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Smoothness of hairs for some entire functions

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1 Preliminaries

Let $f$ be an entire function and $f^n$ denote the $n$-th iterate of $f$. Recall that the Fatou set $F(f)$ and the Julia set $J(f)$ of $f$ are defined as follows:

\[
F(f) := \{ z \in \mathbb{C} \mid \{f^n\}_{n=1}^{\infty} \text{ is a normal family in a neighborhood of } z \},
\]

\[
J(f) := \mathbb{C} \setminus F(f).
\]

By definition, $F(f)$ is open and $J(f)$ is closed in $\mathbb{C}$. Also $J(f)$ is compact if $f$ is a polynomial, while it is non-compact if $f$ is transcendental. This is due to the fact that $\infty$ is an essential singularity of $f$.

The purpose of this paper is to construct so-called hairs, which are subsets of the Julia set $J(f)$, and to show their smoothness for a certain class of transcendental entire functions. Devaney and Krych first constructed hairs for exponential family $E_{\lambda}(z) = \lambda e^z (\lambda \in \mathbb{C} \setminus \{0\})$ in 1984 ([DK]). Here we briefly explain their results. Define

\[
B_l := \{ z \mid (2l-1)\pi < \text{Im } z + \theta < (2l+1)\pi \}, \quad \theta = \arg \lambda \in [-\pi, \pi), \quad l \in \mathbb{Z}
\]

then we can define itinerary $S(z) := s = (s_0, s_1, \cdots, s_n, \cdots) \in \mathbb{Z}^N$ for a point $z \in \mathbb{C}$ by $E_{\lambda}^n(z) \in B_{s_n}$.

**Theorem 1.1 (Devaney-Krych, 1984).** If $s \in \mathbb{Z}^N$ satisfies the following "growth condition":

\[
3x_0 \in \mathbb{R}, \forall n, (2|s_n| + 1)\pi + |\theta| \leq g^n(x_0), \quad g(t) := |\lambda|e^t,
\]

then there exists $h_s(t) \subset J(E_{\lambda})$ which satisfies the following:

(i) $E_\lambda(h_s(t)) = h_{\sigma(s)}(g(t))$, where $\sigma$ is the shift map on $\mathbb{Z}^N$,

(ii) $E_\lambda^n(h_s(t)) \to \infty (n \to \infty)$ for every $t$.

The curve $h_s(t)$ is called a hair. Viana showed that this hair $h_s(t)$ is a $C^\infty$ curve ([V]).

In this paper we consider the existence and smoothness of hairs under a general setting. In particular we generalize this result for the exponential functions to $f(z) := P(z)e^{Q(z)}$, where

\[
P(z), \quad Q(z)
\]
where $P(z)$ and $Q(z)$ are polynomials. For simplicity, we state the result for the easiest case, that is, for a "fixed" itinerary $s = (s_0, s_0, s_0, \cdots)$. We state our detailed setting and the results of existence in §2. In §3 and §4 we explain the smoothness of hairs. In §5 we state the result for $f(z) := P(z)e^{Q(z)}$ as an application of our general results. Finally in §6 we briefly explain how to construct hairs for general itineraries.

2 $C^0$ a priori estimates — existence of a hair $h(t)$ —

Our setting is as follows:

A: Let $U, V \subset \mathbb{C}$ be unbounded domains, $f : U \rightarrow V$ a holomorphic diffeomorphism and $g : [\tau_*, \infty) \rightarrow \mathbb{R}$ the reference mapping, i.e., an increasing $C^\infty$ function such that $g(t) > t$ for $t \geq \tau_*$. (Hence $g^n(t) \rightarrow \infty$ ($n \rightarrow \infty$).)

B: (Initial curves) : There exist continuous curves $h_0, h_1 : [\tau_*, \infty) \rightarrow \mathbb{C}$ and a continuous increasing function $R : [\tau_*, \infty) \rightarrow \mathbb{R}_+$ and a constant $0 < \frac{3}{2}\kappa < 1$ which satisfy the following:

- $|h_1(t) - h_0(t)| \leq (1 - \kappa)R(t)$ for $t \in [\tau_*, \infty)$;
- If $|w - h_0(g(t))| \leq R(g(t))$ for some $t \in [\tau_*, \infty)$, then $w \in V$ and
  \[ |f'(z)| \frac{R(t)}{R(g(t))} \geq \frac{1}{\kappa}, \text{ where } z = (f|_U)^{-1}(w) \]
  (This is equivalent to that $f : B_f(t) \rightarrow \overline{D(h_0(g(t)), R(g(t)))}$ is a homeomorphism with
  \[ |f'(z)| \frac{R(t)}{R(g(t))} \geq \frac{1}{\kappa}, \ z \in B_f(t), \text{ where } B_f(t) := \{ z \in U : |f(z) - h_0(g(t))| \leq R(g(t)) \}).

