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INDISCERNIBILITY ON TREES

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1. Introduction

The notion of indiscernible sequence plays an essential role in model theory. A reason why the notion is so useful lies in the fact that, given any sequence of tuples that satisfies certain property, we can often choose an indiscernible sequence that still retains that property.

Let us say a sequence $\langle \bar{a}_i \mid i < \omega \rangle$ of tuples is *modelled* by a sequence $\langle \bar{b}_i \mid i < \omega \rangle$ if, for any finite set Δ of $\mathcal{L}$-formulas and any finite sequence $i_1, \cdots, i_d \in \omega$, there exists a finite sequence $j_1, \cdots, j_d \in \omega$ such that (1) the finite sequences $\langle i_1, \cdots, i_d \rangle$ and $\langle j_1, \cdots, j_d \rangle$ have the same order type, and (2) the tuples $(\bar{b}_{i_1}, \cdots, \bar{b}_{i_d})$ and $(\bar{a}_{j_1}, \cdots, \bar{a}_{j_d})$ have the same Δ-type. A routine argument using Ramsey's theorem and compactness yields the following theorem.

Theorem. Any sequence $\langle \bar{a}_i \mid i < \omega \rangle$ can be modelled by some indiscernible sequence $\langle \bar{b}_i \mid i < \omega \rangle$.

Indeed, it is this theorem that often allows us to choose an indiscernible sequence that retains certain desired property. The main idea of this article is that we can generalize the notion of indiscernible sequence to sequences of the form $\langle \bar{a}_i \mid i \in {}^{\beta>}\alpha \rangle$, where α, β are ordinals, and prove a generalized version of the theorem above. The proof relies on Halpern-Läuchli theorem, which is a Ramsey-like theorem for trees. The idea of the proof is essentially due to Shelah and Džamonja [1] who introduced the notions of indiscernibility for sequences indexed by the binary tree ${}^{\omega>}2$. We are also influenced by Lynn Scow who gave a detailed exposition on their proof in her recent PhD thesis [4]. We have revised their proofs (and corrected errors). In doing so, we could significantly clarify the argument by introducing some new notions and terminologies. Our result also generalizes the original result by allowing the index set I in $\langle \bar{a}_i \mid i \in I \rangle$ to be ${}^{\beta>}\alpha$ for any ordinals α and β. We have also been able to apply the result to a couple of classification problems, which we will discuss in the last section.

We do not aim to include all the details of the proofs in this article. Instead, we aim only to give a rough sketch of the ideas and how the argument flows. Interested readers may refer to [3] for full details when it becomes available in print.

Convention & notations: We work in a fixed, sufficiently saturated model $\mathcal{M}$. When we talk about tuples of elements, we shall mean tuples of elements from $\mathcal{M}$, unless specified otherwise. When $\langle \bar{a}_{\eta_1}, \cdots, \bar{a}_{\eta_d} \rangle$ is a finite sequence of tuples, we shall often abbreviate it

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simply as $\bar{a}_{\bar{\eta}}$. When $\bar{\eta}$ is an element of a Cartesian product $({}^{\omega>}n)^k$, we shall often abuse the notation and write it as $\bar{\eta} \in {}^{\omega>}n$.

2. Main Results

Let $\eta, \nu \in {}^{\omega>}n$. Recall $(\eta \cap \nu)$ denotes the greatest common lower bound of η and ν.

Definition 1. Let $\bar{\eta} = \langle \eta_0, \cdots, \eta_{d-1} \rangle \in {}^{\omega>}n$.

(1) $\bar{\eta}$ is 1-*closed* if $\forall\, i, j < d$, $\exists\, k < d$ such that $\eta_i \cap \eta_j = \eta_k$.
(2) $\bar{\eta}$ is 0-*closed* if it is 1-closed, contains the root $\langle\rangle$, and is closed under level-restriction. i.e. $\forall\, i, j < d$, $\exists\, k < d$ such that $\eta_i \lceil_{|\eta_j|} = \eta_k$.

