k$ [14]. Note that the surface diffusion flow $\beta = \partial_s^2 k$ is also area-preserving flow, which is the gradient flow of L in the H^{-1} sense [8]. Moreover, the area is always preserved under the flow $\beta = c\partial_s^m k$ with a constant c and $m = 1, 2, \ldots$. We will show an example using this flow later.

The eikonal equation is known as the gradient flow equation $\beta = -\delta A(\Gamma) = -1$, which comes from $\partial_t A(\Gamma(t)) = \langle \delta A(\Gamma), \beta\rangle$. This is area-decreasing flow such as $\partial_t A(\Gamma(t)) = -L(\Gamma(t))$.

Total-length-preserving flows are described as the form $\beta = \langle k\beta_0\rangle/\langle k\rangle - \beta_0$ formally. This is, however, not a gradient flow in general. The gradient flow of $J(\Gamma(t)) + \lambda L(\Gamma(t))$ is $\beta = -\delta J(\Gamma) - \lambda k$, and from $\partial_t L(\Gamma(t)) = 0$ we have $\lambda = -\langle k\delta J(\Gamma)\rangle/\langle k^2\rangle$. Hence $\beta = \langle k\delta J(\Gamma)\rangle k/\langle k^2\rangle - \delta J(\Gamma)$ is the length-preserving gradient flows of J. In particular, the case $J = W$ relates to the Euler's elastica [11].

The Helfrich flow is the area- and total-length-preserving gradient flow of

$$\frac{1}{2}\int_{\Gamma(t)} (k - c_0)^2\, ds,$$

that is the gradient flow of

$$\begin{aligned} H(\Gamma(t)) &= \frac{1}{2}\int_{\Gamma(t)} (k - c_0)^2\, ds + \lambda_1 L(\Gamma(t)) + \lambda_2 A(\Gamma(t)) \\ &= W(\Gamma(t)) + \left(\frac{c_0^2}{2} + \lambda_1\right) L(\Gamma(t)) + \lambda_2 A(\Gamma(t)) - 2\pi c_0 \end{aligned}$$

with Lagrange multipliers λ_1 and λ_2. Then we have the Helfrich flow $\beta = -\delta H(\Gamma)$, where

$$\delta H(\Gamma) = \delta W(\Gamma) + \lambda_1 k + \lambda_2 + \frac{c_0^2}{2} k,$$

and λ_1 and λ_2 are determined from

$$\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \frac{1}{\langle k\rangle^2 - \langle k^2\rangle}\begin{pmatrix}1 & -\langle k\rangle\\ -\langle k\rangle & \langle k^2\rangle\end{pmatrix}\begin{pmatrix}\langle k\delta W(\Gamma)\rangle + c_0^2\langle k^2\rangle/2\\ \langle \delta W(\Gamma)\rangle + c_0^2\langle k\rangle/2\end{pmatrix}, \quad \langle k\rangle = \frac{2\pi}{L(\Gamma(t))}, \tag{3.1}$$

unless k is a constant, i.e., $\Gamma(t)$ is a circle. See, e.g. [23] and references therein. The Helfrich flow with the case $c_0 = 0$ is nothing but the area- and total-length-preserving Willmore flow related to the shape optimization problem [24].

4 Effect of nontrivial tangential velocities

We recall the local length g and its time derivative $\partial_t g$:

$$g(u,t) = |\partial_u \boldsymbol{x}(u,t)| > 0, \quad \partial_t g = g(\partial_s \alpha + k\beta) \quad (u \in [0,1] \subset \mathbb{R}/\mathbb{Z},\ t \in [0,T)),$$

where $g\partial_s \alpha = \partial_u \alpha$.

