Smoothness of hairs for some entire functions

Masashi KISAKA (木坂 正史)
Department of Mathematical Sciences,
Graduate School of Human and Environmental Studies,
Kyoto University, Kyoto 606-8501, Japan

Mitsuhiro SHISHIKURA (宍戸 光広)
Department of Mathematics,
Faculty of Science,
Kyoto University, Kyoto 606-8502, Japan

1 Preliminaries

Let $f$ be an entire function and $f^n$ denote the $n$-th iterate of $f$. Recall that the Fatou set $F(f)$ and the Julia set $J(f)$ of $f$ are defined as follows:

\[
F(f) := \{ z \in \mathbb{C} \mid \{f^n\}_{n=1}^{\infty} \text{ is a normal family in a neighborhood of } z \},
\]

\[
J(f) := \mathbb{C} \setminus F(f).
\]

By definition, $F(f)$ is open and $J(f)$ is closed in $\mathbb{C}$. Also $J(f)$ is compact if $f$ is a polynomial, while it is non-compact if $f$ is transcendental. This is due to the fact that $\infty$ is an essential singularity of $f$.

The purpose of this paper is to construct so-called hairs, which is subsets of the Julia set $J(f)$, and to show their smoothness for a certain class of transcendental entire functions. Devaney and Krych first constructed hairs for exponential family $E_{\lambda}(z) = \lambda e^z (\lambda \in \mathbb{C} \setminus \{0\})$ in 1984 ([DK]). Here we briefly explain their results. Define

\[
B_l := \{ z \mid (2l-1)\pi < \text{Im } z + \theta < (2l+1)\pi \}, \quad \theta = \arg \lambda \in [-\pi, \pi), \quad l \in \mathbb{Z}
\]

then we can define itinerary $S(z) := s = (s_0, s_1, \cdots, s_n, \cdots) \in \mathbb{Z}^N$ for a point $z \in \mathbb{C}$ by $E^s_{\lambda}(z) \in B_{s_n}$.

**Theorem 1.1** (Devaney-Krych, 1984). If $s \in \mathbb{Z}^N$ satisfies the following “growth condition”:

\[
\exists x_0 \in \mathbb{R}, \forall n, (2|s_n| + 1)\pi + |\theta| \leq g^n(x_0), \quad g(t) := |\lambda|e^t,
\]

then there exists $h_s(t) \subset J(E_{\lambda})$ which satisfies the following:

(i) $E_{\lambda}(h_s(t)) = h_{\sigma(s)}(g(t))$, where $\sigma$ is the shift map on $\mathbb{Z}^N$,

(ii) $E^n_{\lambda}(h_s(t)) \to \infty (n \to \infty)$ for every $t$.

The curve $h_s(t)$ is called a hair. Viana showed that this hair $h_s(t)$ is a $C^\infty$ curve ([V]).

In this paper we consider the existence and smoothness of hairs under a general setting. In particular we generalize this result for the exponential functions to $f(z) := P(z)e^{Q(z)}$, where $P(z), Q(z)$ are polynomials.
where $P(z)$ and $Q(z)$ are polynomials. For simplicity, we state the result for the easiest case, that is, for a “fixed” itinerary $s = (s_0, s_0, s_0, \ldots)$. We state our detailed setting and the results of existence in §2. In §3 and §4 we explain the smoothness of hairs. In §5 we state the result for $f(z) := P(z)e^{Q(z)}$ as an application of our general results. Finally in §6 we briefly explain how to construct hairs for general itineraries.

2 $C^0$ a priori estimates — existence of a hair $h(t)$ —

Our setting is as follows:
A: Let $U, V \subset \mathbb{C}$ be unbounded domains, $f : U \to V$ a holomorphic diffeomorphism and $g : [\tau_*, \infty) \to \mathbb{R}$ the reference mapping, i.e., an increasing $C^\infty$ function such that $g(t) > t$ for $t \geq \tau_*$. (Hence $g^n(t) \to \infty$ ($n \to \infty$).)

B: (Initial curves) : There exist continuous curves $h_0, h_1 : [\tau_*, \infty) \to \mathbb{C}$ and a continuous increasing function $R : [\tau_*, \infty) \to \mathbb{R}_+$ and a constant $0 < \kappa / 3 < 1$ which satisfy the following:

| $h_1(t) - h_0(t) | \leq (1 - \kappa)R(t)$ for $t \in [\tau_*, \infty)$; (1)
| $| w - h_0(g(t)) | \leq R(g(t))$ for some $t \in [\tau_*, \infty)$, then $w \in V$ and
| $| f'(z) | \frac{R(t)}{R(g(t))} \geq \frac{1}{\kappa}$, where $z = (f|\nu)^{-1}(w)$ (This is equivalent to that $f : B_f(t) \to \overline{D(h_0(g(t)), R(g(t)))}$ is a homeomorphism with $|f'(z)| \frac{R(t)}{R(g(t))} \geq \frac{1}{\kappa}, z \in B_f(t)$, where $B_f(t) := \{z \in U : |f(z) - h_0(g(t))| \leq R(g(t))\}$.)