Definition 2. Let $\bar{\eta} := \langle \eta_0, \cdots, \eta_{d-1} \rangle$, $\bar{\nu} := \langle \nu_0, \cdots, \nu_{d-1} \rangle$ be tuples in ${}^{\omega>}n$. We say $\bar{\eta} \approx_0 \bar{\nu}$ if

(1) both $\bar{\eta}$ and $\bar{\nu}$ are 0-closed tuples,
(2) $\forall i, j < d$ and $\forall t < n$,
  (a) $\eta_i \trianglelefteq \eta_j$ iff $\nu_i \trianglelefteq \nu_j$, (Partial order)
  (b) $\eta_i^\frown\langle t \rangle \trianglelefteq \eta_j$ iff $\nu_i^\frown\langle t \rangle \trianglelefteq \nu_j$ (Directionality)
  (c) $|\eta_i| < |\eta_j|$ iff $|\nu_i| < |\nu_j|$. (Length relation)

Definition 3. We say $\bar{\eta} \approx_1 \bar{\nu}$ if, in Definition 2, '0-closed' is replaced by '1-closed', and the length relation condition is omitted.

Definition 4.

(1) We say a sequence $\langle\, \bar{a}_\eta \mid \eta \in {}^{\omega>}n \,\rangle$ is i-fti if $\bar{\eta} \approx_i \bar{\nu}$ implies $tp(\bar{a}_{\bar{\eta}}) = tp(\bar{a}_{\bar{\nu}})$, for all $\bar{\eta}, \bar{\nu} \in {}^{\omega>}n$. $(i = 0, 1.)$
(2) We say a sequence $\langle\, \bar{a}_\eta \mid \eta \in {}^{\omega>}n \,\rangle$ is i-*modelled* by a sequence $\langle\, \bar{b}_\eta \mid \eta \in {}^{\omega>}n \,\rangle$ if, for any i-closed tuple $\bar{\eta} \in {}^{\omega>}n$ and any finite set Δ of $\mathcal{L}$-formulas, there exists $\bar{\nu} \in {}^{\omega>}n$ such that $\bar{\eta} \approx_i \bar{\nu}$ and $tp_\Delta(\bar{b}_{\bar{\eta}}) = tp_\Delta(\bar{a}_{\bar{\nu}})$.

Remark. Clearly, $\bar{\eta} \approx_0 \bar{\nu}$ implies $\bar{\eta} \approx_1 \bar{\nu}$. Hence, 1-$fti$ implies 0-fti.

Our main goal is to prove the following lemma.

Lemma 5 (Main Lemma). $\forall\, i \in \{0, 1\}$, any sequence $\langle\, \bar{a}_\eta \mid \eta \in {}^{\omega>}n \,\rangle$ can be i-modelled by some i-fti sequence $\langle\, \bar{b}_\eta \mid \eta \in {}^{\omega>}n \,\rangle$.

Although 1-fti seems to be a pretty natural way to define indiscernibility on trees, it is rather difficult to handle because $\approx_1$-equivalent tuples are not 'rigid' enough. On the other hand, it turns out that we have just enough control over $\approx_0$-equivalent tuples to apply Halpern-Läuchli theorem, a kind of Ramsey's theorem for trees. Keep in mind that, what we are really interested in is to prove Main Lemma for the case $i = 1$. The $i = 0$ case is an auxiliary, technical notion intended to help us ultimately to prove the $i = 1$ case. We mention that, in [1], Shelah and Džamonja also defined a notion 2-fti (they called it 2-$fbti$, where 'b' comes from the fact that they were working with the binary tree ${}^{\omega>}2$) which is the same as 1-fti except that, in 2-fti, even the directionality condition is omitted. They claimed that any sequence can be 2-modelled by some 2-fti tree. But we suspect that their proof is erroneous. We have tried to find a correct proof for it but, so far, to no avail.

The strategy for proving Main Lemma is to prove the $i = 0$ case first, and then deduce the $i = 1$ case. But first, we need to define a few more technical notions.