Local-length-preserving flows are the case where the tangential velocity α satisfies

$$\partial_s \alpha = -k\beta.$$

This implies $\partial_t g = 0$, and $g(u,t) \equiv g(u,0)$. Hence, the total length of $\Gamma(t)$ is preserved and equals to the initial total length $L_0 = L(\Gamma(0))$:

$$L(\Gamma(t)) = \int_{\Gamma(t)} ds = \int_0^1 g(u,t)\,du = \int_0^1 g(u,0)\,du = \int_{\Gamma(0)} ds = L_0.$$

We, however, note that the total-length-preserving flow is not the same as the local-length-preserving flow. For example, in [34], the area- and local length-preserving Willmore flow is studied:

$$\beta = \langle \delta W(\Gamma) \rangle - \delta W(\Gamma), \quad \partial_s \alpha = -k\beta.$$

On the other hand, as mentioned above, the area- and total-length-preserving Willmore flow is

$$\beta = -\delta W(\Gamma) - \lambda_1 k - \lambda_2 \quad (\lambda_1 \text{ and } \lambda_2 \text{ are defined in (3.1) with } c_0 = 0),$$

which is the Helfrich flow $\beta = -\delta H(\Gamma)$ in the case $c_0 = 0$. Obviously, these two flows are different, but can be unified by means of linear interpolation between the local length in each flow [31].

Another interesting area- and local-length-preserving flow can be made artificially as follows. Assume that $g(u,0) \equiv L_0$. Under the local-length-preserving flow, we have $g(u,t) \equiv L_0$ and $\partial_s = L_0^{-1}\partial_u$, since $\partial_s \mathsf{F}(u,t) = g(u,t)^{-1}\partial_u \mathsf{F}(u,t)$. Let us try the flow $\beta = 2\mu\partial_s k$ with $\alpha = \mu k^2$ subject to the local length constraint. Then the enclosed area is preserved, and by virtue of (2.1) we obtain modified KdV equation formally:

$$\partial_t k - \frac{3\mu}{L_0} k^2 \partial_u k - \frac{2\mu}{L_0^3}\partial_u^3 k = 0 \quad (\text{especially, if } \mu = -\frac{L_0^3}{2},\ L_0 = 2).$$

Relative local-length-preserving flows are also interesting, especially from a numerical point of view. We define the relative local length by

$$r(u,t) = \frac{g(u,t)}{L(\Gamma(t))}.$$

Then relative local-length-preserving flows will state that

$$\partial_s \alpha = \langle k\beta \rangle - k\beta, \tag{4.1}$$

which is derived from

$$\partial_t r = \frac{g}{L}\left(\partial_s \alpha + k\beta - \langle k\beta \rangle\right) \equiv 0 \quad (\forall u).$$

In numerical computation, especially in the direct approach, smooth curves are approximated by polygonal curves, and each vertex evolves in time. Hence distribution of vertices play an important role in computational stability. Under this flow, if $r(u,0) \equiv 1$, then the relation $g(u,t) \equiv L(\Gamma(t))$ keeps in time. This corresponds to the uniform distribution of vertices along the curve, and this makes stable computation. Therefore, the uniform distribution scheme is one of a reasonable method in the direct approach. This topic will be mentioned later.

5 Polygonal analogues

In this section, we will show two kinds of discrete flows which are polygonal analogue to the smooth case.

Crystalline curvature flow is a kind of anisotropic curvature flow with a singular anisotropy, which will be stated as follows. As mentioned above, the gradient flow of the total interface energy $E_\gamma(\Gamma(t))$ was $\beta = -(\sigma(\theta) + \sigma''(\theta))k$. In general, for the characterization of σ, the following Frank diagram is useful

$$F_\sigma = \{\boldsymbol{x} \in \mathbb{R}^2;\ \boldsymbol{x} = \boldsymbol{n}(\theta)/\sigma(\theta),\ \theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}\},$$

since the sign of curvature of F_σ is the same as the one of $\sigma + \sigma''$. Therefore, if the Frank diagram is strictly convex, then the gradient flow $\beta = -(\sigma + \sigma'')k$ could be caught in the framework of the parabolic PDE. Besides the Frank diagram, the following Wulff shape also plays an important role (Historically, the Wulff shape appeared about 60 years before the Frank diagram):

$$W_\sigma = \bigcap_{\theta \in S^1} \{\boldsymbol{x} \in \mathbb{R}^2;\ \boldsymbol{x}.\boldsymbol{n}(\theta) \leq \sigma(\theta)\}.$$

