A generalization of Shelah’s omitting types theorem

竹内 耕太 (Kota Takeuchi)
筑波大学数理物質科学研究科
(Graduate School of Pure and Applied Sciences,
University of Tsukuba)

Abstract
This note gives a generalization of Shelah’s omitting types theorem.

$L$ を可算言語、$T$ を $L$-理論とする。Shelah のタイプ排除定理は、連続濃度未満の完全タイプの集合 $R$ について、$R$ の全てのタイプを排除するモデルが存在することを保証する。一方、Newelski の研究により、濃度 $\omega_1$ のタイプの集合 $R$ で、$R$ の元をすべて排除するようなモデルが存在しないような理論の存在が、ZFC と矛盾しないことも知られている [1]。よって、Shelah のタイプ排除定理における、タイプが完全であるという仮定をどの程度弱められるかという問いは、興味深い問題と言える。本研究では、次の形に拡張された Shelah のタイプ排除定理の証明を行う。

**Theorem** Let $T$ be a theory formulated in a countable language $L$ and $L_0$ a sublanguage of $L$. Let $R$ be a set of nonisolated complete $L_0$-types such that $|R| < 2^\omega$. Let $S$ be a countable set of nonisolated $L$-types. Then there is a model $M \models T$ omitting all the members of $R \cup S$.

**Proof:**
Throughout, $L$ is a countable language and $T$ is a countable first-order theory formulated in $L$. ($T$ may be incomplete.) We always work under $T$. $L$-formulas are denoted by $\varphi, \psi, \theta, \chi, \ldots$. We fix a sublanguage $L_0 \subset L$. $L_0$-formulas are denoted by $\xi, \ldots$. Types are (possibly incomplete) $L$-types over the empty set. We say a type $p(\bar{x})$ is a complete $L_0$-type if $p$ consists of only $L_0$-formulas, and if for every $\xi(\bar{x}) \in L_0$, $\xi$ or $\neg \xi$ is in $p$. 
Definition 1 Let $L_0 \subset L$ and $\varphi_i(\overline{x}) \in L$ satisfiable.

1. We say that two $L$-formulas $\varphi_0(\overline{x})$ and $\varphi_1(\overline{x})$ are $L_0$-separable in $\overline{x}' \subset \overline{x}$ if there are $L_0$-formulas $\xi_0(\overline{x}')$ and $\xi_1(\overline{x}')$ such that $T \models \varphi_k(\overline{x}) \rightarrow \xi_k(\overline{x}')$ ($k = 0, 1$), and $\xi_0$ and $\xi_1$ are incompatible in $T$.

2. We say $\varphi_0(\overline{x})$ and $\varphi_1(\overline{x})$ are essentially $L_0$-separable in $\overline{x}'$ if there are satisfiable $L$-formulas $\varphi_k(\overline{x})$ ($k = 0, 1$) with $T \models \varphi_k(\overline{x}) \rightarrow \varphi_k(\overline{x})$ ($k = 0, 1$) such that $\varphi_0$ and $\varphi_1$ are $L_0$-separable in $\overline{x}'$.

3. Let $\Phi = \varphi_0(\overline{x}), ..., \varphi_n(\overline{x})$ be a sequence of satisfiable $L$-formulas. We say that $\Phi$ is maximally $L_0$-separated if for each $i \neq j$ and each subsequence $\overline{x}' \subset \overline{x}$, whenever $\varphi_i'(\overline{x})$ and $\varphi_j'(\overline{x})$ are essentially $L_0$-separable in $\overline{x}'$ then they are $L_0$-separable in $\overline{x}'$.

A maximally $L_0$-separated sequence $\Phi' = \varphi'_0(\overline{x}), ..., \varphi'_n(\overline{x})$ will be called a maximal $L_0$-separation of $\Phi$ if $T \models \varphi'_i(\overline{x}) \rightarrow \varphi_i(\overline{x})$ ($i = 0, ..., n$).

Lemma 2 Let $\Phi = \varphi_0(\overline{x}), ..., \varphi_n(\overline{x})$ be satisfiable $L$-formulas. Then there are satisfiable $L$-formulas $\varphi_i(\overline{x})$ ($i \leq n$) such that $\Phi' = \varphi'_0(\overline{x}), ..., \varphi'_n(\overline{x})$ is a maximal $L_0$-separation of $\Phi$.

Proof: Let $\bar{y} \subset \overline{x}$ and suppose that $\varphi_i(\overline{y})$ and $\varphi_j(\overline{y})$ are essentially $L_0$-separable in $\overline{y}$. Choose $L$-formulas $\varphi_i'(\overline{x})$ and $L$-formulas $\varphi_j'(\overline{x})$ witnessing the essential $L_0$-separability. Then we replace $\varphi_i(\overline{x})$ and $\varphi_j(\overline{x})$ by $\varphi_i'(\overline{x})$ and $\varphi_j'(\overline{x})$, respectively. We repeat this process (finitely many times) and finally we get a desired maximal $L_0$-separation.

