Gradient Flow for the Helfrich Variational Problem

Takeyuki Nagasawa
Department of Mathematics, Faculty of Science,
Saitama University
Saitama 338-8570, Japan

Abstract

The gradient flow associated to the Helfrich variational problem, called the Helfrich flow, is considered. A local existence result of \( n \)-dimensional Helfrich flow is given for any \( n \). We also discuss known results, related topics, the development of our research group in this decade, and some open problems.

1 The Helfrich variational problem and its background

Let \( \Sigma \subset \mathbb{R}^{n+1} \) be a closed and oriented hypersurface immersed in \( \mathbb{R}^{n+1} \). We do not assume that the inclusion \( \Sigma \subset \mathbb{R}^{n+1} \) is an embedding. The function \( H \) stands for the mean curvature. The integral

\[
\int_{\Sigma} H^2 dS
\]

is called the Willmore functional, in which many mathematicians have been interested.

Now consider a variational problem for a functional related with the Willmore functional under some constraints. Let \( A(\Sigma) \) be the area of \( \Sigma \). The vectors \( f \) and \( \nu \) are the position vector of a point on \( \Sigma \) and the unit normal vector there respectively. Put

\[
V(\Sigma) = -\frac{1}{n+1} \int_{\Sigma} f \cdot \nu dS.
\]
This is the enclosed volume, when $\Sigma$ is an embedded hypersurface and $\nu$ is the inner normal. For given constants $c_0$, $A_0$, and $V_0$, consider critical points of

$$W(\Sigma) = \frac{n}{2} \int_{\Sigma} (H - c_0)^2 dS$$

under the constrains $A(\Sigma) = A_0$, $V(\Sigma) = V_0$.

This problem was firstly proposed by Helfrich [5] as a model of shape transformation theory of human red blood cells. For this case $n$ is 2, and $c_0$ is the spontaneous curvature which is determined by the molecular structure of cell membrane. The surface $\Sigma$ stands for the cell membrane.

For $n = 1$, the functional is

$$\frac{1}{2} \int_{\Sigma} \kappa^2 ds - c_0 \int_{\Sigma} \kappa ds + \frac{1}{2} c_0^2 \int_{\Sigma} ds,$$

where $\kappa(= H)$ is the curvature of the curve $\Sigma$, and $s$ is the arch-length parameter. If we consider the variational problem under the constrain of length $A$ among curves with fixed rotation number, then we can replace the functional with the first integral $\frac{1}{2} \int_{\Sigma} \kappa^2 ds$ only. Because the second and third integrals are respectively constant multiples of rotation number and the length, which are invariant in our problem. According to [3], a shape transformation of a closed loop of plastic tape between two parallel flat plates is governed by the one-dimensional Helfrich variational problem. This problem is also related with the spectral optimization problem for plain domains. Let $\Omega$ be a bounded plane domain, and $\Sigma$ be its boundary. The function $G(x, y, t)$ is the Green function for the heat equation on $\Omega \times (0, T)$. The asymptotic expansion

$$\int_{\Omega} G(x, x, t) dx = \frac{1}{4\pi t} \left( a_0 + a_1 t^{\frac{1}{2}} + a_2 t + a_3 t^{\frac{3}{2}} + \cdots \right) \quad (t \to +0)$$

are well-known as the trace formula. Here

$$a_0 = V(\Sigma), \quad a_1 = -\frac{\sqrt{\pi}}{2} A(\Sigma), \quad a_2 = \frac{1}{3} \int_{\Sigma} \kappa ds, \quad a_3 = \frac{\sqrt{\pi}}{64} \int_{\Sigma} \kappa^2 ds.$$

$a_2$ is determined by the topology of $\Omega$. Hence the one-dimensional Helfrich problem is equivalent to the following problem: For given $a_0$, $a_1$ and $a_2$ find the domain $\Omega$ which minimize $a_3$. This problem was proposed and investigated by Watanabe [19, 20].
2 Known results

