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A note on the superstability of $\mathrm{Th}(\mathbb{F}, \cdot, +, \Gamma, 0, 1, q)$

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Abstract

Let q be a trancendental number in an algebraically closed field $\mathbb{F}$ of characteristic zero. Consider the structure $(\mathbb{F}, \cdot, +, \Gamma, 0, 1, q)$ where Γ is a unary predicate describing the property of the set $q^{\mathbb{Z}}$ sitting in the field $\mathbb{F}$. We show the superstability of the theory of the above structure.

1 Introduction

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero and q be a transcendental element in $\mathbb{F}$. We want to describe the property of the field $\mathbb{F}$ with $q^{\mathbb{Z}}$ sitting in it.

From now on let $\mathbb{F}_\Gamma$ denote the structure $(\mathbb{F}, \cdot, +, \Gamma, 0, 1, q)$. Recall that $\mathbb{F}$ being algebraically closed the theory of $\mathbb{F}$ is strongly minimal. We see that two structures $(\mathbb{Z}, +, 0)$ and $(q^{\mathbb{Z}}, \cdot, 1)$ are *isomorphic* via the exponential law, i.e., $q^{x+y} = q^x \cdot q^y$.

Showing that the theory of $\mathbb{F}_\Gamma$ is superstable is a preliminary step to showing that the theory of a quantum torus is superstable. We discuss this issue in Section 2.

1.1 First-order description of the set $q^{\mathbb{Z}}$ in the field

Introduce a unary predicate $\Gamma(x)$ and a constant symbol q. $\Gamma(x)$ describes the property of the set $q^{\mathbb{Z}}$ as a multiplicative subgroup. It satisfies the following;

Property of Γ

- q is transcendental,
- $\forall x(\Gamma(x) \to (x \text{ is transcendental}))$ (see Remark 1 below),
- for all k and l, $\forall x \forall y \Big((\Gamma(x) \wedge \Gamma(y) \wedge x \cdot y \neq 1) \to x^k \cdot y^l \neq 1\Big)$
- $\Gamma(1)$, $\neg\Gamma(0)$,
- $\forall x \forall y \Big(\Gamma(x) \wedge \Gamma(y) \to \Gamma(x \cdot y)\Big)$,
- $\forall x \forall y \Big((\Gamma(x) \wedge \Gamma(y) \wedge x \neq 0 \wedge x \neq 1 \wedge y \neq 0 \wedge 1 \neq q) \to \neg\Gamma(x+y)\Big)$,
- $\forall x \Big(\Gamma(x) \to \Gamma(q \cdot x)\Big)$,
- $\forall x \, \exists y \Big(\Gamma(x) \to (\Gamma(y) \wedge x = q \cdot y)\Big)$,
- $\forall x \, \exists y \Big(\Gamma(x) \to (\Gamma(y) \wedge 1 = x \cdot y)\Big)$.

Therefore the above sentences are all included in the theory $T_\Gamma = \mathrm{Th}(\mathbb{F}_\Gamma)$.

Remark 1 *$\forall x(\Gamma(x) \to$ (x is transcendental)) cannot be expressible by just one formula. We need to say that for each integer $n \geq 1$,*

$$\forall x(\Gamma(x) \to (x \textit{ is not a solution to any equation of degree up to } n))$$

Set $G = \{x \in \mathbb{F} : \mathbb{F} \models \Gamma(x)\}$. The above sentences expressing the property of $q^{\mathbb{Z}}$ can only assure that $q^{\mathbb{Z}} \subseteq G$. To say that $q^{\mathbb{Z}} = G$ we need to say that for any $x \in G$ there exists an integer n such that $x = q^n$. But this statement cannot be expressible in first-order way.

Remark 2 *Our intention of using the unary predicate Γ is to represent the set $q^{\mathbb{Z}}$ sitting in the field. Unfortunately, however, it is not possible to exclude the possibility of G containing q' another transcendental element of $\mathbb{F}$ such that G is generated by qq' not just q.*

Note that Th$(\mathbb{Z}, +, 0)$ is superstable by Theorem III.5.8 of [Ba] (p. 94). We give here more direct proof of the superstability by counting complete types.

