Title: Smoothness of hairs for some entire functions (Dynamical Systems: Theories to Applications and Applications to Theories)

Author(s): KISAKA, Masashi; SHISHIKURA, Mitsuhiro

Citation: 数理解析研究所講究録 (2011), 1742: 137-144

Issue Date: 2011-05

URL: http://hdl.handle.net/2433/170920

Type: Departmental Bulletin Paper

Textversion: publisher

Kyoto University
Smoothness of hairs for some entire functions

Masashi KISAKA (木坂 正史)
Department of Mathematical Sciences,
Graduate School of Human and Environmental Studies,
Kyoto University, Kyoto 606-8501, Japan

Mitsuhiro SHISHIKURA (宍戸 光広)
Department of Mathematics,
Faculty of Science,
Kyoto University, Kyoto 606-8502, Japan

1 Preliminaries

Let $f$ be an entire function and $f^n$ denote the $n$-th iterate of $f$. Recall that the Fatou set $F(f)$ and the Julia set $J(f)$ of $f$ are defined as follows:

\[ F(f) := \{ z \in \mathbb{C} | \{ f^n \}_{n=1}^{\infty} \text{ is a normal family in a neighborhood of } z \}, \]
\[ J(f) := \mathbb{C} \setminus F(f). \]

By definition, $F(f)$ is open and $J(f)$ is closed in $\mathbb{C}$. Also $J(f)$ is compact if $f$ is a polynomial, while it is non-compact if $f$ is transcendental. This is due to the fact that $\infty$ is an essential singularity of $f$.

The purpose of this paper is to construct so-called hairs, which are subsets of the Julia set $J(f)$, and to show their smoothness for a certain class of transcendental entire functions. Devaney and Krych first constructed hairs for exponential family $E_{\lambda}(z) = \lambda e^z (\lambda \in \mathbb{C} \setminus \{0\})$ in 1984 ([DK]). Here we briefly explain their results. Define

\[ B_l := \{ z | (2l-1)\pi < \text{Im} z + \theta < (2l+1)\pi \}, \quad \theta = \arg \lambda \in [-\pi, \pi), \quad l \in \mathbb{Z} \]

then we can define itinerary $S(z) := s = (s_0, s_1, \cdots, s_n, \cdots) \in \mathbb{Z}^N$ for a point $z \in \mathbb{C}$ by $E_{\lambda}^n(z) \in B_{s_n}$.

\[ g(t) = |\lambda|e^t, \]

**Theorem 1.1** (Devaney-Krych, 1984). If $s \in \mathbb{Z}^N$ satisfies the following “growth condition”:

\[ 3x_0 \in \mathbb{R}, \forall n, (2|s_n| + 1)\pi + |\theta| \leq g^n(x_0), \quad g(t) := |\lambda|e^t, \]

then there exists $h_s(t) \subset J(E_{\lambda})$ which satisfies the following:

(i) $E_{\lambda}(h_s(t)) = h_{\sigma(s)}(g(t))$, where $\sigma$ is the shift map on $\mathbb{Z}^N$,

(ii) $E_{\lambda}^n(h_s(t)) \to \infty (n \to \infty)$ for every $t$.

The curve $h_s(t)$ is called a hair. Viana showed that this hair $h_s(t)$ is a $C^\infty$ curve ([V]).

In this paper we consider the existence and smoothness of hairs under a general setting. In particular we generalize this result for the exponential functions to $f(z) := P(z)e^{Q(z)}$, where $P(z)$ and $Q(z)$ are entire functions.
where $P(z)$ and $Q(z)$ are polynomials. For simplicity, we state the result for the easiest case, that is, for a "fixed" itinerary $s = (s_0, s_0, s_0, \cdots)$. We state our detailed setting and the results of existence in §2. In §3 and §4 we explain the smoothness of hairs. In §5 we state the result for $f(z) := P(z)e^{Q(z)}$ as an application of our general results. Finally in §6 we briefly explain how to construct hairs for general itineraries.

2 $C^0$ a priori estimates — existence of a hair $h(t)$ —

Our setting is as follows:

A: Let $U$, $V \subset \mathbb{C}$ be unbounded domains, $f : U \to V$ a holomorphic diffeomorphism and $g : [\tau_*, \infty) \to \mathbb{R}$ the reference mapping, i.e., an increasing $C^\infty$ function such that $g(t) > t$ for $t \geq \tau_*$. (Hence $g^n(t) \to \infty$ $(n \to \infty)$.)