Definition 2.1. Let $\rho : [\tau, \infty) \to \mathbb{R}_+$ be a continuous increasing function and define a norm $||\psi||_{\rho, \tau}$ for $\psi : [\tau, \infty) \to \mathbb{C}$ by $||\psi||_{\rho, \tau} := \sup_{t \geq \tau} |\psi(t)|\rho(t)$. We call $\rho$ a weight function. Then it is easy to see that the space $X_{\rho, \tau} := \{\psi \mid ||\psi||_{\rho, \tau} < \infty\}$ becomes a Banach space.

Note that if we put $\rho_*(t) := 1/R(t)$ then the condition B (1) can be read as

$||h_1 - h_0||_{\rho_*, \tau_*} \leq 1 - \kappa$.

Under the above setting, we can show the existence of a hair $h(t)$:

Lemma 2.2. Under the assumptions A and B, there exist continuous functions $h_n : [\tau_*, \infty) \to \mathbb{C}$ $(n = 2, 3, \ldots)$ such that for $n = 0, 1, 2, \ldots$,

$||h_n - h_0||_{\rho_*, \tau_*} \leq 1 - \kappa^n$; (3)
$f \circ h_{n+1}(t) = h_n \circ g(t)$ for $t \geq \tau_*$; (4)
$||h_{n+1} - h_n||_{\rho_*, \tau_*} \leq (1 - \kappa)\kappa^n$. (5)
Therefore there exists a continuous function $h(t) = \lim_{n \to \infty} h_n(t)$ satisfying

$$f \circ h(t) = h \circ g(t) \quad \text{for } t \geq \tau_*, \quad \text{and} \quad |h(t) - h_0(t)| \leq R(t).$$

(6)

Of course, $f^n(h(t)) \to \infty (n \to \infty)$ holds for $\forall \ t \geq \tau_*$, since we have $f^n(h(t)) = h(g^n(t))$ and $g^n(t) \to \infty (n \to \infty)$.

3 $C^1$ estimates

From (4), we have

$$\log h_{n+1} = \log h_n' \circ g + \log g' - \log f' \circ h_n.$$  

(7)

So define

$$\psi_n(t) := \log h_n'(t),$$

(8)

Then we have

$$\psi_{n+1} - \psi_n = (\psi_n - \psi_{n-1}) \circ g - (\log f' \circ h_{n+1} - \log f' \circ h_n).$$

(9)

If $\psi_n - \psi_{n-1} \to 0$ as $t \to \infty$, by composing $g$, $(\psi_n - \psi_{n-1}) \circ g$ may go to 0 faster. This can be formulated in terms of $\| \cdot \|_{\rho_0, \tau}$ with an appropriate weight function $\rho_0 : [\tau_*, \infty) \to \mathbb{R}^+$ (which is assumed to be increasing). In fact, for a function $\psi : [\tau_*, \infty) \to \mathbb{C}$ (for our case, $\psi = \psi_n - \psi_{n-1}$), we have

$$\|\psi \circ g\|_{\rho_0, \tau} = \sup_{t \geq \tau} |\psi(g(t))| \rho_0(t) = \sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} \cdot |\psi(g(t))| \rho_0(g(t))$$

$$\leq \left( \sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} \right) \cdot \left( \sup_{t' \geq g(\tau)} |\psi(t')| \rho_0(t') \right) = \left( \sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} \right) \|\psi\|_{\rho_0, \rho_0(g(t))}.$$

(10)

So if $\sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))} < 1$, then $\| \cdot \|_{\rho_0, \tau}$-norm is contracted by composing $g$. This implies the possibility to prove the geometric convergence of (9).

For further estimates ($C^k$, $k = 1, 2, \ldots$), we need to prepare the following.

Definition 3.1. (1) Let $\rho_k$, $\sigma_k : [\tau_*, \infty) \to \mathbb{R}^+$, $(k = 0, 1, 2, \ldots)$ be weight functions with $\sigma_k(t) \leq \rho_k(t)$. These are to measure the norm $\| \cdot \|_{\rho_k, \tau}$ of $\psi_{n+1}^{(k)} - \psi_n^{(k)}$ and the norm $\| \cdot \|_{\sigma_k, \tau}$ of $\psi_n^{(k)}$.