By the method of Lagrange multipliers, the Helfrich variational problem is described as
\[ \delta W(\Sigma) + \lambda_1 \delta A(\Sigma) + \lambda_2 V(\Sigma) = 0. \]
Here \( \delta \) stands for the first variation, and \( \lambda_j \)'s are Lagrange multipliers. According to [4], the above equation becomes
\[ \Delta_g H + (H - c_0) \left\{ \frac{n^2}{2} H(H + c_0) + R \right\} - \lambda_1 n H - \lambda_2 = 0. \]
Here \( \Delta_g \) is the Laplace-Beltrami operator, and \( R \) is the scalar curvature. Regarding \( \Sigma \) as the image \( f(\Sigma_0) \) of a \( (n-1) \)-dimensional manifold \( \Sigma_0 \), we obtain a quasilinear elliptic equation of forth order.

The two-dimensional Helfrich problem has a long history, and there are several known facts. It is easy to see spheres are critical points. In 1977, Jenkins [6] had found bifurcating solutions from spheres numerically. Subsequently Peterson [16] and Ou-Yang-Helfrich [15] formally investigated their stability/instability. Their arguments were justified mathematically by Takagi and the author in [11]. Au-Wan [2] considered critical points far from spheres but with rotational symmetry. Critical points without rotational symmetry were constructed by Takagi and the author [12].

In this article, we consider the associated gradient flow, called the Helfrich flow
\[ v(t) = -\delta W(\Sigma(t)) - \lambda_1 \delta A(\Sigma(t)) - \lambda_2 \delta V(\Sigma(t)). \quad (2.1) \]
The function \( v = \partial_{t}f \cdot \nu \) is the normal velocity of deformation of families of hypersurfaces \( \Sigma(t) \). We shall overview known results about the Helfrich flow in the next section.

3 The Helfrich flow

In considering the flow problem, the multiplies are unknown functions of \( t \). It is natural that they are determined so that \( \frac{d}{dt}A(\Sigma(t)) = \frac{d}{dt}V(\Sigma(t)) = 0 \). We have
\[ \frac{d}{dt}A(\Sigma(t)) = \langle \delta A(\Sigma(t)), v(t) \rangle, \quad \frac{d}{dt}V(\Sigma(t)) = \langle \delta V(\Sigma(t)), v(t) \rangle, \]
where \( \langle \cdot, \cdot \rangle \) is the \( L^2(\Sigma(t)) \)-inner product. It follows from these and (2.1) that
\[ \left( \begin{array}{c} \langle \delta A(\Sigma(t)), \delta A(\Sigma(t)) \rangle \\ \langle \delta A(\Sigma(t)), \delta V(\Sigma(t)) \rangle \\ \langle \delta V(\Sigma(t)), \delta A(\Sigma(t)) \rangle \\ \langle \delta V(\Sigma(t)), \delta V(\Sigma(t)) \rangle \end{array} \right) \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right) = - \left( \begin{array}{c} \langle \delta A(\Sigma(t)), \delta W(\Sigma(t)) \rangle \\ \langle \delta V(\Sigma(t)), \delta W(\Sigma(t)) \rangle \end{array} \right). \quad (3.1) \]
Denote the Gramian of the left-hand side by $G(\Sigma(t))$. If $G(\Sigma(t))$ does not vanish, then the multipliers are uniquely determined by the above relation. In this case we denote

$$\lambda_j = \lambda_j(\Sigma(t)).$$

When $G(\Sigma(t))$ vanishes, the multipliers are not uniquely determined, but we can show that $\lambda_1 \delta A(\Sigma(t)) + \lambda_2 \delta(\Sigma(t))$ is uniquely determined.

**Theorem 3.1** Let $P(\Sigma)$ be the orthogonal projection from $L^2(\Sigma)$ to $(\text{span}_{L^2(\Sigma)} \{\delta A(\Sigma), \delta V(\Sigma)\})^\perp$. Then the equation of Helfrich flow can be written as

$$v(t) = -P(\Sigma(t))\delta W(\Sigma(t)) \quad (t > 0).$$

Solutions, if exist, satisfy

$$\frac{d}{dt} W(\Sigma(t)) \equiv -\|v(t)\|_{L^2(\Sigma(t))}^2, \quad \frac{d}{dt} A(\Sigma(t)) \equiv 0, \quad \frac{d}{dt} V(\Sigma(t)) \equiv 0. \quad (3.3)$$

We get the existence and uniqueness of the initial value problem. Let $\Sigma_0$ be the initial hypersurface, and $h^\alpha$ be the little Hölder space.