Definition 3 Let $\psi(x_1, ..., x_n)$ be an $L$-formula and $s(\overline{y})$ an $L$-type. We say $\psi(x_1, ..., x_n)$ totally omits $s(\overline{y})$ if whenever $M \models T$ and $a_1, ..., a_n \in M$ satisfies $\psi(\overline{a})$ then no tuple from $\{a_1, ..., a_n\}$ realizes $s(\overline{y})$. Let $\Sigma$ be a finite set of formulas. We simply say that $\Sigma$ totally omits $s$ if $\bigwedge \Sigma$ totally omits $s$.

Remark 4

- Let $s(\overline{x})$ be a nonisolated type. Then for every satisfiable $L$-formula $\varphi(\overline{x})$ there is a satisfiable $L$-formula $\varphi'(\overline{x})$ with $T \models \varphi'(\overline{x}) \rightarrow \varphi(\overline{x})$ such that $\varphi'$ and $s$ are inconsistent.

- It is easy to check that for every satisfiable $L$-formula $\varphi(\overline{x})$ and nonisolated type $s(\overline{y})$, there is a satisfiable $L$-formula $\psi(\overline{x})$ with $T \models \psi \rightarrow \phi$ such that $\psi$ totally omits $s$. 
Next lemma is easy but important for our proof of the theorem.

**Lemma 5** Let \( \varphi_0(\overline{x}) \) and \( \varphi_1(\overline{x}) \) be satisfiable \( L \)-formulas such that they are not essentially \( L_0 \)-separable in \( \overline{x}' \subset \overline{x} \). Then \( \varphi_0 \) and \( \varphi_1 \) isolate the same complete \( L_0 \)-type \( p(\overline{x}') \).

**Proof:** Suppose otherwise. Then it is easy to find an \( L_0 \)-formula \( \chi(\overline{x}') \) such that both \( \varphi_0 \land \chi \) and \( \varphi_1 \land \neg \chi \) are satisfiable. Two \( L \)-formulas \( \varphi_0 \land \chi \) and \( \varphi_1 \land \neg \chi \) are \( L_0 \)-separable in \( \overline{x}' \). Since \( T \models \varphi_0 \land \chi \rightarrow \varphi_0 \) and \( T \models \varphi_1 \land \neg \chi \rightarrow \varphi_1 \), this means that \( \varphi_0 \) and \( \varphi_1 \) are essentially \( L_0 \)-separable. A contradiction.

Suppose \( Z = \{z_i|i < \omega\} \) is a fixed countable set of new variables. We denote a sequence \( z_0, z_1, \ldots, z_{i-1} \) by \( \overline{z}_i \). Enumerate \( S \) as \( S = \{s_i(\overline{z}_i) : i \in \omega\} \). We may assume that for each \( s_n(\overline{z}_n), |\overline{x}_n| \leq n \). Let \( \{\theta_i(\overline{z}_i, z_i)\} \) be an enumeration of \( L \)-formulas having the form \( \exists x \varphi(\overline{z}_i, z_i) \rightarrow \varphi(\overline{z}_i, z_i) \).

By induction, we construct a binary tree \( \{\Sigma_\eta(\overline{z}_{\text{len}(\eta)}) | \eta \in 2^{<\omega}\} \) of finite sets of \( L \)-formulas with the following properties: For every \( n \in \omega \) and every \( \eta \in 2^n \),

1. If \( m < n \) then \( \Sigma_{\eta|m} \subset \Sigma_{\eta|n} \);
2. \( \{\wedge \Sigma(\overline{z}_n)\}_{\sigma \in 2^n} \) is maximally separated;
3. \( \Sigma_\eta \) is consistent;
4. \( \Sigma_\eta \) contains \( \theta_n \);
5. \( \Sigma_\eta \) totally omits each of \( s_i \) \( (i \leq n) \).

Let \( \Sigma_\emptyset = \emptyset \) and suppose \( \Sigma_\sigma(\overline{z}_n) \) is defined for every \( \sigma \in 2^n \). Take two copies of \( \Sigma_\sigma(\overline{z}_n) \) and set

\[
\Sigma^{0,k}_\sigma(\overline{z}_n) = \Sigma_\sigma(\overline{z}_n) \; (k = 0, 1). 
\]

Then, by Lemma 2, there is a set \( \{\psi_{\sigma,k}(\overline{z}_n)\}_{\sigma \in 2^n, k = 0, 1} \) which is a maximal \( L_0 \)-separation of \( \{\wedge \Sigma^{0,k}_\sigma(\overline{z}_n)\}_{\sigma \in 2^n, k = 0, 1} \). Set

\[
\Sigma^{1,k}_\sigma(\overline{z}_n) = \Sigma^{0,k}_\sigma(\overline{z}_n) \cup \{\psi_{\sigma,k}(\overline{z}_n)\}. 
\]