(2) For given weight functions $\rho_k$, $\sigma_k$, define

$$\alpha_k(t) := \frac{\rho_k(t)|g'(t)|^k}{\rho_k(g(t))},$$

$$\overline{\alpha}_k(\tau) := \sup_{t \geq \tau} \alpha_k(t),$$

$$D_k(t) := \sup_{z \in B_f(t)} \left | (\log f')^{(k)}(z) \right |, \quad k = 0, 1, 2, \ldots, \quad t, \ \tau \geq \tau_*,$$

$$B_f(t) := \{ z \in U : |f(z) - h_0(g(t))| \leq R(g(t)) \}$$
Now in order to prove that \( h(t) \) is \( C^1 \), we assume that there exist weight functions \( \rho_0, \sigma_0 : [\tau_*, \infty) \to \mathbb{R}_+ \) satisfying the following conditions \( C_0, D_0 \) and \( F_0 \):

\[ C_0: \ h_0, h_1 \text{ are } C^1 \text{ with } h_0'(t), h_1'(t) \neq 0 \text{ and } \phi_0(t) = \log h_0'(t), \phi_1(t) = \log h_1'(t) \text{ satisfy} \]

\[ ||\phi_1 - \phi_0||_{\rho_0, \tau_*} < \infty \quad \text{and} \quad ||\phi_0||_{\sigma_0, \tau_*} < \infty. \]

\[ D_0: \ \lim_{\tau \to \infty} \alpha_0(\tau) = \limsup_{t \to \infty} \frac{\rho_0(t)}{\rho_0(g(t))} < 1. \]

\[ F_0: \ K_0 := \sup_{t \geq \tau_*} D_1(t) R(t) \rho_0(t) < \infty. \]

**Lemma 3.2.** Suppose \( A, B, C_0, D_0 \) and \( F_0 \) are satisfied. Then \( h_n \) are \( C^1 \) \( (n = 2, 3, \ldots) \) and there exists \( \kappa_0 < 1 \) and \( C_0 \) such that \( \phi_n(t) = \log h_n'(t) \) satisfy

\[ ||\phi_{n+1} - \phi_n||_{\rho_0, \tau_*} \leq C_0 \kappa_0^n \quad (n = 0, 1, 2, \ldots). \]

Therefore the limit \( h(t) \) is also \( C^1 \) and \( \phi(t) = \log h'(t) \) satisfies

\[ ||\phi - \phi_0||_{\rho_0, \tau_*} \leq \frac{C_0}{1 - \kappa_0} \quad \text{and} \quad ||\phi||_{\sigma_0, \tau_*} \leq \frac{C_0}{1 - \kappa_0} + ||\phi_0||_{\sigma_0, \tau_*} < \infty. \]

\[ \square \]

4 Higher order derivatives — estimate for \( \phi_n^{(k)} (k = 1, 2, \ldots) \) —

Differentiating (7) and using \( h_{n+1}' = e^{\phi_{n+1}} \), we have

\[ \phi_{n+1}' = (\phi_n' \circ g) \cdot g' + (\log g')' - ((\log f')' \circ h_{n+1}) e^{\phi_{n+1}}, \]

\[ \phi_{n+1}'' = (\phi_n'' \circ g') + (\phi_n' \circ g) \cdot g'' + (\log g')'' - ((\log f')'' \circ h_{n+1}) e^{2\phi_{n+1}} - ((\log f')' \circ h_{n+1}) e^{\phi_{n+1}} \phi_{n+1}'. \]

More generally, the following holds:

**Lemma 4.1.** For \( k = 1, 2, \ldots \), we have

\[ \phi_{n+1}^{(k)} = (\phi_n^{(k)} \circ g') + \sum_{1 \leq \ell \leq k} \text{const} (\phi_n^{(\ell)} \circ g) g^{(j_1)} \cdots g^{(j_k)} + (\log g')^{(k)} \]

\[ - \sum_{1 \leq \ell \leq k, 0 \leq \nu} \text{const} (\log f')^{(\ell)} \circ h_{n+1}) e^{\ell \phi_{n+1}} \phi_{n+1}^{(j_1)} \cdots \phi_{n+1}^{(j_\nu)}, \]

where the coefficients "const" are some constants depending the indices \( \ell, j_1, j_2, \ldots \).

\[ \square \]

Note that in the right hand side of (14), only the first term contains \( k \)-th derivative of \( \phi_n \) and all other terms involve lower order derivatives of \( \phi_n \) (or none). Therefore if lower order derivatives are "under control," it is expected that we can proceed as in the previous section.
For the exponential map \( f(z) = \lambda e^z \) and \( g(t) = |\lambda|e^t \), we have \((\log f')' \equiv 1\) and \((\log f^{(\ell)})' \equiv 0 \) \((\ell > 1)\). So the formula (14) simplifies substantially. Moreover \( g^{(j_1)} \ldots g^{(j_k)} \) is a constant multiple of \( g(t)^\ell \) which also simplifies the expression.