**Theorem 3.2** (i) Assume that $\Sigma_0$ is in the class of $h^{3+\alpha}$ for some $\alpha \in (0, 1)$, and that $G(\Sigma_0) \neq 0$. Then there exists $T > 0$ such that there uniquely exists the solution $\{\Sigma(t)\}_{0 \leq t < T}$ of (3.2) satisfying $\Sigma(0) = \Sigma_0$.

(ii) Assume that $G(\Sigma_0) = 0$. $H_0$ and $R_0$ are the mean curvature and the scalar curvature of $\Sigma_0$ respectively. Put

$$\bar{H}_0 = \frac{1}{A(\Sigma_0)} \int_{\Sigma_0} H_0 dS, \quad \bar{R}_0 = R_0 - \frac{1}{A(\Sigma_0)} \int_{\Sigma_0} R_0 dS.$$

If $(\bar{H}_0 - c_0) \bar{R}_0 \equiv 0$, then there exists a global solution $\{\Sigma(t)\}_{t \geq 0}$ of (3.2) satisfying $\Sigma(0) = \Sigma_0$.

**Remark 3.1** The uniqueness is uncertain in the case (ii). We, however, can show the uniqueness when $n = 1$. See Theorem 5.1.

Sketches of proofs shall be given in the next two sections. For details, see [13].

The low-dimensional Helfrich flow has been considered in [7] (for $n = 2$) and in [9] (for $n = 1$).

In [7], the multiplier $\lambda_j$'s are not determined as above, but are given as "known" constants. That is, for given $\{\lambda_1, \lambda_2, \Sigma_0\}$ as the data, solutions of (2.1) were constructed. Of course, solutions do not satisfy $\frac{d}{dt} A(\Sigma(t)) \equiv 0,$
\[ \frac{d}{dt}V(\Sigma(t)) \equiv 0, \text{ and we cannot expect the global existence. Indeed, there exist solutions blowing up in finite/infinite time. The problem is shifted to find triples } \{ \lambda_1, \lambda_2, \Sigma_0 \} \text{ so that the solution can extend globally in time. In [7], the existence of such triples near spheres. Furthermore, such triples form a finite dimensional center manifold. The class of initial surfaces is } h^{2+\alpha} \text{ for some } \alpha \in (0, 1), \text{ which is wider than ours. In our formulation } \nabla_{g}H \text{ appears in the concrete expression of } \lambda_j(\Sigma(t)), \text{ and therefore we need extra regularity of } \Sigma_0 \text{ than [7]. See Remark 5.1 below.}

In [9], the gradient flow \{ \Sigma_{\epsilon}(t) \} associated with the functional
\[
W(\Sigma_{\epsilon}) + \frac{1}{2\epsilon}(A(\Sigma_{\epsilon}) - A_0)^2 + \frac{1}{2\epsilon}(V(\Sigma_{\epsilon}) - V_0)^2
\]
was constructed. The solution of (2.1) was obtained as the limit of \{ \Sigma_{\epsilon}(t) \} as \( \epsilon \to +0 \). This is a global solution, and satisfies (3.3). The class of initial curves is \( C^\infty \), but the uniqueness was uncertain.