Definition 2.1. Let $\rho : [\tau, \infty) \rightarrow \mathbb{R}_+$ be a continuous increasing function and define a norm $||\psi||_{\rho, \tau}$ for $\psi : [\tau, \infty) \rightarrow \mathbb{C}$ by
  \[ ||\psi||_{\rho, \tau} := \sup_{t \geq \tau} |\psi(t)| \rho(t). \]
We call $\rho$ a weight function. Then it is easy to see that the space $X_{\rho, \tau} := \{ \psi \mid ||\psi||_{\rho, \tau} < \infty \}$ becomes a Banach space.

Note that if we put $\rho_*(t) := 1/R(t)$ then the condition B (1) can be read as
  \[ ||h_1 - h_0||_{\rho_*, \tau_*} \leq 1 - \kappa. \]

Under the above setting, we can show the existence of a hair $h(t)$:

Lemma 2.2. Under the assumptions A and B, there exist continuous functions $h_n : [\tau_*, \infty) \rightarrow \mathbb{C}$ ($n = 2, 3, \ldots$) such that for $n = 0, 1, 2, \ldots$,

- $||h_n - h_0||_{\rho_*, \tau_*} \leq 1 - \kappa^n$;  \[ ||h_n - h_0||_{\rho_*, \tau_*} \leq 1 - \kappa^n; \]  \[ f \circ h_{n+1}(t) = h_n \circ g(t) \text{ for } t \geq \tau_*; \]
- $||h_{n+1} - h_n||_{\rho_*, \tau_*} \leq (1 - \kappa)\kappa^n$.  \[ ||h_{n+1} - h_n||_{\rho_*, \tau_*} \leq (1 - \kappa)\kappa^n. \]
Therefore there exists a continuous function \( h(t) = \lim_{n \to \infty} h_n(t) \) satisfying
\[
f \circ h(t) = h \circ g(t) \quad \text{for } t \geq \tau_* \quad \text{and} \quad |h(t) - h_0(t)| \leq R(t).
\]

Of course, \( f^n(h(t)) \to \infty \) (\( n \to \infty \)) holds for \( \forall t \geq \tau_* \), since we have \( f^n(h(t)) = h(g^n(t)) \) and \( g^n(t) \to \infty \) (\( n \to \infty \)).

### 3 \( C^1 \) estimates

From (4), we have
\[
\log h'_{n+1} = \log h'_{n} \circ g + \log g' - \log f' \circ h_{n+1}.
\]

So define
\[
\psi_n(t) := \log h'_n(t),
\]

Then we have
\[
\psi_{n+1} - \psi_n = (\psi_n - \psi_{n-1}) \circ g - (\log f' \circ h_{n+1} - \log f' \circ h_n).
\]

If \( \psi_n - \psi_{n-1} \to 0 \) as \( t \to \infty \), by composing \( g \), \( (\psi_n - \psi_{n-1}) \circ g \) may go to 0 faster. This can be formulated in terms of \( \| \cdot \|_{\rho_0, \tau} \) with an appropriate weight function \( \rho_0 : [\tau_*, \infty) \to \mathbb{R}^+ \) (which is assumed to be increasing). In fact, for a function \( \psi : [\tau_*, \infty) \to \mathbb{C} \) (for our case, \( \psi = \psi_n - \psi_{n-1} \)), we have
\[
\|\psi \circ g\|_{\rho_0, \tau} = \sup_{t \geq \tau} |\psi(g(t))| \rho_0(t) = \left( \sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} \right) \left( \sup_{t' \geq \rho_0(\tau)} |\psi(t')| \rho_0(t') \right) = \left( \sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} \right) \|\psi\|_{\rho_0, \rho_0(\tau)}.
\]

So if \( \sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} < 1 \), then \( \| \cdot \|_{\rho_0, \tau} \)-norm is contracted by composing \( g \). This implies the possibility to prove the geometric convergence of (9).

For further estimates \( (C^k, k = 1, 2, \ldots) \), we need to prepare the following.

**Definition 3.1.**
(1) Let \( \rho_k, \sigma_k : [\tau_*, \infty) \to \mathbb{R}_+ \), \( (k = 0, 1, 2, \ldots) \) be weight functions with \( \sigma_k(t) \leq \rho_k(t) \). These are to measure the norm \( \| \cdot \|_{\rho_0, \tau} \) of \( \psi_{n+1}^{(k)} - \psi_n^{(k)} \) and the norm \( \| \cdot \|_{\sigma_0, \tau} \) of \( \psi_n^{(k)} \).