Definition 6. For $m < \omega$ and a tuple $\bar{\eta} := \langle \eta_0, \cdots, \eta_{d-1} \rangle \in {}^{\omega>}n$,

(1) $L(\bar{\eta}) := \{ |\eta_i| \mid i < d \}$,
(2) $u_m(\bar{\eta}) := \{ i \in L(\bar{\eta}) \mid i > m \}$

Definition 7. Let $\bar{\eta} = \langle \eta_0, \cdots, \eta_{d-1} \rangle$ and $\bar{\nu} = \langle \nu_0, \cdots, \nu_{d-1} \rangle$ be tuples in ${}^{\omega>}n$. For $m, s < \omega$, we say $\bar{\eta} \approx_{(m,s)} \bar{\nu}$ if

(1) $\bar{\eta} \approx_0 \bar{\nu}$,
(2) $m \in L(\bar{\eta}) \cap L(\bar{\nu})$,
(3) $|u_m(\bar{\eta})| = |u_m(\bar{\nu})| \leq s$,
(4) $|\eta_i| \leq m$ iff $|\nu_i| \leq m$, for each $i < |\bar{\eta}|$. And if both sides of the biconditional are true, then $\eta_i = \nu_i$

Definition 8. For $m, s < \omega$ and a finite set Δ of $\mathcal{L}$-formulas, a sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is said to be (m, s, Δ)-*indiscernible* if $\bar{\eta} \approx_{(m,s)} \bar{\nu}$ implies $tp_\Delta(\bar{a}_{\bar{\eta}}) = tp_\Delta(\bar{a}_{\bar{\nu}})$, for all $\bar{\eta}, \bar{\nu} \in {}^{\omega>}n$.

We shall use the notation $(< \omega, s, \Delta)$-*indiscernible* to mean '(m, s, Δ)-indiscernible for every $m < \omega$.'

Definition 9. Let $T := \langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ and $S := \langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$ be sequences (viewed as functions ${}^{\omega>}n \to \mathcal{M}$). We say $S \leq^m T$ (for $m < \omega$) if there exists a 1-1 map $h : {}^{\omega>}n \to {}^{\omega>}n$ such that $S = T \circ h$ and $\forall \eta, \nu \in {}^{\omega>}n$ and $\forall t < n$,

(1) $\eta \trianglelefteq \nu$ iff $h(\eta) \trianglelefteq h(\nu)$, (Partial order)
(2) $\eta^\frown\langle t \rangle \trianglelefteq \nu$ iff $h(\eta)^\frown\langle t \rangle \trianglelefteq h(\nu)$, (Directionality)
(3) $|\eta| < |\nu|$ iff $|h(\eta)| < |h(\nu)|$, (Length relation)
(4) if $|\eta| \leq m$ then $h(\eta) = \eta$. (Fixing up to m-th level)

Note any $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is trivially $(< \omega, 0, \Delta)$-indiscernible. And being $(< \omega, s, \Delta)$-indiscernible for every $s < \omega$ and Δ is equivalent to being 0-*fti*.

The following is the key technical lemma on which the whole argument of this article is based. Its proof relies on a Ramsey-like theorem called Halpern-Läuchli theorem (whose precise statement will be given at the end of this section) and is rather long and technical, so we omit it. (Interested readers may refer to [3] when it becomes available in print, or [4] or [1].)

Lemma 10 (Key Technical Lemma). Suppose $T := \langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is a $(< \omega, s, \Delta)$-indiscernible sequence. Then, $\forall m < \omega$, there exists a $(m, s{+}1, \Delta)$-indiscernible sequence $S := \langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$ such that $S \leq^m T$.

For convenience, let us call sequences of the form $\langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$ *parameterized trees.*

Corollary 11. Suppose a sequence $T = \langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is $(< \omega, s, \Delta)$-indiscernible. Then there exists a $(< \omega, s{+}1, \Delta)$-indiscernible sequence $S = \langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$ such that $S \leq^0 T$.

Proof. Suppose $T = \langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is a $(< \omega, s, \Delta)$-indiscernible sequence. By applying Key Technical Lemma, we can build a sequence $T_0, T_1, \cdots$ of parameterized trees satisfying

(1) $\cdots \leq^3 T_2 \leq^2 T_1 \leq^1 T_0 \leq^0 T$,

(2) each T_i is $(\leq i, s{+}1, \Delta)$-indiscernible.

Condition (1) allows us to define $S := \lim_{i\to\infty} T_i$. Then clearly $S \leq^0 T$ and S is a desired $(< \omega, s+1, \Delta)$-indiscernible tree. □

Recall that any $\langle\, \bar{a}_\eta \mid \eta \in {}^{\omega>}n \,\rangle$ is trivially $(< \omega, 0, \Delta)$-indiscernible. We immediately obtain the following corollary.