4 Proof of Theorem 3.1

Theorem 3.1 is a special case of general theory of projected gradient flow [18]. We denote \( \Sigma(t) \) simply by \( \Sigma \). \( \| \cdot \| \) stands for the \( L^2(\Sigma) \)-norm. Put
\[
\tilde{H} = H - \frac{1}{A(\Sigma)} \int_{\Sigma} H \, dS, \quad H_* = \begin{cases} \frac{\tilde{H}}{\| \tilde{H} \|} & \text{if } \tilde{H} \not\equiv 0, \\ 0 & \text{if } \tilde{H} \equiv 0, \end{cases} \quad 1_* = \frac{1}{\| 1 \|}.
\]
Note that \( \langle H_*, 1_* \rangle = 0 \). Since \( \delta A(\Sigma) = -nH \) and \( \delta V(\Sigma) = -1 \), we have
\[
\text{span}_{L^2(\Sigma)} \{ \delta A(\Sigma), \delta V(\Sigma) \} = \text{span}_{L^2(\Sigma)} \{ H, 1 \} = \text{span}_{L^2(\Sigma)} \{ H_*, 1_* \}.
\]
Hence (2.1) becomes
\[
v = -\delta W(\Sigma) - \lambda_1 \delta A(\Sigma) - \lambda_2 \delta V(\Sigma) = -\delta W(\Sigma) - \mu_1 1_* - \mu_2 H_* \quad (4.1)
\]
for some \( \mu_j \). It follows from \( \frac{d}{dt}A(\Sigma) = \frac{d}{dt}V(\Sigma) = 0 \) that
\[
\langle 1_*, v \rangle = \langle H_*, v \rangle = 0.
\]
Taking the \( L^2(\Sigma) \)-inner product (4.1) and \( 1_*, H_* \), we get
\[
0 = \langle 1_*, v \rangle = \langle 1_*, \delta W(\Sigma) \rangle - \mu_1, \quad 0 = \langle H_*, v \rangle = \langle H_*, \delta W(\Sigma) \rangle - \mu_2 \| H_* \|^2.
\]
In spite of \( H_* \equiv 0 \) or not, it holds that
\[
-\mu_1 1_* - \mu_2 H_* = \langle 1_*, \delta W(\Sigma) \rangle 1_* + \langle H_*, \delta W(\Sigma) \rangle H_*.
\]
Consequently we obtain (3.2).

It holds for solutions to (3.2) that

\[ \frac{d}{dt} W(\Sigma) = \langle \delta W(\Sigma), v \rangle = \langle \delta W(\Sigma), -P(\Sigma)\delta W(\Sigma) \rangle = -\|P(\Sigma)\delta W(\Sigma)\|^2 = -\|v\|^2. \]

Since \( v \in (\text{span} \{\delta A(\Sigma), \delta V(\Sigma)\})^\perp \), we have

\[ \frac{d}{dt} A(\Sigma) = \langle \delta A(\Sigma), v \rangle = 0, \quad \frac{d}{dt} V(\Sigma) = \langle \delta V(\Sigma), v \rangle = 0. \]

\[ \square \]

5 Sketch of Proof of Theorem 3.2

The local existence for the case \( G(\Sigma_0) \neq 0 \) is in a similar manner to [7]. If the Helfrich flow with \( \Sigma(0) = \Sigma_0 \) exists, and if \( \Sigma \) is close to \( \Sigma_0 \) in \( C^2 \)-sense for small \( t > 0 \), then \( G(\Sigma) \neq 0 \). It follows from (3.1) that

\[ \begin{pmatrix} \lambda_1(\Sigma) \\ \lambda_2(\Sigma) \end{pmatrix} = -\frac{1}{G(\Sigma)} \begin{pmatrix} \langle \delta V(\Sigma), \delta V(\Sigma) \rangle & -\langle \delta V(\Sigma), \delta A(\Sigma) \rangle \\ -\langle \delta A(\Sigma), \delta V(\Sigma) \rangle & \langle \delta A(\Sigma), \delta A(\Sigma) \rangle \end{pmatrix} \begin{pmatrix} \langle \delta A(\Sigma), \delta W(\Sigma) \rangle \\ \langle \delta V(\Sigma), \delta W(\Sigma) \rangle \end{pmatrix}. \] (5.1)