Note that the curvature of the boundary of the Wulff shape ∂W_σ is given by $(\sigma + \sigma'')^{-1}$, and then the gradient flow is described as $\beta = -\dfrac{\text{the curvature of } \Gamma}{\text{the curvature of } W_\sigma}$.

In the case where the Frank diagram is a convex polygon, σ is called **crystalline energy**, and the gradient flow can not be handled in the usual manner, since the Wulff shape becomes also convex polygon and the curvature can not be defined. To overcome this difficulty, smooth curves are restricted in an admissible class and have introduced the crystalline curvature defined on each edge. This strategy was proposed by Taylor [38, 39, 40, 41] and independently by Angenent and Gurtin [2]. Also one can find essentially the same method as a numerical scheme for curvature flow equation in Roberts [35]. We refer the reader Almgren and Taylor [1] for detailed history.

Let σ be the crystalline energy with the Frank diagram being N_σ-sided convex polygon. Hence the corresponding Wulff shape is also N_σ-sided convex polygon which can be described as

$$W_\sigma = \bigcap_{j=1}^{N_\sigma} \left\{ \boldsymbol{x} \in \mathbb{R}^2;\ \boldsymbol{x}.\boldsymbol{n}(\varphi_j) \leq \sigma(\varphi_j) \right\},$$

where φ_j is the outward normal angle of the j-th edge.

Let Γ be an N-sided Jordan polygonal curve with the vertices $\{\boldsymbol{x}_i\}_{i=1}^N$:

$$\Gamma = \bigcap_{i=1}^{N} \Gamma_i, \quad \Gamma_i = [\boldsymbol{x}_{i-1}, \boldsymbol{x}_i] \quad (\boldsymbol{x}_0 = \boldsymbol{x}_N).$$

By θ_i we denote the outward normal angle θ_i on the i-th edge. We call curve Γ **admissible** if $\{\varphi_j\}_{j=1}^{N_\sigma} = \{\theta_i\}_{i=1}^N$ and any adjacent two angles in the set $\{\theta_i\}_{i=1}^N$ are also adjacent in the set $\{\varphi_j\}_{j=1}^{N_\sigma}$. That is, for any $\{\theta_k, \theta_{k+1}\} \subset \{\theta_i\}_{i=1}^N$ there exist l such that $\{\theta_k, \theta_{k+1}\} = \{\varphi_l, \varphi_{l+1}\} \subset \{\varphi_j\}_{j=1}^{N_\sigma}$ $(\varphi_{N_\sigma+1} = \varphi_1,\ \theta_{N+1} = \theta_1)$.

For an admissible curve $\Gamma(t)$, the total crystalline energy on $\Gamma(t)$ can be defined as $E_\sigma(\Gamma(t)) = \sum_{i=1}^N \sigma(\theta_i)|\Gamma_i(t)|$. Hence we have

$$\partial_t E_\sigma(\Gamma(t)) = \sum_{i=1}^{N} \Lambda_i \beta_i |\Gamma_i(t)|, \quad \Lambda_i = \chi_i \frac{l_\sigma(\theta_i)}{|\Gamma_i(t)|},$$

where $l_\sigma(\theta_i)$ is the length of the Wulff shape whose edge has normal angle θ_i, and χ_i is the transition number which takes $+1$ (resp. -1) if Γ is convex (resp. concave) around Γ_i in the inward normal direction, otherwise $\chi_i = 0$. Then the i-th outward normal velocity $\beta_i = -\Lambda_i$ is the gradient flow of E_σ in the discrete L^2 sense. This flow is called crystalline curvature flow.