B: (Initial curves) : There exist continuous curves $h_0, h_1 : [\tau_*, \infty) \to \mathbb{C}$ and a continuous increasing function $R : [\tau_*, \infty) \to \mathbb{R}_+$ and a constant $0 < \frac{3}{\kappa} < 1$ which satisfy the following:

- $|h_1(t) - h_0(t)| \leq (1 - \kappa)R(t)$ for $t \in [\tau_*, \infty)$; \hfill (1)
- If $|w - h_0(g(t))| \leq R(g(t))$ for some $t \in [\tau_*, \infty)$, then $w \in V$ and
  \[ |f'(z)| \frac{R(t)}{R(g(t))} \geq \frac{1}{\kappa}, \text{ where } z = (f|_U)^{-1}(w) \] \hfill (2)

(This is equivalent to that $f : B_f(t) \to \overline{D(h_0(g(t)), R(g(t)))}$ is a homeomorphism with
\[ |f'(z)| \frac{R(t)}{R(g(t))} \geq \frac{1}{\kappa}, \text{ for } z \in B_f(t), \text{ where } B_f(t) := \{z \in U : |f(z) - h_0(g(t))| \leq R(g(t))\}.)

Definition 2.1. Let $\rho : [\tau, \infty) \to \mathbb{R}_+$ be a continuous increasing function and define a norm $||\psi||_{\rho,\tau}$ for $\psi : [\tau, \infty) \to \mathbb{C}$ by
\[ ||\psi||_{\rho,\tau} := \sup_{t \geq \tau} |\psi(t)|\rho(t). \]

We call $\rho$ a weight function. Then it is easy to see that the space $X_{\rho,\tau} := \{\psi \mid ||\psi||_{\rho,\tau} < \infty\}$ becomes a Banach space.

Note that if we put $\rho_*(t) := 1/R(t)$ then the condition B (1) can be read as
\[ ||h_1 - h_0||_{\rho_*,\tau_*} \leq 1 - \kappa. \]

Under the above setting, we can show the existence of a hair $h(t)$:

Lemma 2.2. Under the assumptions A and B, there exist continuous functions $h_n : [\tau_*, \infty) \to \mathbb{C}$ $(n = 2, 3, \ldots)$ such that for $n = 0, 1, 2, \ldots$,
\[ ||h_n - h_0||_{\rho_*,\tau_*} \leq 1 - \kappa^n; \] \hfill (3)
\[ f \circ h_{n+1}(t) = h_n \circ g(t) \text{ for } t \geq \tau_*; \] \hfill (4)
\[ ||h_{n+1} - h_n||_{\rho_*,\tau_*} \leq (1 - \kappa)\kappa^n. \] \hfill (5)
Therefore there exists a continuous function $h(t) = \lim_{n \to \infty} h_n(t)$ satisfying
\[
f \circ h(t) = h \circ g(t) \quad \text{for} \quad t \geq \tau_* \quad \text{and} \quad |h(t) - h_0(t)| \leq R(t).
\] (6)

Of course, $f^n(h(t)) \to \infty (n \to \infty)$ holds for $t \geq \tau_*$, since we have $f^n(h(t)) = h(g^n(t))$ and $g^n(t) \to \infty (n \to \infty)$.

3 \quad C^1 \text{ estimates}

From (4), we have
\[
\log h_{n+1}' = \log h_n' \circ g + \log g' - \log h_{n+1}.
\] (7)

So define
\[
\psi_n(t) := \log h_n'(t),
\] (8)

Then we have
\[
\psi_{n+1} - \psi_n = (\psi_n - \psi_{n-1}) \circ g - (\log f' \circ h_{n+1} - \log f' \circ h_n).
\] (9)

If $\psi_n - \psi_{n-1} \to 0$ as $t \to \infty$, by composing $g$, $(\psi_n - \psi_{n-1}) \circ g$ may go to 0 faster. This can be formulated in terms of $|| \cdot ||_{\rho_0,\tau}$ with an appropriate weight function $\rho_0 : [\tau_*, \infty) \to \mathbb{R}^+$ (which is assumed to be increasing). In fact, for a function $\psi : [\tau_*, \infty) \to \mathbb{C}$ (for our case, $\psi = \psi_n - \psi_{n-1}$), we have
\[
|| \psi \circ g ||_{\rho_0,\tau} = \sup_{t \geq \tau} |\psi(g(t))| \rho_0(t) = \sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} \cdot |\psi(g(t))| \rho_0(g(t))
\] \leq \left( \sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} \right) \cdot \left( \sup_{t' \geq g(\tau)} |\psi(t')| \rho_0(t') \right) = \left( \sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} \right) ||\psi||_{\rho_0,\psi(G(\tau))}. \quad (10)

So if $\sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} < 1$, then $|| \cdot ||_{\rho_0,\tau}$-norm is contracted by composing $g$. This implies the possibility to prove the geometric convergence of (9).