Taking into the first variation formulas of \( A, V \), and \( W \) (see [4]), we have

\[ \langle \delta A(\Sigma), \delta A(\Sigma) \rangle = n^2 \int_{\Sigma} H^2 dS, \quad \langle \delta A(\Sigma), \delta V(\Sigma) \rangle = n \int_{\Sigma} H dS, \]

\[ \langle \delta V(\Sigma), \delta V(\Sigma) \rangle = \int_{\Sigma} dS, \]

\[ \langle \delta A(\Sigma), \delta W(\Sigma) \rangle = n \int_{\Sigma} \left( |\nabla_g H|^2 - \frac{1}{2} n^2 H^4 + H^2 R - c_0 HR + \frac{1}{2} n c_0^2 H^2 \right) dS, \]

\[ \langle \delta V(\Sigma), \delta W(\Sigma) \rangle = \int_{\Sigma} \left( -\frac{1}{2} n^2 H^3 + HR - c_0 R + \frac{1}{2} n c_0^2 H \right) dS, \]

\[ G(\Sigma) = \int_{\Sigma} n^2 H^2 dS \int_{\Sigma} dS - \left( \int_{\Sigma} nH dS \right)^2 = n^2 \int_{\Sigma} dS \int_{\Sigma} \tilde{H}^2 dS. \] (5.2)

Inserting these into (5.1), we have the concrete expression of \( \lambda_j(\Sigma) \)'s. Thus we get
Proposition 5.1 When $G(\Sigma) \neq 0$, the Lagrange multiplies $\lambda_j(\Sigma)$ are written by

$$\int_{\Sigma} |\nabla_g H|^2 dS, \quad \int_{\Sigma} H^p dS \quad (p = 0, 1, 2, 3, 4), \quad \int_{\Sigma} H^q R dS \quad (q = 0, 1, 2),$$

on which the multipliers analytically depend.

In order to prove Theorem 3.2 (i), we regard $\Sigma$ as the perturbation of $\Sigma_0$ in the normal direction with signed distance $\rho$. It is possible for a short time interval. Let $\bigcup_{\ell=1}^{m} U_{\ell}$ be an open covering of $\Sigma_0$. We denote the inner unit normal vector fields of $\Sigma_0$ by $\nu_0$. The mapping $X_\ell : U_\ell \times (-a, a) \ni (s, r) \mapsto s + r\nu_0(s) \in \mathbb{R}^{n+1}$ is a $C^\infty$-diffeomorphism from $U_\ell \times (-a, a)$ to $\mathcal{R}_\ell = \text{Im}(X_\ell)$ provided $a > 0$ is sufficiently small. Let denote the inverse mapping $X_\ell^{-1}$ by $(S_\ell, \Lambda_\ell)$, where $S_\ell(X_\ell(s, r)) = s \in U_\ell$, and $\Lambda_\ell(X_\ell(s, r)) = r \in (-a, a)$.

When $\Sigma(t)$ is sufficiently close to $\Sigma_0$ for small $t > 0$, we can represent it as a graph of a function on $\Sigma_0$ as

$$\Sigma_{\rho(t)} = \Sigma(t) = \bigcup_{\ell=1}^{m} \text{Im}(X_\ell : U_\ell \rightarrow \mathbb{R}^{n+1}, [s \mapsto X_\ell(s, \rho(s, t))] ).$$

Conversely for a given function $\rho : \Sigma_0 \times [0, T) \rightarrow (-a, a)$ we define the mapping $\Phi_{\ell, \rho}$ from $\mathcal{R}_\ell \times [0, T)$ to $\mathbb{R}$ by

$$\Phi_{\ell, \rho}(x, t) = \Lambda_\ell(x) - \rho(S_\ell(x), t).$$

Then $(\Phi_{\ell, \rho}(\cdot, t))^{-1}(0)$ gives the surface $\Sigma_{\rho(t)}$.

The velocity in the direction of inner normal vector field of $\Sigma = \{\Sigma_{\rho(t)} | t \in [0, T]\}$ at $(x, t) = (X_\ell(s, \rho(s, t)), t)$ is given by

$$v(s, t) = -\frac{\partial_t \Phi_{\ell, \rho}(x, t)}{||\nabla_x \Phi_{\ell, \rho}(x, t)||_{x=X_\ell(s, \rho(s, t))}} = \frac{\partial_t \rho(s, t)}{||\nabla_x \Phi_{\ell, \rho}(x, t)||_{x=X_\ell(s, \rho(s, t))}}.$$

