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On atomic AEC and quasi-minimality

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Abstract

Atomic abstract elementary class have been researched in connection with the model theory of infinitary logic. In recent years, the results were summarized by J.T.Baldin [1]. In that book, categoricity problem of atomic AEC is discussed mainly. I tried some local argument around the problem.

Apology In this note, I do not have exact references to the papers in which the results are originally proved.

1. Atomic AEC and splitting

We recall some definitions.

Definition 1 A class of structures $(\mathbf{K}, \prec_{\mathbf{K}})$ (of a language L) is an *abstract elementary class* (*AEC*) if the class $\mathbf{K}$ and class of pairs satisfying the binary relation $\prec_{\mathbf{K}}$ are each closed under isomorphism and satisfy the following conditions ;
A1. If $M \prec_{\mathbf{K}} N$, then $M \subseteq N$.
A2. $\prec_{\mathbf{K}}$ is a partial order on $\mathbf{K}$.
A3. If $\{ A_i : i < \delta \}$ is a $\prec_{\mathbf{K}}$-increasing chain :
(1) $\bigcup_{i<\delta} A_i \in \mathbf{K}$
(2) for each $j < \delta$, $A_j \prec_{\mathbf{K}} \bigcup_{i<\delta} A_i$
(3) if each $A_i \prec_{\mathbf{K}} M \in \mathbf{K}$, then $\bigcup_{i<\delta} A_i \prec_{\mathbf{K}} M$.
A4. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ *and* $A \subseteq B$, then $A \prec_{\mathbf{K}} B$.
A5. There is a Löwenheim-Skolem number LS($\mathbf{K}$) such that if $A \subseteq B \in \mathbf{K}$, there is an $A' \in \mathbf{K}$ with $A \subseteq A' \prec_{\mathbf{K}} B$ and $|A'| \leq |A| + LS(\mathbf{K})$.

Definition 2 We say an AEC $(\mathbf{K}, \prec_{\mathbf{K}})$ is *atomic* if $\mathbf{K}$ is the class of atomic models of a countable complete first order theory and $\prec_{\mathbf{K}}$ is first order elementary submodel.

In the following, $\mathbf{K}$ denotes an atomic AEC.

Definition 3 Let T be a countable first order theory.

A set A contained in a model M of T is *atomic* if every finite sequence in A realizes a principal type over the empty set.

Let A be an atomic set.

$S_{at}(A)$ is the collection of $p \in S(A)$ such that if $a \in \mathcal{M}$ realizes p, Aa is atomic (where $\mathcal{M}$ is the big model).

We refer to a $p \in S_{at}(A)$ as an *atomic type.*

We consider the notion of stability for atomic types.

Definition 4 The atomic class $\mathbf{K}$ is $\lambda -$ *stable* if for every $M \in \mathbf{K}$ of cardinality λ, $|S_{at}(M)| = \lambda$.

Example 5 ([1]) 1. Let $\mathbf{K}_1$ be the class of atomic models of the theory of dense linear order without endpoints. Then $\mathbf{K}_1$ is not $\omega-$stable.

2. Let $\mathbf{K}_2$ be the class of atomic models of the theory of the ordered Abelian group of rationals. Then $\mathbf{K}_2$ is $\omega-$stable.

The notion of independence by splitting is available in this context.

Definition 6 A complete type p over B *splits over* $A \subset B$ if there are b, $c \in B$ which realize the same type over A and a formula $\phi(x, y)$ such that $\phi(x, b) \in p$ and $\neg\phi(x, c) \in p$.

Let A, B, C be atomic.

We write $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and say A is *independent from* B *over* C if for any finite sequence $a \in A$, tp(a/B) does not split over some finite set of C.

Fact 7 ([1]) *Under the atomic* $\omega-$*stable assumption of* $(\mathbf{K}, \prec_{\mathbf{K}})$ *(and some assumption of parameters), the independence relation by splitting (over models) satisfies almost all forking axioms.*

Theorem 8 ([1]) *If* $\mathbf{K}$ *is* $\omega-$*stable and has a model of power* $\aleph_1$*, then it has a model of power* $\aleph_2$*.*

I considered the same problems under some weaker condition.

Definition 9 Let $\mathbf{K}$ be an atomic AEC and $M \in \mathbf{K}$.