(2) For given weight functions \( \rho_k, \sigma_k \), define
\[
\alpha_k(t) := \frac{\rho_k(t)}{\rho_k(g(t))} |g'(t)|^k,
\]
\[
\bar{\alpha}_k(\tau) := \sup_{t \geq \tau} \alpha_k(t),
\]
\[
D_k(t) := \sup_{z \in B_f(t)} |(\log f')^{(k)}(z)|, \quad k = 0, 1, 2, \ldots, \quad t, \quad \tau \geq \tau_*.
\]
\[
B_f(t) := \{ z \in U : |f(z) - h_0(g(t))| \leq R(g(t)) \}.
\]
Now in order to prove that $h(t)$ is $C^1$, we assume that there exist weight functions $\rho_0, \sigma_0 : [\tau_*, \infty) \to \mathbb{R}_+$ satisfying the following conditions $C_0$, $D_0$ and $F_0$:

$C_0$: $h_0, h_1$ are $C^1$ with $h_0'(t), h_1'(t) \neq 0$ and $\psi_0(t) = \log h_0'(t), \psi_1(t) = \log h_1'(t)$ satisfy
\[
||\psi_1 - \psi_0||_{\rho_0, \tau_*} < \infty \quad \text{and} \quad ||\psi_0||_{\sigma_0, \tau_*} < \infty.
\]

$D_0$: $\lim_{\tau \to \infty} \alpha_0(\tau) = \lim_{t \to \infty} \sup_{\infty} \frac{\rho_0(t)}{\rho_0(g(t))} < 1$.

$F_0$: $K_0 := \sup_{t \geq \tau_*} D_1(t)R(t)\rho_0(t) < \infty$.

Lemma 3.2. Suppose $A$, $B$, $C_0$, $D_0$ and $F_0$ are satisfied. Then $h_n$ are $C^1$ $(n = 2, 3, \ldots)$ and there exists $\kappa_0 < 1$ and $C_0$ such that $\psi_n(t) = \log h_n'(t)$ satisfy
\[
||\psi_{n+1} - \psi_n||_{\rho_0, \tau_*} \leq C_0\kappa_0^n \quad (n = 0, 1, 2, \ldots).
\]

Therefore the limit $h(t)$ is also $C^1$ and $\psi(t) = \log h'(t)$ satisfies
\[
||\psi - \psi_0||_{\rho_0, \tau_*} \leq \frac{C_0}{1 - \kappa_0} \quad \text{and} \quad ||\psi||_{\sigma_0, \tau_*} \leq \frac{C_0}{1 - \kappa_0} + ||\psi_0||_{\sigma_0, \tau_*} < \infty.
\]

\[\square\]

4 Higher order derivatives — estimate for $\psi^{(k)}_n (k = 1, 2, \ldots)$ —

Differentiating (7) and using $h_{n+1}' = e^{\psi_{n+1}}$, we have
\[
\psi_{n+1}' = (\psi_n' \circ g) \cdot g' + (\log g')' - ((\log f')' \circ h_{n+1}) e^{\psi_{n+1}},
\]
\[
(12)
\]
\[
\psi_{n+1}'' = (\psi_n'' \circ g) \cdot (g')^2 + (\psi_n' \circ g) \cdot g'' + (\log g'')' - ((\log f'')' \circ h_{n+1}) e^{2\psi_{n+1}} - ((\log f')' \circ h_{n+1}) e^{\psi_{n+1}} \psi_{n+1}'
\]
\[
(13)
\]

More generally, the following holds:

Lemma 4.1. For $k = 1, 2, \ldots$, we have
\[
\psi_{n+1}' = (\psi_n' \circ g) (g')^k + \sum_{1 \leq \ell \leq k} \text{const} \left(\psi_n^{(\ell)} \circ g\right) g^{(j_1)} \cdots g^{(j_\ell)} + (\log g')^{(k)}
\]
\[
- \sum_{1 \leq \ell \leq k, 0 \leq \nu} \text{const} \left((\log f')^{(\ell)} \circ h_{n+1}\right) e^{\ell\psi_{n+1}} \psi_{n+1}^{(j_1)} \cdots \psi_{n+1}^{(j_\nu)},
\]
\[
(14)
\]

where the coefficients "const" are some constants depending the indices $\ell, j_1, j_2, \ldots$. □

Note that in the right hand side of (14), only the first term contains $k$-th derivative of $\psi_n$ and all other terms involve lower order derivatives of $\psi_n$ (or none). Therefore if lower order derivatives are "under control," it is expected that we can proceed as in the previous section.
For the exponential map $f(z) = \lambda e^z$ and $g(t) = |\lambda|e^t$, we have $(\log f)'(z) \equiv 1$ and $(\log f^{(\ell)}) \equiv 0$ ($\ell > 1$). So the formula (14) simplifies substantially. Moreover $g^{(j_1)} \ldots g^{(j_\ell)}$ is a constant multiple of $g(t)^\ell$ which also simplifies the expression.