We see first that there are continuum many 1-types over a countable set of parameters, e.g., the natural numbers, as follows;

Let $\sigma \in 2^{\omega}$. Denote $\sigma = \langle \sigma(0), \sigma(1), \cdots, \sigma(i), \cdots \rangle$. Define

$$\sigma \restriction 0 = \emptyset, \quad \text{and } \sigma \restriction k = \langle \sigma(o), \cdots, \sigma(k-1) \rangle.$$

The main idea is that for each σ we define a type $t_\sigma(x)$ which specifies the property of the number realzing the type. Say, suppose $\sigma = \langle 1, 0, 0, 1, \cdots \rangle$. Then the formulas in $t_\sigma(x)$ asserts that

- x is a number of the form $2k_0$ for some k_0,
- x is a number of the form $2^2 k_1 + 2$ for some k_1,
- x is a number of the form $2^3 k_2 + 2$ for some k_2,
- x is a number of the form $2^4 k_4 + 10$ for some k_3,
- and so on.

To make the above description presice, we define a mapping f which associates a natural number to each initial segment of σ;

$$f(\langle \sigma(0) \rangle) = \begin{cases} 2 & \text{if } \sigma(0) = 1 \\ 1 & \text{if } \sigma(0) = 0 \end{cases}$$

Suppose $f(\sigma \restriction i) = l$ has been defined, then

$$f(\sigma \restriction (i+1)) = \begin{cases} l + 2^i & \text{if } \sigma(i+1) = 1 \\ l & \text{if } \sigma(i+1) = 0 \end{cases}$$

With this function f we now define a 1-type $t_\sigma(x)$ corresponding to σ such that for each i

$$\exists y\, (x = \underbrace{y + \cdots + y}_{2^k \text{ times}} + l) \in t_\sigma(x) \iff f(\sigma \restriction i) = 2^k + l$$

To be more precise, the type $t_\sigma(x)$ is the completion of the type having all the formula "$\exists y\, (x = \underbrace{y + \cdots + y}_{2^k \text{ times}} + l)$" above.

Remark 3 *Note that for any natural number* $(\mathbb{Z}, +, 0) \simeq (k\mathbb{Z}, +, 0)$ *as additive infinite cyclic groups having one generator. Therefore* Th$(k\mathbb{Z}, +, 0)$ *is also superstable for each* k.

Proposition 4 *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, and q be a transcendental element of $\mathbb{F}$. Γ is a unary predicate satisfying the properties listed above. Then the first-order theory T_Γ is superstable.*

Proof: First we classify the 1-types over the empty set in this theory.

1) Without Γ, there are only two kinds of 1-types; algebraic ones and a transcendental one. Algebraic 1-types are isolated by the minimal polynomial of the element realizing the type. On the other hand, there is only one transcendental type.

2) With Γ , one type of x can say that for each n x^n is trancendental and $\Gamma(x^n)$ holds.

3) There are continuum may 1-types describing the property of integers due to the superstability of the theory $\mathrm{Th}(\mathbb{Z}, +, 0)$, Let $t(x)$ be one of them. Suppose

$$\exists y\,(x = \underbrace{y + \cdots + y}_{2^k \text{ times}} + l) \in t(x).$$

Corresponding to this type $t(x)$, we define the type $t^*(x)$ such that

$$\exists y_1 \cdots \exists y_k\, \exists u \Big(\Gamma(u) \wedge u \neq 1 \wedge y_1 = u \cdot u \wedge y_2 = y_1 \cdot y_1 \wedge \cdots$$
$$\wedge y_k = y_{k-1} \cdot y_{k-1} \wedge x = y_k \cdot y_k \cdot \overbrace{u \cdots u}^{l \text{ times}}\Big) \in t^*(x)$$