M has no infinite splitting chain if for any $p \in S_{at}(M)$ which is realized outside M, there is no increasing sequence $\{A_i\}_{i<\omega}(\subset M)$ such that $p \lceil A_{i+1}$ splits over A_i for all $i < \omega$.

We can prove the next facts.

Fact 10 *If* $\mathbf{K}$ *is* $\omega-$*stable, then no model of* $\mathbf{K}$ *has infinite splitting chain.*

Fact 11 *Let* $\mathbf{K}$ *have no infinite splitting chain (i.e. every* $M \in \mathbf{K}$ *has no infinite splitting chain).*
If $\mathbf{K}$ *has a model of power* $\aleph_1$*, then it has a model of power* $\aleph_2$*.*

Fact 12 *Under the assumption that* $(\mathbf{K}, \prec_{\mathbf{K}})$ *has no infinite splitting chain, the independence relation by splitting (over models) satisfies almost all forking axioms except symmetry.*

At present, I do not have the definitive result about symmetry of splitting. But we can prove the next fact.

Definition 13 Let $\mathbf{K}$ be an atomic AEC and $M \in \mathbf{K}$.
M has infinite splitting left − chain if there is a sequence $\{B_i\}_{i<\omega} \subset M$ and b (outside M), and $A \subset M$ such that $\mathrm{tp}_{at}(B_i/Ab\{B_j : j < i\})$ splits over $A \cup \{B_j : j < i\}$ for all $i < \omega$.

Fact 14 *Let* $\mathbf{K}$ *have no infinite splitting chain. Suppose that any countable atomic set is extended to a countable model in* $\mathbf{K}$*.*
If the independence by splitting over models is not symmetry, then there is an infinite splitting left-chain.

2. *−excellent AEC and categoricity

We recall some definitions.

Definition 15 The atomic AEC $\mathbf{K}$ is $*-$*excellent* if
A1. $\mathbf{K}$ has arbitrarily large models,
A2. $\mathbf{K}$ is $\omega-$stable,
A3. $\mathbf{K}$ satisfies the amalgamation property,
A4. Let p be a complete type over a model $M \in \mathbf{K}$ such that $p\lceil C$ is realized in M for each finite $C \subset M$, then there is a model $N \in \mathbf{K}$ with N primary over Ma such that p is realized by a in N.

Definition 16 Let $M \in \mathbf{K}$ and $A \subset M$.
The type $p \in S_{at}(A)$ is *big* if for any $M' \supset A$ with $M' \in \mathbf{K}$, there exists an N' such that $M' \prec_{\mathbf{K}} N'$ and p has a realization in $N' - M'$.
A triple (M, N, ϕ) is called a *Vaughtian triple* if $\phi(M) = \phi(N)$ where $M \prec_{\mathbf{K}} N$ with $M \neq N$ and $L(M)-$formula ϕ is big.

The next theorem is the analogous result of Morley's categoricity theorem for atomic AEC.

Theorem 17 ([1]) *Suppose* $\mathbf{K}$ *is an* $*-$*excellent atomic AEC.*
Then the following are equivalent.

(1) **K** *is categorical in some uncountable cardinality.*
(2) **K** *has no Vaughtian triple.*
(3) **K** *is categorical in every uncountable cardinality.*

Theorem 18 ([1]) *For each $2 \leq k < \omega$, there is an $L_{\omega_1,\omega}$–sentence ϕ_k such that :*

ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$, and
ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$.

In the proof of Theorem 17, the geometry of quasi-minimal formula plays the important role. I considered that the argument in [5] makes the proof concise.

Definition 19 The type $p \in S_{at}(A)$ is *quasi – minimal* if p is big and for any M containing A, p has a unique extension to a type over M which is not realized in M.

Lemma 20 ([1]) *Let* **K** *be ω–stable.*
Then for any $M \in$ **K**, *there is a $c \in M$ and a formula $\phi(x, c)$ which is quasi-minimal.*

Definition 21 Let X be an infinite set and cl a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the set of all subsets of X. If the function cl satisfies the following properties, we say (X, cl) is a *pregeometry*.
(I) $A \subset B \Longrightarrow A \subset \mathrm{cl}(A) \subset \mathrm{cl}(B)$,
(II) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,
(III) (Finite character) $b \in \mathrm{cl}(A) \Longrightarrow b \in \mathrm{cl}(A_0)$ for some finite $A_0 \subset A$,
(IV) (Exchange axiom)
$b \in \mathrm{cl}(A \cup \{c\}) - \mathrm{cl}(A) \Longrightarrow c \in \mathrm{cl}(A \cup \{b\})$.