Suppose weight functions $\rho_k, \sigma_k : [\tau_*, \infty) \to \mathbb{R}_+$ are given. We require the following conditions:

- **C_k**: $h_0, h_1$ are $C^{k+1}$ and $\psi_0 = \log h'_0$ and $\psi_1 = \log h'_1$ satisfy
  \[ ||\psi_1^{(k)} - \psi_0^{(k)}||_{\rho_k, \tau_*} < \infty \quad \text{and} \quad ||\psi_0^{(k)}||_{\sigma_k, \tau_*} < \infty. \]

- **D_k**: $\lim_{\tau \to \infty} \alpha_k(\tau) < 1$.

- **E_k**: For $1 \leq \ell < k$ and $j_1, \ldots, j_\ell \geq 1$ with $j_1 + \cdots + j_\ell = k$,
  \[ \sup_{t \geq \tau_*} \rho_\ell(t) g^{(j_1)}(t) \cdots g^{(j_\ell)}(t) < \infty. \]

- **F_k**: For $1 \leq \ell \leq k$, $\nu \geq 0$, $j_1, \ldots, j_\nu \geq 1$ with $\ell + j_1 + \cdots + j_\nu = k$,
  \[ \sup_{t \geq \tau_*} D_{\ell+1}(t) R(t) \frac{\rho_k(t)}{\sigma_{j_1}(t) \cdots \sigma_{j_\nu}(t)} < \infty; \]
  \[ \sup_{t \geq \tau_*} D_\ell(t) \frac{\rho_k(t)}{\rho_0(t) \sigma_{j_1}(t) \cdots \sigma_{j_\nu}(t)} < \infty; \]
  \[ \text{if } \nu \geq 1, \text{ then for } 1 \leq i \leq \nu, \sup_{t \geq \tau_*} D_i(t) \frac{\rho_k(t)}{\sigma_{j_1}(t) \cdots \sigma_{j_\nu}(t) \rho_{j_i}(t)} < \infty. \]

Here if $\nu = 0$, set $\sigma_{j_1}(t) \cdots \sigma_{j_\nu}(t) = 1$.

Under these assumptions, we can show the following:

**Lemma 4.2.** Let $k \geq 1$. Suppose A, B, C_j ($0 \leq j \leq k$), D_j ($0 \leq j \leq k$), E_j ($1 \leq j \leq k$) and F_j ($0 \leq j \leq k$) are satisfied. Then $h_n$ are $C^{k+1}$ ($n = 2, 3, \ldots$) and there exist constants $0 < \kappa_k < 1$ and $C_k$ such that

\[ ||\psi_{n+1}^{(k)} - \psi_n^{(k)}||_{\rho_k, \tau_*} \leq C_k \kappa_k^n \quad (n = 0, 1, 2, \ldots). \] (15)

Therefore the limit $h(t)$ is also $C^{k+1}$ and $\psi = \log h'$ satisfies

\[ ||\psi^{(k)} - \psi_0^{(k)}||_{\rho_k, \tau_*} \leq \frac{C_k}{1 - \kappa_k} \quad \text{and} \quad ||\psi_0^{(k)}||_{\sigma_k, \tau_*}, ||\psi^{(k)}||_{\sigma_k, \tau_*} \leq C'_k. \]

\[
\square
\]

5 **Examples**

As an application of our results, we consider the following function:

\[ f(z) = P(z)e^{Q(z)}, \quad P(z) = b_m z^m + \cdots + b_0, \quad Q(z) = a_d z^d + \cdots + a_1 z + a_0 \]

$m = \deg P \geq 0, \quad d = \deg Q \geq 1, \quad (a_d \neq 0, \quad b_m \neq 0).$

By a linear change of coordinate and multiplying $P$ by $e^{a_0}$, we may assume that $a_d = 1$ and $a_0 = 0$. Let $g(t) = t^m e^{t^d}$ be the “reference function” to compare.
Lemma 5.1. For any $\varepsilon > 0$, there exists $R > 0$ such that for $t \in \mathbb{C}$ with $|t| \geq R$, there exists a unique $w = w(t)$ such that $|w| < \varepsilon$, $P(t(1 + w))e^{O(t(1 + w))} = t^m e^{t^d}$ and $|tw| \leq C$, where $C$ is a constant.