We can write down the Laplace-Beltrami operator, the mean curvature, the scalar curvature, and the Lagrange multipliers in terms of the function $\rho$ and its derivatives, denoted $\Delta_\rho$, $H(\rho)$, $R(\rho)$, and $\lambda_j(\rho)$ respectively. Then the equation (3.2) is represented as

$$\partial_t \rho = L_\rho \left( -\Delta_\rho H(\rho) - \frac{1}{2} n^2 H^3(\rho) + H(\rho) R(\rho) - c_0 R(\rho) + \frac{1}{2} nc_0^2 H(\rho) \right) + \lambda_1(\rho) n H(\rho) + \lambda_2(\rho),$$

(5.3)
where

\[ L_\rho = \| \nabla_x \Phi_{\ell,\rho}(x, t) \|_{x=x(s,\rho(s,t))}. \]

We can find the expression of not only \( \Delta_\rho, H( \rho ) \) but also the Gaussian curvature \( K( \rho ) \) in [7] for the case \( n = 2 \). In our case the expression of \( \Delta_\rho \) and \( H( \rho ) \) is the same as in [7], and we can get that of \( R( \rho ) \) in a similar way. In particular \( \lambda_j( \rho ) \) can be written in terms of \( \rho \) and its derivatives up to third order. Combining Proposition 5.1, we can see that the right-hand side of (5.3) is linear with respect to the fourth-order derivative of \( \rho \), but not linear with respect to lower derivatives. The principal term \(-L_\rho \Delta_\rho H( \rho )\) is the same as the equation dealt with [7, (2.1)]. Let \( h^\gamma( \Sigma_0 ) \) be the little Hölder space on \( \Sigma_0 \) of order \( \gamma \). We fix \( 0 < \alpha < \beta < 1 \). Then, for \( \beta_0 \in (\alpha, \beta) \) and \( a > 0 \), put

\[ U = \{ \rho \in h^{3+\beta_0}( \Sigma_0 ) \mid \| \rho \|_{C^2( \Sigma_0 )} < a \}. \]

For two Banach spaces \( E_0 \) and \( E_1 \) satisfying \( E_1 \hookrightarrow E_0 \), the set \( \mathcal{H}( E_1, E_0 ) \) is the class of \( A \in \mathcal{L}( E_1, E_0 ) \) such that \(-A\), considered as an unbounded operator in \( E_0 \), generates a strongly continuous analytic semigroup on \( E_0 \).

**Proposition 5.2** There exist

\[ Q \in C^\infty( U, \mathcal{H}( h^{4+\alpha}( \Sigma_0 ), h^\alpha( \Sigma_0 ) ) ), \quad F \in C^\infty( U, h^{\beta_0}( \Sigma_0 ) ) \]

such that the equation (5.3) is in the form

\[ \rho_t + Q( \rho ) \rho + F( \rho ) = 0. \]

Applying [1, Theorem 12.1] with \( X_\beta = U, E_1 = h^{4+\alpha}( \Sigma_0 ), E_0 = h^\alpha( \Sigma_0 ), \) and \( E_\gamma = h^{\beta_0}( \Sigma_0 ) \), we get the assertion (i) in Theorem 3.2.

**Remark 5.1** The equation dealt with in [7] is a similar fourth-order equation, but linear with respect to the third order derivative of \( \rho \). Therefore it was solvable for initial data in the class \( h^{2+\alpha} \).

Now consider the assertion (ii) in Theorem 3.2. Before going to prove, we see an example of \( \Sigma_0 \) satisfying \( G( \Sigma_0 ) = 0 \) and \( \{ \tilde{H}_0 - c_0 \} \tilde{R}_0 \equiv 0 \). A typical example is a sphere. Indeed, spheres have constant mean curvature, and there for \( G( \Sigma_0 ) = 0 \) (see (5.2)). Since the scalar curvature is also constant, we have \( \tilde{R}_0 = 0 \). Furthermore spheres are stationary solutions to (3.2).

To show the assertion (ii), it is enough to see that \( \Sigma_0 \) is a stationary solution.