In the proof of Theorem 17, the closure operator cl is defined as follows.

Definition 22 Let $c \in M \in$ **K**. And suppose $\phi(x, c)$ determines a quasi-minimal type over M.

For any elementary extension $N(\in$ **K**$)$ of M, *cl* is defined on the set of realizations of $\phi(x, c)$ in N by $a \in \mathrm{cl}(A)$ if $\mathrm{tp}(a/Ac)$ is not big.

We recall some result from [5].

Definition 23 Let M be an uncountable structure and $p(x) \in S_1(M)$.
For all subsets $A \subset M$, the operator cl'_p is defined by $\mathrm{cl}'_p(A) = \{a \in M :$ a is not a realization of $p\lceil A\}$.
The *n-th closure* $\mathrm{cl}^n_p(A)$ of A is inductively defined as follows :
$\mathrm{cl}^0_p(A) = A$ and $\mathrm{cl}^{n+1}_p(A) = \mathrm{cl}'_p(\mathrm{cl}^n_p(A))$
We put $\mathrm{cl}_p(A) = \bigcup_{n\in\omega} \mathrm{cl}^n_p(A)$.

Theorem 24 ([5]) *Let N be a quasi-minimal model and $p(x) \in S_1(N)$. Suppose that $p(x)$ does not split over A for some countable $A \subset N$ (and $N \neq cl_p(a)$ for some finite $a \in N$).*

Then cl_p is a closure operator and exactly one of the following two holds ;
1. Every cl_p−free sequence over A is totally indiscernible. In this case, (N, cl_p) is a pregeometry, and p is definable over A.
2. Oterwise. In this case, there is a finite extension A_0 of A and an A_0−definable partial order $\leq$ such that every cl_p−free sequence over A_0 is strictly increasing.

We can deduce the next lemma.

Lemma 25 *Let $\mathbf{K}$ be ω−stable and have a sufficiently large model. And let $M \in \mathbf{K}$.*

Then there is a finite $c \in M$ and a formula $\phi(x, c)$ such that $\phi(x, c)$ determines a quasi-minimal $p(x) \in S_{at}(M)$ and $p(x)$ does not split over c, and cl defines a pregeometry in $(p\lceil c)(M)$.

3. P−closure in atomic AEC

I considered P-closure in the quasi-minimal set $\phi(M, c)$ above. The *P-closure* is the collection of realizations of types that is P-analysable and co-foreign to P. In this note, I omits the P-analysable assumption, resulting in a larger P-closure.

In this section, $A \mathop{\smile\!\!\!\!\!\!|}_C B$ means tp(A/BC) *does not split over* C.

Assumptions

Let $\mathbf{K}$ be an ω−stable atomic AEC and $M \in \mathbf{K}$.

$\phi(x, c)$ determines a quasi-minimal $p(x) \in S_{at}(M)$ and $p(x)$ does not split over c. And we may assume that $c = \emptyset$.

The set P of types is defined by
$P = \{q \in S(A') : q$ is a conjugate of $p\lceil A$ for some finite $A \subset M\}$.

Definition 26 In this definition, parameters are finite subset of $\phi(M, c)$ above and types are atomic types.

Now P is an $\emptyset$-invarint family of types.

A complete type $q \in S_{at}(A)$ is *foreign* to P if for all $a \models q$, $A \subset B$ with $a \mathop{\smile\!\!\!\!\!\!|}_A B$, and realizations $\bar{c}$ of extensions of types in P over B, we always have $a \mathop{\smile\!\!\!\!\!\!|}_A \bar{c}B$.

A partial type q is *co − foreign* to P if every type in P is foreign to q.

The *P − closure* cl$_P(A)$ of a set A is the collection of all element a such that tp(a/A) is co-foreign to P.

We can prove the next fact.

Fact 27 *Let* $\mathbf{K}$ *be* $\omega-$*stable and have a sufficiently large model. And let* $M \in \mathbf{K}$ *and* $\phi(x, c)$ *be a quasi-minimal formula that determines* $p(x) \in S_{at}(c)$ *for some* $c \in M$ *as above.*

For any $A \subset p(M)$, *cl(A)=cl*$_P$*(A).*

And $(p(M), cl_P)$ *is a pregeometry.*

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