数理解析研究所講究録
第 1742 巻 2011 年 68-81

FIXED POINT THEOREM, PERIODICITY THEOREM AND BINOMINAL AND TRINOMINAL SEQUENCES FOR ITERATION DYNAMICAL SYSTEMS OF DISCRETE LAPLACIANS ON THE PLANE LATTICE
(離散ラプラス作用素の反復力学系における不動点定理,周期性定理と二項・三項係数))

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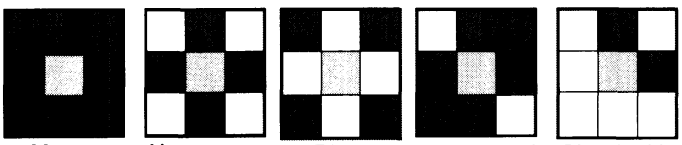
Key words: dynamical system, discrete Laplacians, fixed point theorem, periodicity theorem

We treat mathematical problems for iteration dynamical systems of discrete Laplacians on the plane lattice. At first we prove a fixed point theorem for even neighborhoods and a periodicity theorem for odd neighborhoods of the dynamical systems, respectively. Then we are concerned with binominal and trinomial sequences associated the dynamical systems and comment some observations.

1. ITERATION DYNAMICAL SYSTEM OF DISCRETE LAPLACIAN

We choose the plane lattice which is generated by two families of lines which are orthogonal to each other. We identify a cell of the lattice with one point of the cell, e.g., the midpoint. We call a set of cells which are attached with the reference cell a neighborhood. We call neighborhood even (or odd) if the number of the cells is even (respectively odd). We give several examples. Some of them are well known ([1]):



Moore Neumann Diag Neumann Hexagonal Sierpinski

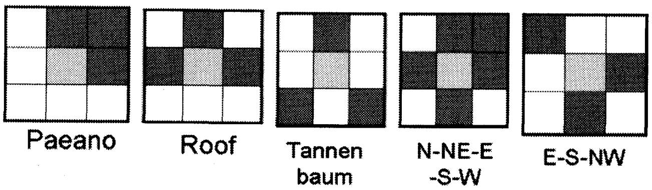


Figure 1. Examples of neighborhoods

We take the space **F** of $\{0,1\}$ valued functions on the plane lattice. Choosing a neighborhood U_p, we define the Laplacian operation as follows:

$$\Delta_{U_p} f(p) = \sum_{q \in U_p} (f(q) - f(p))$$

For an initial function $f_0 \in$ **F**, we consider the dynamical system

$$\{f_n\}, f_n = \Delta_U f_{n-1} \ (n = 1,2, \dots).$$

We call the dynamical system of a given neighborhood, the neighborhood dynamical system. For example, when we choose the Moore neighborhood we call it Moore dynamical system. For other neighborhoods, we call them following the neighborhoods in an analogous manner.

2. COMPUTER SIMULATION

Choosing sources and neighborhoods, we can realize a wide class of phenomena by these iteration dynamical systems. We call a point Q a source of the dynamical system $\{f_n\}$ if $f_n(Q) = 1$ for any step n. In case of sources we apply the discrete Laplacian only at all points except the sources. We give now some examples of computer simulations ([1], [2]).

(1) Ice crystals.

We can generate ice crystals under suitable conditions. We can realize them by use of the dynamical system of the hexagonal neighborhood. The realizations can be obtained systematically and we may expect to describe its mathematical theory in terms of the dynamical system.

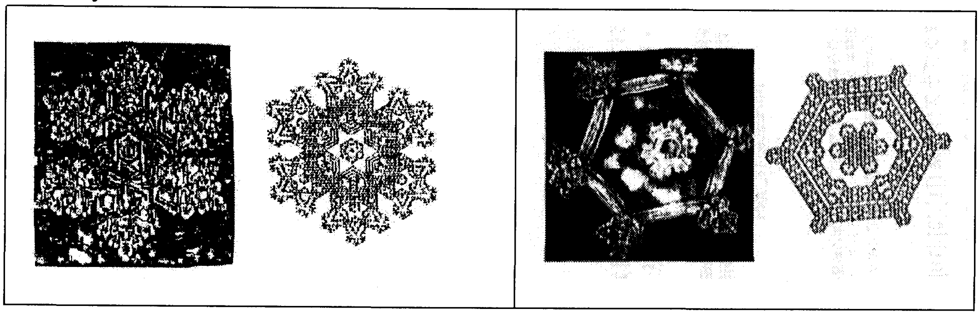


Figure 2. Ice crystals (see also [5])

(2) The evolution of extinct animals ([3]).