Next, for each \( \sigma \in 2^n \), take a satisfiable \( L \)-formula \( \chi_{\sigma,k}(\overline{z}_n) \models \Sigma^{1,k}_\sigma(\overline{z}_n) \) such that \( \chi_{\sigma,k} \) totally omits \( s_i(\overline{x}_i) \) for every \( i \leq n \). (Such formula exists by Remark 4.) Set

\[
\Sigma^{2,k}_\sigma(\overline{z}_n) = \Sigma^{1,k}_\sigma(\overline{z}_n) \cup \{\chi_{\sigma,k}(\overline{z}_n)\}.
\]
Finally set $\Sigma^* = \Sigma^2_k(\bar{z}_n) \cup \{\theta_n(\bar{z}_n, z_n)\}$. It is easy to check that $\{\Sigma^*(\bar{z}_{n+1})\}_{n \in 2^{n+1}}$ satisfies the required conditions 1-5 (with $n$ replaced by $n + 1$). So we have succeeded to construct all $\Sigma_\eta$'s. Now, for a path $\eta \in 2^\omega$, we define $\Sigma_\eta(Z)$ by $\Sigma_\eta = \bigcup_{n \in \omega} \Sigma_{\eta|n}$. Recall that $\theta_n$ has the form $\exists \bar{x} \varphi(\bar{z}_n, x) \rightarrow \varphi(\bar{z}_n, z_n)$. So, by the condition 4, every $M_\eta$ realizing $\Sigma_\eta(Z)$ is a model of $T$. By the condition 5, $M_\eta$ omits all types in $S$.

**Claim A** For each $p \in R$, $\{\eta \in 2^\omega \mid M_\eta \models \exists \bar{x}p(\bar{x})\}$ is countable.

We fix $p(\bar{x}) \in R$ and $\bar{z} \subset Z$ with $|\bar{z}| = |\bar{z}|$. Suppose $\Sigma_\eta(Z) \cup p(\bar{z})$ is consistent. Take any $\eta' \neq \eta$. If $\Sigma_{\eta'}(Z) \cup p(\bar{z})$ is also consistent, then $\Sigma_{\eta|n}$ and $\Sigma_{\eta'|n}$ are not essentially $L_0$-separable in $\bar{z}$, where $n$ is chosen so that $\bar{z} \subset \bar{z}_n$. Hence $p$ must be isolated by a $L$-formula, by Lemma 5. But $R$ is a set of nonisolated types, a contradiction. So, for each $p \in R$ and $\bar{z} \subset Z$, $\{\eta \in 2^\omega \mid \Sigma_\eta(Z) \cup p(\bar{z}) \text{ consistent}\}$ has at most one element. This proves the claim, since there are only countably many possible choices of $\bar{z} \subset Z$. (End of Proof of Claim)

Finally, by the claim above and the assumption that $|R| < 2^\omega$, we can find a path $\eta \in 2^\omega$ such that $M_\eta$ omits $R$.

**Corollary 6** Suppose $\alpha < 2^\omega$. Let $T_0$ be a complete $L$-theory and $p, q_i \in S(T)$ ($i < \alpha$). If for every $i < \alpha$ there is a model $M$ such that $M$ omits $q_i$ and $M$ realizes $p$, then there is a model $N$ such that $N$ omits all $q_i$'s but $N$ realizes $p$.

**Definition 7** Let $M$ be an $L$-structure. We say that $M$ is finitely generated if there is a tuple $\bar{a} \in M$ such that $M = acl_M(\bar{a})$.

In [4], Tsuboi generalized Steinhorn's omitting types theorem. He showed the next result in his paper,

**Theorem 8** Let $M$ be an $L$-structure. Suppose $M$ is not finitely generated. Let $p(\bar{x})$ be a type that is not realized in $M$. Then $p(\bar{x})$ is not isolated in the theory $Th_{L(M)}(M) \cup \{y = a \mid a \in M\}$. So, there is a proper elementary extension $N$ of $M$, which omits $p$.

We also have a generalization of Tsuboi's result, by our generalization of Shelah's omitting types theorem.
**Corollary 9** Let $M$ be an $L$-structure and $\kappa < 2^{\omega}$. Suppose $M$ is not finitely generated. Let $p_\eta(\bar{x}_\eta) \in S(T)$ be a complete type that is not realized in $M$, for each $\eta < \kappa$. Then, there is a proper elementary extension $N$ of $M$, which omits $p_\eta$ for all $\eta < \kappa$.

**References**