Suppose weight functions \( \rho_k, \sigma_k : [\tau_*, \infty) \to \mathbb{R}_+ \) are given. We require the following conditions:

**C\( \_k \):** \( h_0, h_1 \) are \( C^{k+1} \) and \( \psi_0 = \log h_0' \) and \( \psi_1 = \log h_1' \) satisfy

\[
||\psi_1^{(k)} - \psi_0^{(k)}||_{\rho_k, \tau_*} < \infty \quad \text{and} \quad ||\psi_0^{(k)}||_{\sigma_k, \tau_*} < \infty.
\]

**D\( \_k \):** \( \lim_{\tau \to \infty} \sigma_k(\tau) < 1 \).

**E\( \_k \):** For \( 1 \leq \ell < k \) and \( j_1, \ldots, j_{\ell} \geq 1 \) with \( j_1 + \cdots + j_{\ell} = k \),

\[
\sup_{t \geq \tau_*} \frac{\rho_\ell(t)|g^{(j_1)}(t) \cdots g^{(j_{\ell})}(t)|}{\rho_\ell(g(t))} < \infty.
\]

**F\( \_k \):** For \( 1 \leq \ell \leq k \), \( \nu \geq 0 \), \( j_1, \ldots, j_{\nu} \geq 1 \) with \( \ell + j_1 + \cdots + j_{\nu} = k \),

\[
\sup_{t \geq \tau_*} \frac{\rho_\ell(t)}{\sigma_{j_1}(t) \cdots \sigma_{j_{\nu}}(t)} < \infty; \quad \sup_{t \geq \tau_*} \frac{\rho_\ell(t)}{\sigma_{j_1}(t) \cdots \sigma_{j_{\nu}}(t)} < \infty;
\]

if \( \nu \geq 1 \), then for \( 1 \leq i \leq \nu \),

\[
\sup_{t \geq \tau_*} \frac{\rho_\ell(t)}{\sigma_{j_1}(t) \cdots \sigma_{j_{\nu}}(t)} < \infty.
\]

Here if \( \nu = 0 \), set \( \sigma_{j_1}(t) \cdots \sigma_{j_{\nu}}(t) = 1 \).

Note that the last condition should be satisfied only when \( \nu \geq 1 \).

Under these assumptions, we can show the following:

**Lemma 4.2.** Let \( k \geq 1 \). Suppose \( A, B, C_j (0 \leq j \leq k) \), \( D_j (0 \leq j \leq k) \), \( E_j (1 \leq j \leq k) \) and \( F_j (0 \leq j \leq k) \) are satisfied. Then \( h_n \) are \( C^{k+1} \) \((n = 2, 3, \ldots)\) and there exist constants \( 0 < \kappa_k < 1 \) and \( C_k \) such that

\[
||\psi_{n+1}^{(k)} - \psi_n^{(k)}||_{\rho_k, \tau_*} \leq C_k \kappa_k^n \quad (n = 0, 1, 2, \ldots).
\]

Therefore the limit \( h(t) \) is also \( C^{k+1} \) and \( \psi = \log h' \) satisfies

\[
||\psi^{(k)} - \psi_0^{(k)}||_{\rho_k, \tau_*} \leq \frac{C_k}{1 - \kappa_k} \quad \text{and} \quad ||\psi_0^{(k)}||_{\sigma_k, \tau_*}, \quad ||\psi^{(k)}||_{\sigma_k, \tau_*} < C'_k.
\]

5 Examples

As an application of our results, we consider the following function:

\[
f(z) = P(z)e^{Q(z)}, \quad P(z) = b_mz^m + \cdots + b_0, \quad Q(z) = a_dz^d + \cdots + a_1z + a_0
\]

\[
m = \deg P \geq 0, \quad d = \deg Q \geq 1, \quad (a_d \neq 0, \ b_m \neq 0).
\]

By a linear change of coordinate and multiplying \( P \) by \( e^{a_0} \), we may assume that \( a_d = 1 \) and \( a_0 = 0 \). Let \( g(t) = t^me^{t^d} \) be the “reference function” to compare.
Lemma 5.1. For any $\epsilon > 0$, there exists $R > 0$ such that for $t \in \mathbb{C}$ with $|t| \geq R$, there exists a unique $w = w(t)$ such that $|w| < \epsilon$, $P(t(1+w))e^{Q(t(1+w))} = t^m e^{t^d}$ and $|tw| \leq C$, where $C$ is a constant.