Assume that \( G( \Sigma ) = 0 \). It follows from (5.2) that \( \Sigma \) has a constant mean curvature \( H = \tilde{H} \). Hence

\[ \text{span}_{L^2( \Sigma )} \{ \delta A( \Sigma ), \delta V( \Sigma ) \} = \text{span}_{L^2( \Sigma )} \{ 1 \}, \]
and

\[ P(\Sigma)\phi = \phi - \frac{1}{A(\Sigma)} \int_\Sigma \phi \, dS \]

for \( \phi \in L^2(\Sigma) \). Therefore at the time when \( G(\Sigma(t)) = 0 \), the equation (3.2) becomes

\[
\begin{align*}
v(t) &= -\delta W(\Sigma) + \frac{1}{A(\Sigma)} \int_\Sigma \delta W(\Sigma) \, dS \\
&= -\Delta_g \tilde{H} - \frac{1}{2} n^2 \tilde{H}^3 + \tilde{H} R - c_0 R + \frac{1}{2} nc_0^2 \tilde{H} \\
&\quad + \frac{1}{A(\Sigma)} \int_\Sigma \left( \frac{1}{2} n^2 \tilde{H}^3 - \tilde{H} R + c_0 R - \frac{1}{2} nc_0^2 \tilde{H} \right) \, dS \\
&= - \left( \tilde{H} - c_0 \right) \tilde{R},
\end{align*}
\]

where

\[ \tilde{R} = R - \frac{1}{A(\Sigma)} \int_\Sigma R \, dS. \]

Consequently if the hypersurface \( \Sigma_0 \) satisfies \( G(\Sigma_0) = 0 \) and \( (\tilde{H}_0 - c_0) \tilde{R}_0 \equiv 0 \), then it is a stationary solution of (3.2).

We do not know the uniqueness in case of Theorem 3.2 (ii), expect for \( n = 1 \).

**Theorem 5.1** Consider the one-dimensional Helfrich flow. If \( \Sigma_0 \) satisfies \( G(\Sigma_0) = 0 \), then \( \{ \Sigma(t) \equiv \Sigma_0 \} \) is the unique global solution with \( \Sigma(0) = \Sigma_0 \).

**Remark 5.2** When \( n = 1 \), the scalar curvature is zero by its definition, and therefore the condition \( (\tilde{H}_0 - c_0) \tilde{R}_0 \equiv 0 \) is automatically satisfied.

**Proof.** When \( n = 1 \), the integral \( \int_\Sigma H \, dS \) is a constant multiple of the rotation number. Therefore it does not depend on \( t \). Consequently we have

\[
\frac{d}{dt} G(\Sigma) = A_0 \frac{d}{dt} \int_\Sigma H^2 \, dS = 2A_0 \frac{d}{dt} W(\Sigma) = -2A_0 \|v\|^2 \leq 0.
\]

Combining this with \( G(\Sigma) \geq 0 \) (see (5.2)), it holds that \( G(\Sigma) \equiv 0 \) provided \( G(\Sigma_0) = 0 \). Using the above relation again, we have \( v \equiv 0 \), that is, \( \Sigma \) is stationary. \( \square \)
6 Gramian estimates

Assume that \( G(\Sigma_{0}) \neq 0 \), then we may do \( G(\Sigma) \neq 0 \) for small \( t > 0 \). Since \((G(\Sigma))^{-1}\) appears in the equation, it is desirable for proving global existence of solutions to have some a propri estimates of \( G(\Sigma) \). It follows from (5.2) that \( G(\Sigma) \geqq 0 \), which is algebraically trivial since it is a Gramian. Now we consider lower bounds of \( G \).