Suppose $t_0(x)$ and $t_1(x)$ are distinct 1-types in $\mathrm{Th}(\mathbb{Z}, +, 0)$. We see that they determine pairwise inconsistent 1-types $t_0^*(x)$ and $t_1^*(x)$ in $\mathrm{Th}(\mathbb{F}_\Gamma)$. If otherwise there were a number α realizing $t_0(x)$ and $t_1(x)$. It follows that there exist u_0 and u_1 such that for some k and l $u_0^k = u_1^l$. Without loss of generality we may assume that $k \leq l$. This implies that

$$1 = (u_0^{-1} u_1)^k \cdot u_1^{l-k}$$

contradictiong the property of $\mathrm{Th}(\mathbb{F}_\Gamma)$.

In this way we see that there are continuum many complete 1-types. It follows that the theory $\mathrm{Th}(\mathbb{F}_\Gamma)$ is superstable since the cardinality of the complete types is stable once the cardinality of the parameters reaches $2^{\aleph_0}$. ∎

2 Theory of a quantum torus

The purpose of showing the superstability of the theory of $\mathbb{F}_\Gamma$ is that the theory of quantum torus defined over the field $\mathbb{F}$ is superstable. This is a part of a Zilber's project of finding exemples of analytic Zariski structures among quantum algebraic structures.

A candidate for this project is a quantum torus. Quantum tori are geometric objects associated with non-commutative algebras $\mathcal{A}_q$ with q generating a multiplicative cyclic group.

When q is a root of unity, we have a quantum torus which is a Zariski structure (Zilber's result).

In this note, however, we explain very briefly that with q generating an infinite cyclic group the resulting structure gives rise to a quantum torus. The details are written in [IZ].

2.1 Description of the torus $T_q^2(\mathbb{C})$

In this subsection we give more concrete description of quantum torus by taking the complex numbers $\mathbb{C}$ not just any algebraically closed $\mathbb{F}$.

Consider a $\mathbb{C}$-algebra $\mathcal{A}_q^2$ generated by operators U, U^{-1}, V, V^{-1} satisfying

$$VU = qUV$$

where $q = e^{2\pi i h}$ with $h \in \mathbb{R}$. Let $\Gamma_q = q^{\mathbb{Z}}$ be a multiplicative subgroup of $\mathbb{C}^*$ generated by q.

The quantum 2-torus $T_q^2(\mathbb{C})$ associated with the algebra $\mathcal{A}_q^2$ and the group Γ_q is the 3-sorted structure $(\mathbf{U}_\phi, \mathbf{V}_\phi, \mathbb{C}^*)$ with the actions U and V satisfying

$$\begin{array}{rcl} U & : & \mathbf{u}(\gamma u, v) \mapsto \gamma u \mathbf{u}(\gamma u, v) \\ V & : & \mathbf{u}(\gamma u, v) \mapsto v \mathbf{u}(q^{-1}\gamma u, v) \end{array} \tag{1}$$

and

$$\begin{array}{rcl} U & : & \mathbf{v}(\gamma v, u) \mapsto u \mathbf{v}(q\gamma v, u) \\ V & : & \mathbf{v}(\gamma v, u) \mapsto \gamma v \mathbf{v}(\gamma v, u) \end{array} \tag{2}$$

Two operators U and V are acting on $\mathbb{C}^*\mathbf{U}$ and $\mathbb{C}^*\mathbf{V}$. We view both $\mathbb{C}^*\mathbf{U}$ and $\mathbb{C}^*\mathbf{V}$ as the following equivalence classes;

$$\mathbb{C}^*\mathbf{U} \simeq (\mathbb{C} \times \mathbf{U})/E$$

where for $(x, y), (x', y') \in \mathbb{C} \times \mathbf{U}$ define

$$(x, y) \sim_E (x', y') \iff \exists \gamma \left(\gamma \in \Gamma_q \wedge (y' = \gamma y \wedge x' = x\gamma^{-1})\right) \tag{3}$$

Similary for $\mathbb{C}^*\mathbf{V}$.