Corollary 12. Let $T := \langle\, \bar{a}_\eta \mid \eta \in {}^{\omega>}n \,\rangle$ be any sequence. Then, given any finite set Δ of $\mathcal{L}$-formulas, there exists a sequence $S_1^\Delta, S_2^\Delta, \cdots$ of parameterized trees such that,

(1) $\cdots \leq^0 S_2^\Delta \leq^0 S_1^\Delta \leq^0 S_0^\Delta := T$,
(2) each S_i^Δ is $(< \omega, i, \Delta)$-indiscernible.

Proof of Main Lemma (The $i = 0$ case). Recall that being 0-fti is equivalent to being $(< \omega, s, \Delta)$-indiscernible for every s and Δ. Use Corollary 12 and compactness. □

Now, it remains to prove the $i = 1$ case of Main Lemma.

For technical reasons, let us say a tuple $\bar{\eta} \in {}^{\omega>}n$ is 1^*-*closed* if it is 1-closed and contains the root $\langle\rangle$. By replacing '1-closed' by '1^*-closed', we can define analogous notions of $\approx_1^*$-equivalence, 1^*-fti, and 1^*-modelling property.

Now let us recursively define a sequence $\langle\, h_m : {}^{m\geq}n \to {}^{\omega>}n \mid m < \omega \rangle$ of maps as follows: Define $h_0(\langle\rangle) = \langle\rangle$. For the recursion step, define $h_{m+1}(\langle\rangle) := \langle\rangle$ and

$$h_{m+1}(\langle t\rangle^\frown \eta) := \underbrace{\langle t\rangle^\frown \cdots {}^\frown \langle t\rangle}_{2(t+1)+t\cdot k_m}{}^\frown h_m(\eta)$$

for all $t < n$ and $\eta \in {}^{m\geq}n$, where $k_m := \max\{\, |h_m(\eta)| \mid \eta \in {}^{m\geq}n \,\}$.

Let us define a linear order $<_{lex}$ in ${}^{\omega>}n$ as follows: $\eta <_{lex} \nu$ iff either $\eta \lhd \nu$, or η and ν are incomparable such that $(\eta \cap \nu)^\frown\langle t_1\rangle \unlhd \eta$ and $(\eta \cap \nu)^\frown\langle t_2\rangle \unlhd \nu$ where $t_1 < t_2 < n$.

Note 13.

(1) If $\bar{\eta} \approx_1 \bar{\nu} \in {}^{\omega>}n$ then, $\forall\, i, j < |\bar{\eta}|$, $\eta_i <_{lex} \eta_j \;\Leftrightarrow\; \nu_i <_{lex} \nu_j$.
(2) Each map h_m preserves partial order and directionality.
(3) $\eta <_{lex} \nu \;\Leftrightarrow\; |h_m(\eta)| < |h_m(\nu)|$, for any $\eta, \nu \in {}^{m\geq}n$.
(4) If $\bar{\eta} \approx_1^* \bar{\nu} \in {}^{m\geq}n$ then $h_m(\bar{\eta}) \approx_1^* h_m(\bar{\nu})$ and, $\forall\, i, j < |\bar{\eta}|$,

$$|h_m(\eta_i)| < |h_m(\eta_j)| \;\Leftrightarrow\; \eta_i <_{lex} \eta_j \;\Leftrightarrow\; \nu_i <_{lex} \nu_j \;\Leftrightarrow\; |h_m(\nu_i)| < |h_m(\nu_j)|$$

These properties ensure that, if $\bar{\eta} \approx_1^* \bar{\nu} \in {}^{m\geq}n$ then the tuples $h_m(\bar{\eta})$ and $h_m(\bar{\nu})$ agree on partial order, directionality and length relation. Hence we can *almost* say that $\bar{\eta} \approx_1^* \bar{\nu} \in {}^{m\geq}n$ implies $h_m(\bar{\eta}) \approx_0 h_m(\bar{\nu})$. The only thing that is preventing us from saying it is that the tuples $h_m(\bar{\eta})$ and $h_m(\bar{\nu})$ may not be 0-closed. But this can be easily remedied by taking the 'level-closures' of $h_m(\bar{\eta})$ and $h_m(\bar{\nu})$. Let us define $cl(h_m(\bar{\eta}))$ to be the smallest 0-closed tuple (ordered in some fixed, arbitrary manner) containing $h_m(\bar{\eta})$. Elementary arguments can show that, if $\bar{\eta} \approx_1^* \bar{\nu} \in {}^{m\geq}n$ then indeed we have $cl(h_m(\bar{\eta})) \approx_0 cl(h_m(\bar{\nu}))$. Hence we have the following corollary.