**Proposition 6.1** We have

\[
G(\Sigma) \geqq \frac{n^{2}\left\{ A(\Sigma)^{2} - (n + 1)V(\Sigma) \int_{\Sigma} H \, dS \right\}^{2}}{A(\Sigma) \int_{\Sigma} (\tilde{f} \cdot \nu)^{2} \, dS},
\]

where

\[
\tilde{f} = f - \frac{1}{A(\Sigma)} \int_{\Sigma} f \, dS.
\]

**Proof.** It follows from \( \delta A = -nH \), \( \delta V = -1 \) and scaling argument that

\[
\langle \delta A, \tilde{f} \cdot \nu \rangle = nA, \quad \langle \delta A, \tilde{f} \cdot \nu \rangle = (n + 1)V.
\]

Therefore we obtain

\[
n |A - (n + 1)\bar{H}V| = \left| \langle \delta A - n\bar{H}\delta V, \tilde{f} \cdot \nu \rangle \right| = \left| \langle n\bar{H}, \tilde{f} \cdot \nu \rangle \right| \leqq n\|\bar{H}\|\|\tilde{f} \cdot \nu\|
\]

Combining (5.2), we get the assertion. \( \square \)

This is an a priori lower bound of \( G(\Sigma) \) when \( n = 1 \). To see this, putting \( \tilde{f} = (\tilde{f}_{1}, \tilde{f}_{2}) \), we have

\[
|\tilde{f}_{i}|^{2} \leqq A(\Sigma) \int_{\Sigma} |\partial_{s}f_{i}|^{2} ds = A(\Sigma) \int_{\Sigma} \tau_{i}^{2} ds.
\]

Summing up with respect to \( i \), we get

\[
\|\tilde{f}\|_{\infty} \leqq A(\Sigma)
\]

Therefore Proposition 6.1 implies

\[
G(\Sigma) \geqq \left( 1 - \frac{2V(\Sigma)}{A(\Sigma)^{2}} \int_{\Sigma} \kappa \, ds \right)^{2}.
\]
Since $A(\Sigma)$, $V(\Sigma)$, and $\int_{\Sigma} \kappa \, ds$ are invariant, the estimate is a priori.

Let $n \geq 2$, and let $L_1(\Sigma)$ be the first eigenvalue of $-\Delta_g$. Putting $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)$, we have

$$\int_{\Sigma} (\tilde{f} \cdot \nu)^2 \, dS \leq \sum_i \int_{\Sigma} |\tilde{f}_i|^2 \, dS$$

$$\leq L_1^{-1}(\Sigma) \sum_i \int_{\Sigma} |\nabla f_i|^2 \, dS = L_1^{-1}(\Sigma) \sum_i \int_{\Sigma} g^{jk} \partial_j f_i \partial_k f_i \, dS$$

$$= L_1^{-1}(\Sigma) \int_{\Sigma} g^{jk} \partial_j f \cdot \partial_k f \, dS = L_1^{-1}(\Sigma) \int_{\Sigma} g^{jk} g_{jk} \, dS = nA(\Sigma)L_1^{-1}(\Sigma).$$

Combining Proposition 6.1, we have a lower estimate of $G(\Sigma)$, but it is not a priori. Because $\int_{\Sigma} H \, dS$ and $L_1(\Sigma)$ may depend on $t$.

7 Related and open problems

Okabe [14] considered the gradient flow associated with

$$\int_{\Sigma} \kappa^2 \, ds$$

under constraints

$$A(\Sigma) = A_0, \quad \gamma(\Sigma) = 1.$$

Here $\gamma$ is the local length defined as below. Let $f(\theta)$ be a family of curves, where $\theta$ is a fixed coordinate. The local length is given by

$$\gamma = \|\partial_\theta f\|_{\mathbb{R}^2}.$$

It is a function on the curve, hence the corresponding multiplier is pointwise. Since $\gamma$ depends on the choice of coordinate, it is not a geometrical quantity. Consequently there is a tangential component in the equation. For the gradient flow with one constraint

$$\gamma(\Sigma) = 1,$$

see [8]. For the comparison Okabe’s result with the one-dimensional Helfrich flow, see [10].

In [9], the global existence of one-dimensional Helfrich flow, however, the global solvability of multi-dimensional Helfrich flow is still open. The asymptotic behavior has not been investigated yet.

In connection with the global existence, it is interesting to show a priori estimate of $G(\Sigma)$ for the case $n \geq 2$, for example, an estimate in terms of $A(\Sigma)$, $V(\Sigma)$, and $\int_{\Sigma} K \, dS$, which are invariant. Here $K$ is the Gauß curvature.
References