For further estimates ($C^k$, $k = 1, 2, \ldots$), we need to prepare the following.

**Definition 3.1.** (1) Let $\rho_k, \sigma_k : [\tau_*, \infty) \to \mathbb{R}^+$, ($k = 0, 1, 2, \ldots$) be weight functions with $\sigma_k \leq \rho_k(t)$. These are to measure the norm $|| \cdot ||_{\rho,\tau}$ of $\psi_{n+1}^{(k)} - \psi_n^{(k)}$ and the norm $|| \cdot ||_{\sigma,\tau}$ of $\psi_n^{(k)}$.

(2) For given weight functions $\rho_k, \sigma_k$, define
\[
\alpha_k(t) := \frac{\rho_k(t)|g'(t)|^k}{\rho_k(g(t))},
\]
\[
\overline{\alpha}_k(\tau) := \sup_{t \geq \tau} \alpha_k(t),
\]
\[
D_k(t) := \sup_{z \in B_f(t)} \left| \left( \log f' \right)^{(k)}(z) \right|, \quad k = 0, 1, 2, \ldots, \quad t, \tau \geq \tau_*,
\]
\[
B_f(t) := \{ z \in U : |f(z) - h_0(g(t))| \leq R(g(t)) \}\]
Now in order to prove that \( h(t) \) is \( C^1 \), we assume that there exist weight functions \( \rho_0, \sigma_0 : [\tau_*, \infty) \to \mathbb{R}_+ \) satisfying the following conditions \( C_0, D_0 \) and \( F_0 \):

**C0**: \( h_0, h_1 \) are \( C^1 \) with \( h_0'(t), h_1'(t) \neq 0 \) and \( \psi_0(t) = \log h_0'(t), \psi_1(t) = \log h_1'(t) \) satisfy
\[
||\psi_1 - \psi_0||_{\rho_0, \tau_*} < \infty \quad \text{and} \quad ||\psi_0||_{\sigma_0, \tau_*} < \infty.
\]

**D0**: \( \lim_{\tau \to \infty} \alpha_0(\tau) = \lim_{t \to \infty} \sup_{\tau \geq \tau_*} \frac{\rho_0(t)}{\rho_0(g(t))} < 1. \)

**F0**: \( K_0 := \sup_{t \geq \tau_*} D_1(t) R(t) \rho_0(t) < \infty. \)

**Lemma 3.2.** Suppose \( A, B, C_0, D_0 \) and \( F_0 \) are satisfied. Then \( h_n \) are \( C^1 \) (\( n = 2, 3, \ldots \)) and there exists \( \kappa_0 < 1 \) and \( C_0 \) such that \( \psi_n(t) = \log h_n'(t) \) satisfy
\[
||\psi_{n+1} - \psi_n||_{\rho_0, \tau_*} \leq C_0 \kappa_0^n \quad (n = 0, 1, 2, \ldots).
\]

Therefore the limit \( h(t) \) is also \( C^1 \) and \( \psi(t) = \log h'(t) \) satisfies
\[
||\psi - \psi_0||_{\rho_0, \tau_*} \leq \frac{C_0}{1 - \kappa_0} \quad \text{and} \quad ||\psi||_{\sigma_0, \tau_*} \leq \frac{C_0}{1 - \kappa_0} + ||\psi_0||_{\sigma_0, \tau}. \]

\[\square\]

4 **Higher order derivatives — estimate for \( \psi_n^{(k)}(k = 1, 2, \ldots) \)**

Differentiating (7) and using \( h_{n+1}' = e^{\psi_{n+1}} \), we have
\[
\psi_{n+1}' = (\psi_n' \circ g) \cdot g' + (\log g')' - ((\log f')' \circ h_{n+1}) e^{\psi_{n+1}},
\]
\[
\psi_{n+1}'' = (\psi_n'' \circ g) \cdot (g')^2 + (\psi_n' \circ g) \cdot g'' + (\log g'')''
- ((\log f'')' \circ h_{n+1}) e^{2\psi_{n+1}} - ((\log f')' \circ h_{n+1}) e^{\psi_{n+1}} \psi_n' + \ldots.
\]

More generally, the following holds:

**Lemma 4.1.** For \( k = 1, 2, \ldots \), we have
\[
\psi_{n+1}^{(k)} = (\psi_n^{(k)} \circ g) (g')^k + \sum_{1 \leq \ell \leq k, j_1 \geq \ldots \geq j_\ell \geq 1} \text{const} \ (\psi_n^{(\ell)} \circ g) g^{(j_1)} \ldots g^{(j_\ell)} + (\log g')^{(k)}
- \sum_{1 \leq \ell \leq k, 0 \leq \nu} \text{const} \ ((\log f')^{(\nu)} \circ h_{n+1}) e^{\psi_{n+1}} \psi_{n+1}^{(j_1)} \ldots \psi_{n+1}^{(j_\nu)},
\]

where the coefficients "const" are some constants depending the indices \( \ell, j_1, j_2, \ldots \). \[\square\]

Note that in the right hand side of (14), only the first term contains \( k \)-th derivative of \( \psi_n \) and all other terms involve lower order derivatives of \( \psi_n \) (or none). Therefore if lower order derivatives are "under control," it is expected that we can proceed as in the previous section.
For the exponential map $f(z) = \lambda e^z$ and $g(t) = |\lambda|e^t$, we have $(\log f')' \equiv 1$ and $(\log f^{(\ell)}) \equiv 0$ ($\ell > 1$). So the formula (14) simplifies substantially. Moreover $g^{(j_1)} \cdots g^{(j_\ell)}$ is a constant multiple of $g(t)^\ell$ which also simplifies the expression.

Suppose weight functions $\rho_k, \sigma_k : [\tau_*, \infty) \to \mathbb{R}_+$ are given. We require the following conditions:

**C_k**: $h_0, h_1$ are $C^{k+1}$ and $\psi_0 = \log h_0'$ and $\psi_1 = \log h_1'$ satisfy

$$||\psi^{(k)}_1 - \psi^{(k)}_0||_{\rho_k, \tau_*} < \infty \quad \text{and} \quad ||\psi^{(k)}_0||_{\sigma_k, \tau_*} < \infty.$$  

**D_k**: $\lim_{\tau \to \infty} \alpha_k(\tau) < 1$.

**E_k**: For $1 \leq \ell < k$ and $j_1, \ldots, j_\ell \geq 1$ with $j_1 + \cdots + j_\ell = k$,

$$\sup_{t \geq \tau_*} \frac{\rho_k(t)|g^{(j_1)}(t)\cdots g^{(j_\ell)}(t)|}{\rho_\ell(g(t))} < \infty.$$  

Here if $\nu = 0$, set $\sigma_{j_1}(t) \cdots \sigma_{j_\nu}(t) = 1$.

Note that the last condition should be satisfied only when $\nu \geq 1$.

Under these assumptions, we can show the following:

**Lemma 4.2.** Let $k \geq 1$. Suppose $A, B, C_j (0 \leq j \leq k)$, $D_j (0 \leq j \leq k)$, $E_j (1 \leq j \leq k)$ and $F_j (0 \leq j \leq k)$ are satisfied. Then $h_n$ are $C^{k+1}$ ($n = 2, 3, \ldots$) and there exist constants $0 < \kappa_k < 1$ such that

$$||\psi^{(k)}_{n+1} - \psi^{(k)}_n||_{\rho_k, \tau_*} \leq C_k \kappa_k^n \quad (n = 0, 1, 2, \ldots). \quad (15)$$

Therefore the limit $h(t)$ is also $C^{k+1}$ and $\psi = \log h'$ satisfies

$$||\psi^{(k)} - \psi^{(k)}_0||_{\rho_k, \tau_*} \leq \frac{C_k}{1 - \kappa_k} \quad \text{and} \quad ||\psi^{(k)}_0||_{\sigma_k, \tau_*}, ||\psi^{(k)}||_{\sigma_k, \tau_*} \leq C_k.$$  

\[\square\]

5 Examples

As an application of our results, we consider the following function:

$$f(z) = P(z)e^{Q(z)}, \quad P(z) = b_m z^m + \cdots + b_0, \quad Q(z) = a_d z^d + \cdots + a_1 z + a_0$$

$$m = \deg P \geq 0, \quad d = \deg Q \geq 1, \quad (a_d \neq 0, b_m \neq 0).$$

By a linear change of coordinate and multiplying $P$ by $e^{a_0}$, we may assume that $a_d = 1$ and $a_0 = 0$. Let $g(t) = t^m e^t$ be the "reference function" to compare.
Lemma 5.1. For any $\epsilon > 0$, there exists $R > 0$ such that for $t \in \mathbb{C}$ with $|t| \geq R$, there exists a unique $w = w(t)$ such that $|w| < \epsilon$, $P(t(1 + w))e^{O(t(1 + w))} = t^{m}e^{t^{d}}$ and $|tw| \leq C$, where $C$ is a constant.