Polygonal curvature flow is formulated by using a system of ODEs [5]. Solution polygonal curves belong to a prescribed polygonal class, which is similar to admissible class used in the crystalline curvature flow. Actually, if the initial curve is a convex polygon, then our polygonal curvature flow is nothing but the crystalline curvature flow. However, if the initial polygon is not convex and does not belong to any admissible class, then the polygonal curvature flow can not be regarded as a crystalline curvature flow. Because the prescribed polygonal class is determined by the initial polygon and one can take any polygon as the initial data. On the other hand, in the framework of the crystalline curvature flow, the initial polygon should be taken from the admissible class. The polygonal curvature flow provides polygonal analogue of several moving boundary problems including curvature flow, the area- and/or length-preserving flow, Hele-Shaw flow, advected flow, etc. From a numerical computation viewpoint, the ODEs are discretized implicitly in time keeping a given constant area or length speed, while the solution polygonal curve exists in the prescribed polygonal class.

6 Curvature adjusted tangential velocity

As mentioned in the first section, the tangential velocity functional α has no effect of the shape of evolving curves [12, Proposition 2.4], and the shape is determined by the value of the normal velocity β only. Hence the simplest setting $\alpha \equiv 0$ can be chosen. Dziuk [10] studied a numerical scheme for $\beta = -k$ in this case. In the case general β, however, such a choice of α may lead to various numerical instabilities caused by either undesirable concentration and/or extreme dispersion of numerical grid points. Therefore, to obtain stable numerical computation, several choices of a nontrivial tangential velocity have been emphasized and developed by many authors. We will present a brief review of development of nontrivial tangential velocities.

Kimura [21, 22] proposed a uniform redistribution scheme in the case $\beta = -k$ by using a special choice of α which satisfies discretization of an average condition and the uniform distribution condition:

$$\text{(U)} \quad r(u,t) = \frac{g(u,t)}{L(\Gamma(t))} \equiv 1 \quad (\forall u).$$

Hou, Lowengrub and Shelley [17] utilized condition (U) directly (especially for $\beta = -k$) starting from $r(u,0) \equiv 1$, and derived (4.1). It was proposed independently by Mikula and Ševčovič [27]. In [17, Appendix 2] Hou et al. also pointed out generalization of (4.1) as follows:

$$\frac{\partial_s(\varphi(k)\alpha)}{\varphi(k)} = \frac{\langle f\rangle}{\langle\varphi(k)\rangle} - \frac{f}{\varphi(k)}, \quad f = \varphi(k)k\beta - \varphi'(k)\left(\partial_s^2\beta + k^2\beta\right) \tag{6.1}$$

for a given function φ. If $\varphi \equiv 1$, then this is nothing but (4.1). (6.1) is derived from the following calculation. Let a generalized relative local length be

$$r_\varphi(u,t) = r(u,t)\frac{\varphi(k(u,t))}{\langle\varphi(k(\cdot,t))\rangle}.$$

Then preserving condition $\partial_t r_\varphi(u,t) \equiv 0$ leads (6.1).

As mentioned above, in the paper [27] the authors arrived (4.1) in general frame work of the so-called intrinsic heat equation for $\beta = \beta(\theta,k)$. This result was improvement of [26] in which satisfactory results were obtained only in the case $\beta = \beta(k)$ being linear and sublinear with respect to k. After these results, in the series of the paper [28, 29, 30], they proposed method of asymptotically uniform redistribution, i.e., derived

$$\partial_s\alpha = \langle k\beta\rangle - k\beta + \left(r^{-1} - 1\right)\omega(t) \tag{6.2}$$

for quite general normal velocity $\beta = \beta(\boldsymbol{x}, \theta, k)$, where $\omega \in L^1_{loc}[0, T)$ is a relaxation function satisfying $\int_0^T \omega(t)\,dt = +\infty$. Their method succeeded and was applied to geodesic curvature flows and image segmentation, etc.