When we describe the property of the quantum 2-torus $T_q^2(\mathbb{C})$ we treat both operators U and V as 4-ary relations. The actions are characterised as follows;

1. $\forall \mathbf{u} \in \mathbf{U}\, \exists u \in \mathbb{C}^*\, (U : \mathbf{u} \mapsto u\mathbf{u})$ and
   $\forall \mathbf{u} \in \mathbf{U}\, \exists v \in \mathbb{C}^*\, \exists \mathbf{u}' \in \mathbf{U}\, (V : \mathbf{u} \mapsto v\mathbf{u}' \wedge U : \mathbf{u}' \mapsto q^{-1}u\mathbf{u}')$
2. $\forall \mathbf{v} \in \mathbf{V}\, \exists v \in \mathbb{C}^*\, (V : \mathbf{v} \mapsto v\mathbf{v})$ and
   $\forall \mathbf{v} \in \mathbf{V}\, \exists u \in \mathbb{C}^{:}\, \exists \mathbf{v}' \in \mathbf{V}\, (U : \mathbf{v} \mapsto qu\mathbf{v}' \wedge V : \mathbf{v} \mapsto v\mathbf{v})$

We need to translate the above properties into first-order formulas. First we express simply that $\mathbf{U}$ and $\mathbf{V}$ are acting on both $\mathbb{C}^*\mathbf{U}$ and $\mathbb{C}^*\mathbf{U}$ as follows.

- $\forall x_1 \forall u_1 \forall x_2 \forall u_2\, \forall x_1' \forall u_1' \forall x_2' \forall u_2' \forall x_1' \Big(U(x_1, u_1, x_2, u_2) \to (x_1 \in \mathbb{C}^* \wedge u_1 \in \mathbf{U} \wedge x_1 \in \mathbb{C}^* \wedge u_2 \in \mathbf{U})\Big) \bigwedge \Big((U(x_1, u_1, x_2, u_2) \wedge U(x_1', u_1', x_2', u_2' \wedge (x_1, u_1) \sim_E (x_1', u_1')) \to (x_2, u_2) \sim_E (x_2', u_2')\Big)$

  Here $\sim_E$ is the equivalence relation defined in (3). This formula corresponds to $U : \mathbb{C}^*\mathbf{U} \to \mathbb{C}^*\mathbf{U}$. We need three more similar formulas expressing $V : \mathbb{C}^*\mathbf{U} \to \mathbb{C}^*\mathbf{U}$, $U : \mathbb{C}^*\mathbf{V} \to \mathbb{C}^*\mathbf{V}$ and $V : \mathbb{C}^*\mathbf{V} \to \mathbb{C}^*\mathbf{V}$.

Here is a summary of the intuitive ideas of $\mathbf{U}$, $\mathbf{V}$ and operations U and V.

- Both $\mathbf{U}$ and $\mathbf{V}$ are two dimensional objects.
- Both $\mathbf{U}$ and $\mathbf{V}$ are bases for an ambient module which we do not give any formal description in the theory.
- The operator U moves each element (vector) of $\mathbf{U}$ on its fibre, say *vertically*. On the other hand the operator V moves each element of $\mathbf{U}$ to another element of $\mathbf{U}$, say *holizontally*.
- The operator V does the same actions on $\mathbf{U}$ and $\mathbf{V}$.

Definable subsets in a model of the theory of this 3-sorted structure $(\mathbf{U}, \mathbf{V}, \mathbb{F})$ are determined by the actions U and V on each sort $\mathbf{U}$ and $\mathbf{V}$.

What we can say about the operations U and V are basically the number of times we apply these operations, thus this part can be expressed by positive quantifier free formulas.

However we need an existential quantifer in order to express the equivalence relation $\sim_E$. So the complexity of the definable sets is the boolean combination of positive quantifier free formula modulo existential quantifier. (Near model complete).

Once we have writen down all the property of the quantum torus in first-order way, we see that the resuting theory is superstable since the stability theoretic property is almost same as the theory $\mathrm{Th}(\mathbb{F}_\Gamma)$ described in Section 1. For details, see [IZ].

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