By using this function $w(t)$, we define $h_{0}(t)$ and start constructing $h_{n}(t)$.

**Proposition 5.2.** There exist $\tau_{*} > 0$ and $C^{\infty}$-function $h_{0} : [\tau_{*}, \infty) \rightarrow \mathbb{C}$ such that $h_{0}'(t) \neq 0$ and

$$ f \circ h_{0}(t) = g(t) (= t^{m}e^{t^{d}}) \quad (16) $$

$$ h_{0}(t) := t(1 + w(t)) = t + O(1) \quad (as \ t \rightarrow \infty) \quad (17) $$

$$ (\log h_{0}'(t))^{(k)} = O\left(\frac{1}{t^{k+2}}\right) \quad (k = 0, 1, 2, \ldots). \quad (18) $$

Moreover $h_{0}$, $h_{1} := f^{-1}(h_{0} \circ g)$ satisfies A and B with $R(t) = \frac{\text{const}}{t^{d-1}g(t)}$.

**Proposition 5.3.** Let $\sigma_{k}(t) = t^{k+2} \ (k = 0, 1, 2, \ldots)$. Suppose that $\rho_{k}(t) \ (k = 0, 1, 2, \ldots)$ satisfy

1. $\sigma_{k}(t) \leq \rho_{k}(t)$
2. $\limsup_{t \rightarrow \infty} \frac{\rho_{k}(t)(t^{k+1})(g(t))^k}{\rho_{k}(g(t))} < 1$
3. $\rho_{k}(t) \leq \text{const} \frac{\rho_{\ell}(g(t))}{t^{k(d-1)}} \quad (1 \leq \ell < k)$
4. $\rho_{k}(t) \leq \text{const} \cdot t^{k}g(t)$
5. $\rho_{k}(t) \leq \text{const} \frac{\rho_{j}(t)}{t^{d+j-1}} \quad (1 \leq j < k)$

Then $C_{j} (0 \leq j \leq k)$, $D_{j} (0 \leq j \leq k)$, $E_{j} (1 \leq j \leq k)$ and $F_{j} (0 \leq j \leq k)$ are satisfied.

**Corollary 5.4.** For a suitable choice of const and $\mu_{k} > 0$, $\rho_{k}(t) = \text{const} \frac{e^{t^{d}}}{t^{\mu_{k}}}$ satisfies the hypothesis.

**6 General cases**

In this section we briefly explain how to construct hairs for general itineraries. We consider the following general setting:

**Setting:** Let $U_{l}$, $V_{l} \subset \mathbb{C}$ be unbounded domains and $f_{l} : U_{l} \rightarrow V_{l}$ be holomorphic diffeomorphisms $(l = 0, 1, 2, \cdots)$. Let $g : [\tau_{*}, \infty) \rightarrow \mathbb{R}$ be a reference mapping, i.e., an increasing $C^{\infty}$ function such that $g(t) > t$ for $t \geq \tau_{*}$. (Hence $g'(t) \rightarrow \infty \ (l \rightarrow \infty)$). Set $\tau_{l} := g'(\tau_{*})$. 


Our goal is to construct $h_l : \tau_l, \infty) \rightarrow \mathbb{C}, \quad (l = 0, 1, 2, \cdots)$ such that
\[ f_l \circ h_l(t) = h_{l+1} \circ g(t). \]

**Strategy:** Construct initial curves $h_{l,l} (l = 0, 1, 2, \cdots)$. Then define $h_{n,l} : \tau_l, \infty) \rightarrow \mathbb{C}, \quad (0 \leq l < n)$ by lifting successively so that
\[ f_l \circ h_{n,l}(t) = h_{n,l+1} \circ g(t). \]

See the figure and the diagram below:

![Diagram](image)

Under the similar assumptions as in the previous sections, we can show the existence and smoothness of hairs $h_l(t) (l = 0, 1, 2, \cdots)$. We omit the details. Since the function $f(z) = P(z)e^{Q(z)}$ is structurally finite, we can define the itinerary $s \in \{0, 1, \cdots, d-1\} \times \mathbb{Z}^N$, where $d = \deg Q$. For the details, see [Ki]. So by taking $f_l$ to be the restriction of $f$ to a suitable domain according to $s$, we can apply our results for general setting and obtain the smooth hair $h_l(t)$ corresponding to $s$. 
References