Besides these uniform distribution method, under the so-called crystalline curvature flow, grid points are distributed dense (resp. sparse) on the subarc where the absolute value of curvature is large (resp. small). Although this redistribution is far from uniform, numerical computation is quite stable. One of the reason is that polygonal curves are restricted in an admissible class. To apply the essence of crystalline curvature flow to a general discretization model of motion of smooth curves, the tangential velocity $\alpha = \partial_s\beta/k$ was extracted, which is utilized in crystalline curvature flow equation implicitly [42].

The asymptotically uniform redistribution is quite effective and valid for wide range of application. However, from approximation point of view, unless solution curve is a circle, there is no reason to take uniform redistribution. Hence the redistribution will be desired in a way of taking into account the shape of evolution curves, i.e., depending on size of curvatures. In the paper [36, 37], it is proposed that a method of redistribution which takes into account the shape of limiting curve such as

$$\frac{\partial_s(\varphi(k)\alpha)}{\varphi(k)} = \frac{\langle f \rangle}{\langle \varphi(k) \rangle} - \frac{f}{\varphi(k)} + \left(r_\varphi^{-1} - 1\right)\omega(t).$$

If $\varphi \equiv 1$, then this is nothing but (6.2), and if $\varphi = k$ and Γ is convex, we have $\alpha = \partial_s\beta/k$ in the case $\omega = 0$. Therefore, this is a combination of method of asymptotic uniform redistribution and the crystalline tangential velocity as mentioned above. Notice that this method was applied to an image segmentation and nice results were confirmed [4].

To complete the overview of various tangential redistribution method we also mention a locally dependent tangential velocity. For the case $\beta = -k$ it was proposed by Deckelnick [9] who used $\alpha = -\partial_u(g^{-1})$. Then the evolution equation becomes a simple parabolic PDE $\partial_t\boldsymbol{x} = g^{-2}\partial_u^2\boldsymbol{x}$.

As far as 3D implementation of tangential redistribution is concerned, in a recent paper by Barrett, Garcke and Nürnberg [3] the authors proposed and studied a new efficient numerical scheme for evolution of surfaces driven by the Laplacian of the mean curvature. It turns out, that their numerical scheme has implicitly built in a uniform redistribution tangential velocity vector.

7 Image segmentation

The gradient flow $\beta = -\gamma(\boldsymbol{x})k - \nabla\gamma(\boldsymbol{x}).\boldsymbol{n}$ is utilized for image segmentation as follows. Let an image intensity function be $I : \mathbb{R}^2 \supset \Omega \to [0, 1]$. Here $I = 0$ (resp. $I = 1$) corresponds to black (resp. white) color and $I \in (0, 1)$ corresponds to gray colors. For simplicity, we assume that our target figures are given in white color with black background. Then the image outline or edge correspond to the region where $|\nabla I(\boldsymbol{x})|$ is quite large. Let us introduce an auxiliary function $\gamma(\boldsymbol{x}) = f(|\nabla I(\boldsymbol{x})|)$ where f is a smooth edge detector function like e.g. $f(s) = 1/(1+s^2)$ or $f(s) = e^{-s}$. Hence the solution curve $\Gamma(t)$ of $\beta = -\gamma(\boldsymbol{x})k - \nabla\gamma(\boldsymbol{x}).\boldsymbol{n}$ makes the energy $E_F(\Gamma(t))$ smaller and smaller, in other words, its moves toward the edge where $|\nabla I(\boldsymbol{x})|$ is large. This is a fundamental idea of image segmentation, and it has developed to a sophisticated scheme [29, 30].

The following scheme is more simple [4]. In the following computations, the target figure is given by a digital gray scale bitmap image represented by integer values between 0 and 255 on some prescribed pixels. The values 0 and 255 correspond to black and white colors, respectively, whereas the values between 0 and 255 correspond to gray colors.