Corollary 14. Let $\langle\, \bar{a}_\eta \mid \eta \in {}^{\omega>}n \,\rangle$ be a 0-fti sequence. Then, for each $m < \omega$, $\langle\, \bar{a}_{h_m(\eta)} \mid \eta \in {}^{m\geq}n \,\rangle$ is 1^*-fti.

Applying compactness, we obtain:

Corollary 15. Any 0-fti sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ can be 1*-modelled by some 1*-fti sequence $\langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$.

Finally, we can prove the $i = 1$ case of Main Lemma.

Proof of Main Lemma (The $i = 1$ case). By the $i = 0$ case of Main Lemma, $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ can be 0-modelled by some 0-fti sequence $\langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$. And, by the preceding corollary, $\langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$ can be 1*-modelled by some 1*-fti sequence $\langle \bar{c}_\eta \mid \eta \in {}^{\omega>}n \rangle$. Then, $\langle \bar{c}_{\langle 0 \rangle ^\frown \eta} \mid \eta \in {}^{\omega>}n \rangle$ is a 1-fti sequence 1-modelling $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$. □

Note 16. The notions $\approx_0$, $\approx_1$, 0-modelling and 1-modelling clearly all make sense even for sequences $\langle \bar{a}_\eta \mid \eta \in {}^{\kappa>}\lambda \rangle$ for any ordinals $\kappa \geq \omega$ and $\lambda \geq 2$. Hence, Main Lemma can be extended to this context by compactness.

We end this section by stating Halpern-Läuchli theorem which plays a crucial role in the proof of Main Technical Lemma. Recall that a partially ordered set $(T, \trianglelefteq)$ is called a *tree* if, for every $x \in T$, $\mathrm{Pred}(x) := \{y \in T \mid y \triangleleft x\}$ is linearly ordered. A tree T is called *finitistic* if (1) T has a least element, (2) $|\mathrm{Pred}(x)| < \omega$ for every $x \in T$, and (3) $T[n] := \{x \in T \mid |\mathrm{Pred}(x)| = n\}$ is a finite set for every $n < \omega$.

Definition 17. Let T be a finitistic tree. A subset $S \subseteq T$ is called a *strong subtree of T witnessed by a subset $A \subseteq \omega$* if

(1) A is an infinite set,
(2) S has a least element,
(3) $S \subseteq \bigcup_{n \in A} T[n]$,
(4) $S \cap T[n] \neq \emptyset$, $\forall n \in A$,
(5) if $n < m$ are successive elements in A and
  (a) if $x \in S \cap T[n]$ and y is an immediate successor of x in T, then $\exists! z \in S \cap T[m]$ such that $y \trianglelefteq z$
  (b) if $y \in S \cap T[m]$, then there exists $x \in S \cap T[n]$ such that $x \trianglelefteq y$.

Theorem 18 (Halpern-Läuchli, strong subtree version [6]). Let $\prod_{i<d} T_i$ be a finite Cartesian product of finitistic trees without maximal elements. Then, for every finite partition of $\prod_{i<d} T_i$, there exists a piece P of the partition and a sequence $\langle S_i \subseteq T_i \mid i < d \rangle$ of strong subtrees, all witnessed by the same infinite subset of ω, such that $\bigcup_{n \in \omega} (\prod_{i<d} S_i[n]) \subseteq P$.

Remark.

(1) Our definition of strong subtree is slightly stronger than the one given in [6]. But this doesn't affect the validity of Halpern-Läuchli theorem.
(2) There are several different versions of Halpern-Läuchli theorem. We refer interested readers to [6] for more details on these equivalent versions. The original version by Halpern and Läuchli can be found in [2].

3. Applications

In this section, we report two examples (Claims 20 and 24) where we have been able to successfully apply the results of the previous section.