By using this function $w(t)$, we define $h_0(t)$ and start constructing $h_n(t)$.

Proposition 5.2. There exist $\tau_* > 0$ and $C^\infty$-function $h_0 : [\tau_*, \infty) \to \mathbb{C}$ such that $h_0'(t) \neq 0$ and

\[ f \circ h_0(t) = g(t) \quad (= t^m e^{t^d}) \]  
(16)

\[ h_0(t) := t(1 + w(t)) = t + O(1) \quad (\text{as } t \to \infty) \]  
(17)

\[ (\log h_0'(t))^k = O\left(\frac{1}{t^{k+2}}\right) \quad (k = 0, 1, 2, \ldots). \]  
(18)

Moreover $h_0, h_1 := f^{-1}(h_0 \circ g)$ satisfies $A$ and $B$ with $R(t) = \frac{c\text{const}}{t^{d-1}g(t)}$.

Proposition 5.3. Let $\sigma_k(t) = t^{k+2}$ ($k = 0, 1, 2, \ldots$). Suppose that $\rho_k(t)$ ($k = 0, 1, 2, \ldots$) satisfy

\[ \sigma_k(t) \leq \rho_k(t) \]  
(19)

\[ \limsup_{t \to \infty} \frac{\rho_k(t) t^{k(d-1)} (g(t))^k}{\rho_k(g(t))} < 1 \]  
(20)

\[ \rho_k(t) \leq \text{const} \frac{\rho_\ell(g(t))}{t^{k(d-1)}(g(t))^\ell} \quad (1 \leq \ell < k) \]  
(21)

\[ \rho_k(t) \leq \text{const} \cdot t^k g(t) \]  
(22)

\[ \rho_k(t) \leq \text{const} \frac{\rho_0(t)}{t^{d-k}} \quad (k \geq 1) \]  
(23)

\[ \rho_k(t) \leq \text{const} \frac{\rho_j(t)}{t^{d+j-1}} \quad (1 \leq j < k). \]  
(24)

Then $C_j$ ($0 \leq j \leq k$), $D_j$ ($0 \leq j \leq k$), $E_j$ ($1 \leq j \leq k$) and $F_j$ ($0 \leq j \leq k$) are satisfied.

Corollary 5.4. For a suitable choice of $\text{const}$ and $\mu_k > 0$, $\rho_k(t) = \text{const} \frac{e^{t^d}}{t^{\mu_k}}$ satisfies the hypothesis.

6 General cases

In this section we briefly explain how to construct hairs for general itineraries. We consider the following general setting:

Setting: Let $U_l$, $V_l \subset \mathbb{C}$ be unbounded domains and $f_l : U_l \to V_l$ be holomorphic diffeomorphisms ($l = 0, 1, 2, \cdots$). Let $g : [\tau_*, \infty) \to \mathbb{R}$ be a reference mapping, i.e., an increasing $C^\infty$ function such that $g(t) > t$ for $t \geq \tau_*$. (Hence $g'(t) \to \infty (l \to \infty)$) Set $\tau_l := g'(\tau_*)$. 
Our goal is to construct $h_l : [\tau_l, \infty) \to \mathbb{C}$, $(l = 0, 1, 2, \cdots)$ such that

$$f_l \circ h_l(t) = h_{l+1} \circ g(t).$$

**Strategy:** Construct initial curves $h_{l,l}$ $(l = 0, 1, 2, \cdots)$. Then define $h_{n,l} : [\tau_l, \infty) \to \mathbb{C}$, $(0 \leq l < n)$ by lifting successively so that

$$f_l \circ h_{n,l}(t) = h_{n,l+1} \circ g(t).$$

See the figure and the diagram below:

Under the similar assumptions as in the previous sections, we can show the existence and smoothness of hairs $h_l(t)$ $(l = 0, 1, 2, \cdots)$. We omit the details. Since the function $f(z) = P(z)e^{Q(z)}$ is structurally finite, we can define the itinerary $s \in \{0, 1, \cdots, d - 1\} \times \mathbb{Z}^N$, where $d = \deg Q$. For the details, see [Ki]. So by taking $f_l$ to be the restriction of $f$ to a suitable domain according to $s$, we can apply our results for general setting and obtain the smooth hair $h_n(t)$ corresponding to $s$. 
References