By using this function $w(t)$, we define $h_0(t)$ and start constructing $h_n(t)$.

Proposition 5.2. There exist $\tau_* > 0$ and $C^\infty$-function $h_0 : [\tau_*, \infty) \rightarrow \mathbb{C}$ such that $h_0'(t) \neq 0$ and

\begin{align*}
&f \circ h_0(t) = g(t) \ (= t^m e^{t^d}) \\
h_0(t) := t(1 + w(t)) = t + O(1) \quad \text{(as } t \rightarrow \infty) \\
& (\log h_0'(t))^{(k)} = O \left( \frac{1}{t^{k+2}} \right) \quad (k = 0, 1, 2, \ldots)
\end{align*}

Moreover $h_0$, $h_1 := f^{-1}(h_0 \circ g)$ satisfies A and B with $R(t) = \frac{\text{const}}{t^{d-1}g(t)}$. \hfill \Box

Proposition 5.3. Let $\sigma_k(t) = t^{k+2} \ (k = 0, 1, 2, \ldots)$. Suppose that $\rho_k(t) \ (k = 0, 1, 2, \ldots)$ satisfy

\begin{align*}
&\sigma_k(t) \leq \rho_k(t) \\
&\limsup_{t \rightarrow \infty} \frac{\rho_k(t) t^{k(d-1)}(g(t))^k}{\rho_k(g(t))} < 1 \\
&\rho_k(t) \leq \text{const} \frac{\rho_\ell(g(t))}{t^{k(d-1)}(g(t))^{\ell}} \quad (1 \leq \ell < k) \\
&\rho_k(t) \leq \text{const} \cdot t^k g(t) \\
&\rho_k(t) \leq \text{const} \frac{\rho_0(t)}{t^{d-k}} \quad (k \geq 1) \\
&\rho_k(t) \leq \text{const} \frac{\rho_j(t)}{t^{d+j-1}} \quad (1 \leq j < k).
\end{align*}

Then $C_j \ (0 \leq j \leq k)$, $D_j \ (0 \leq j \leq k)$, $E_j \ (1 \leq j \leq k)$ and $F_j \ (0 \leq j \leq k)$ are satisfied. \hfill \Box

Corollary 5.4. For a suitable choice of const and $\mu_k > 0$, $\rho_k(t) = \text{const} \frac{e^{t^d}}{t^{\mu_k}}$ satisfies the hypothesis. \hfill \Box

6 General cases

In this section we briefly explain how to construct hairs for general itineraries. We consider the following general setting:

Setting: Let $U_l$, $V_l \subset \mathbb{C}$ be unbounded domains and $f_l : U_l \rightarrow V_l$ be holomorphic diffeomorphisms ($l = 0, 1, 2, \cdots$). Let $g : [\tau_*, \infty) \rightarrow \mathbb{R}$ be a reference mapping, i.e., an increasing $C^\infty$ function such that $g(t) > t$ for $t \geq \tau_*$. (Hence $g'(t) \rightarrow \infty \ (l \rightarrow \infty)$). Set $\tau_l := g'(\tau_*)$. 

Our goal is to construct $h_l : [\tau_l, \infty) \to \mathbb{C}$, ($l = 0, 1, 2, \cdots$) such that

$$f_l \circ h_l(t) = h_{l+1} \circ g(t).$$

**Strategy:** Construct initial curves $h_{l,l}$ ($l = 0, 1, 2, \cdots$). Then define $h_{n,l} : [\tau_l, \infty) \to \mathbb{C}$, ($0 \leq l < n$) by lifting successively so that

$$f_l \circ h_{n,l}(t) = h_{n,l+1} \circ g(t).$$

See the figure and the diagram below:

![Diagram](image-url)

Under the similar assumptions as in the previous sections, we can show the existence and smoothness of hairs $h_l(t)$ ($l = 0, 1, 2, \cdots$). We omit the details. Since the function $f(z) = P(z)e^{Q(z)}$ is structurally finite, we can define the itinerary $s \in \{0, 1, \cdots, d - 1\} \times \mathbb{Z}^N$, where $d = \deg Q$. For the details, see [Ki]. So by taking $f_l$ to be the restriction of $f$ to a suitable domain according to $s$, we can apply our results for general setting and obtain the smooth hair $h_s(t)$ corresponding to $s$. 
References