Definition 19. We say a theory T has k-TP1 ($k \geq 2$) if it allows an $\mathcal{L}$-formula $\varphi(\bar{x}\,\bar{y})$ to witness a sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}\omega \rangle$ satisfying:

(1) If $\eta_0 \trianglelefteq \cdots \trianglelefteq \eta_{d-1} \in {}^{\omega>}\omega$ then $\bigcap_{i<d} \varphi(\bar{x}\,\bar{a}_{\eta_i})$ is consistent;
(2) If $\eta_0, \cdots, \eta_{k-1} \in {}^{\omega>}\omega$ are pairwise incomparable elements then $\bigcap_{i<k} \varphi(\bar{x}\,\bar{a}_{\eta_i})$ is not consistent.

Claim 20. A theory T has 2-TP1 iff it has k-TP1 for some $k \geq 2$.

The crucial assumption used in the proof is that, if a sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ and a formula $\varphi(\bar{x}\bar{y})$ witness k-TP1 then Main Lemma allows us to assume $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is 1-*fti*.

Remark.

(1) In proving this claim, we used an idea of Shelah and Usvyatsov who proved a similar theorem [5].
(2) Our definition of k-TP1 is a generalization of TP1 defined by Shelah.

Let us move on to the second application. First, we need some terminology.

Definition 21. We say $\eta_0, \cdots, \eta_{k-1} \in {}^{\omega>}\omega$ are

(1) *siblings* if they are distinct elements sharing the same immediate predecessor. (i.e. there exist $\nu \in {}^{\omega>}\omega$ and distinct $t_0, \cdots, t_{k-1} < \omega$ such that $\nu^\frown\langle t_i \rangle = \eta_i$ for each $i < k$.)
(2) *distant siblings* if there exist $\nu \in {}^{\omega>}\omega$ and distinct $t_0, \cdots, t_{k-1} < \omega$ such that $\nu^\frown\langle t_i \rangle \trianglelefteq \eta_i$ for each $i < k$.

Definition 22. We say a theory T has *weak k-TP1* ($k \geq 2$) if it allows an $\mathcal{L}$-formula $\varphi(\bar{x}\,\bar{y})$ to witness a sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}\omega \rangle$ satisfying:

(1) If $\eta_0 \trianglelefteq \cdots \trianglelefteq \eta_{d-1} \in {}^{\omega>}\omega$ then $\bigcap_{i<d} \varphi(\bar{x}\,\bar{a}_{\eta_i})$ is consistent;
(2) If $\eta_0, \cdots, \eta_{k-1} \in {}^{\omega>}\omega$ are distant siblings then $\bigcap_{i<k} \varphi(\bar{x}\,\bar{a}_{\eta_i})$ is not consistent.

The following definition is due to Shelah and Džamonja [1].

Definition 23. A theory T is said to have SOP_1 if it allows an $\mathcal{L}$-formula to witness a sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}2 \rangle$ satisfying:

(1) If $\eta_0 \trianglelefteq \cdots \trianglelefteq \eta_{d-1} \in {}^{\omega>}2$ then $\bigcap_{i<d} \varphi(\bar{x}\,\bar{a}_{\eta_i})$ is consistent;
(2) If $\eta^\frown\langle 0 \rangle \trianglelefteq \nu \in {}^{\omega>}2$ then $\varphi(\bar{x}\,\bar{a}_{\eta^\frown\langle 1 \rangle}) \wedge \varphi(\bar{x}\,\bar{a}_\nu)$ is not consistent.

Claim 24. If a theory T has weak k-TP1 for some $k \geq 2$, then T has SOP_1.

Again, the crucial assumption used in the proof is that, if a sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ and a formula $\varphi(\bar{x}\bar{y})$ witness weak k-TP1 then we may assume $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is 1-*fti*.

In [1], Shelah and Džamonja also defined the notion SOP_2, which turns out to be equivalent to k-TP1 ($\Leftrightarrow$ 2-TP1). Hence we have the following picture:

$$\mathrm{SOP}_2(\Leftrightarrow k\text{-TP1}) \Rightarrow \text{Weak}\, k\text{-TP1} \Rightarrow \mathrm{SOP}_1 \Rightarrow \mathrm{TP}$$

where TP denotes the tree property characterizing non-simple theories. Shelah and Usvyatsov showed that the implication $\mathrm{SOP}_1 \Rightarrow$ TP can not be reversed [5]. However, it still remains unknown whether any of the other implications above is reversible.

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