Given a figure, we can construct its image intensity function $I : \mathbb{R}^2 \supset \Omega \to \{0, \ldots, 255\} \subset \mathbb{Z}$. Note that $I(\boldsymbol{x})$ is piecewise constant in each pixel.

We consider the flow $\beta = -k + F$ and define the forcing term $F(\boldsymbol{x})$ as follows:

$$F(\boldsymbol{x}) = (F_{max} - F_{min})\frac{I(\boldsymbol{x})}{255} - F_{max} \quad (\boldsymbol{x} \in \Omega),$$

where $F_{max} > 0$ corresponds to purely black color (background) and $F_{min} < 0$ corresponds to purely white color (the object to be segmented). Maximal and minimal values determine the final shape because in general $1/F$ is equivalent to the minimal radius the curve can attain. The choice of small values of F causes the final shape to be rounded or the curve can not pass through narrow gaps.

8 Modeling of negative crystal growth

When a block of ice is exposed to solar beams or other radiation, internal melting of ice occurs. That is, internal melting starts from some interior points of ice without melting the exterior portions, and each water region forms a flower of six petals, which is called "Tyndall figure". The figure is filled with water except for a vapor bubble. This phenomenon was first observed by Tyndall (1858). When Tyndall figure is refrozen, the vapor bubble remains in the ice as a hexagonal disk. This hexagonal disk is filled with water vapor saturated at that temperature and surrounded by ice. McConnel found these disks in the ice of Davos lake [25]. Nakaya called this hexagonal disk "vapor figure" and investigated its properties precisely [33]. Adams and Lewis (1934) called it "negative crystal". Although Nakaya said "this term does not seem adequate" with a certain reason, hereafter we use the term negative crystal for avoiding confusion.

Negative crystal is useful to determine the structure and orientation of ice or solids. Because, within a single ice crystal, all negative crystals are similarly oriented, that is, corresponding edges of hexagon are parallel each other. Furukawa and Kohata made hexagonal prisms experimentally in a single ice crystal, and investigated the habit change of negative crystals with respect to the temperatures and the evaporation mechanisms of ice surfaces [13].

To the best of author's knowledge, after the Furukawa and Kohata's experimental research, there have been no published results on negative crystals, and there are no dynamical model equations describing the process of formation of negative crystals. In the present talk, we will focus on the process of formation of negative crystals after Tyndall figures are refrozen, and try to propose a model equation of interfacial motion which tracks the deformation of negative crystals in time.

The refrozen process may be summarized as the phrase "Negative crystal changes the shape from oval to hexagon". Then our model will be assumed that (1) water vapor region is simply connected and bounded region in the plane $\mathbb{R}^2$ (we denote it by Ω); (2) Ω is surrounded by a single ice crystal (i.e., ice region is $\mathbb{R}^2\backslash\Omega$); (3) moving boundary $\partial\Omega$ is interface of the water vapor region and the single ice crystal; and (4) c-axis (main axis) of the single ice crystal is perpendicular to the plane.

The evolution law of moving interface is similar to the growth of snow crystal, since deformation of negative crystal is regarded as crystal growth in the air. As a model of snow crystal growth, we refer the Yokoyama-Kuroda model [43], which is based on the diffusion process and the surface kinetic process by Burton-Cabrera-Frank (BCF) theory [7]. Meanwhile, we assume the existence of interfacial energy (density) on the boundary $\partial\Omega$. The equilibrium shape of negative crystal is a regular hexagon, and if the region Ω is very close to a regular hexagon, then the evolution process may be described as a gradient flow of total interfacial energy subject to a fixed enclosed area. Therefore, we can divide the process of formation of negative crystals into two stages as follows: (Stage 1) in the former stage by the diffusion and the surface kinetic, oval Ω changes to a hexagon; and (Stage 2) in the latter stage by a gradient flow of interfacial energy, the hexagon converges to a regular hexagon.

A simple modeling of these stages is proposed as an area-preserving crystalline curvature flow for "negative" polygonal curves. See [18, 19, 20] in detail.

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