We present a computer simulation of one of extinct animals which is called echinoderms. The left side is the real data which is given by Sepkoski ([6]) and the right side is a computer simulation which is given by a dynamical system of Moore neighborhood with a single source. We have done the time evolutions of extinct animals systematically.

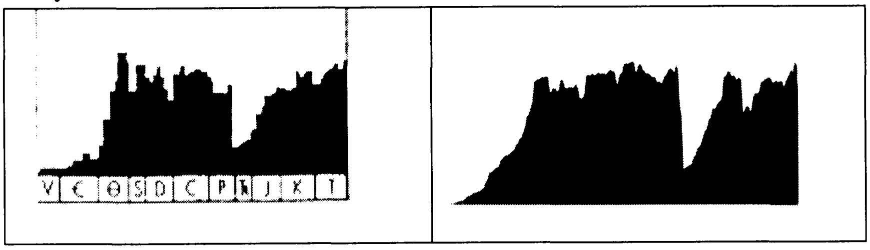


Figure 3. Evolution of an extinct animal

(3) Design patterns ([4]).

We can produce many kinds of design patterns including carpets and embroidery. We give some examples of simulations.

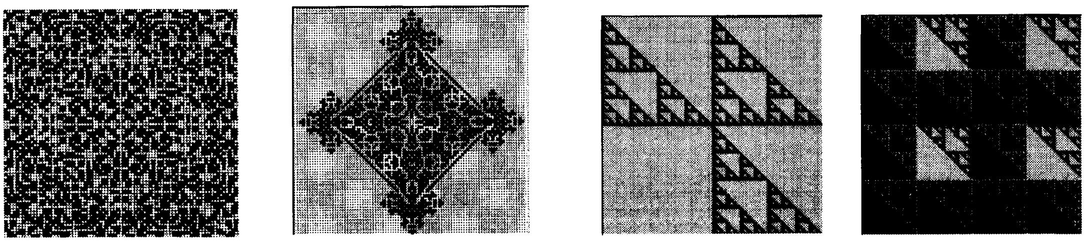


Figure 4. Generation of design patterns

(4) Flower patterns.

Next we proceed to the realizations problems of design patterns of flowers. We see that we have possibilities of realizations of flowers by use of our dynamical systems. This will be performed and published in a near future .

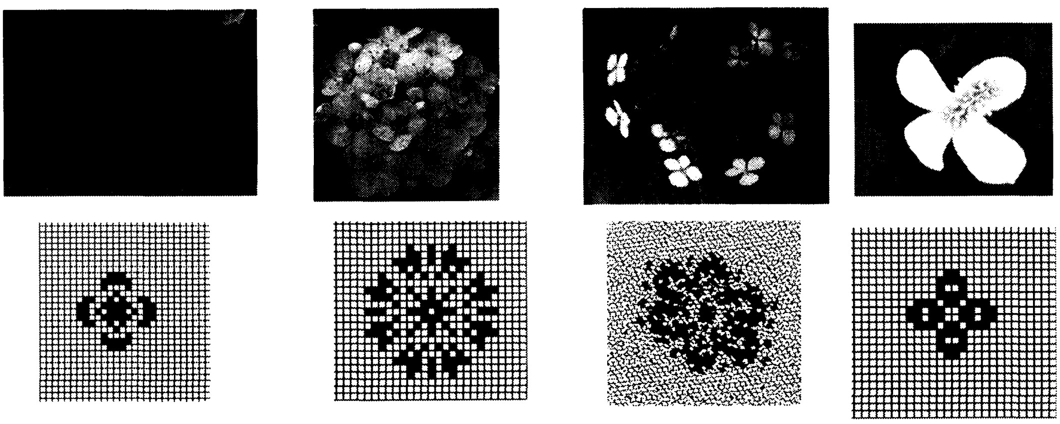


Figure 5. Generation of design patterns

(5) Design patterns of butterfly wings.

We can also try to realize design patterns of butterfly wings. We notice that we can realize so called "snake eye pattern" and "long tail patterns" and we can compare the body construction scheme at the DNA level. We give several computer simulations. These results will be presented in the annual meeting on Biological Mathematics and its application in Kyoto 2010.

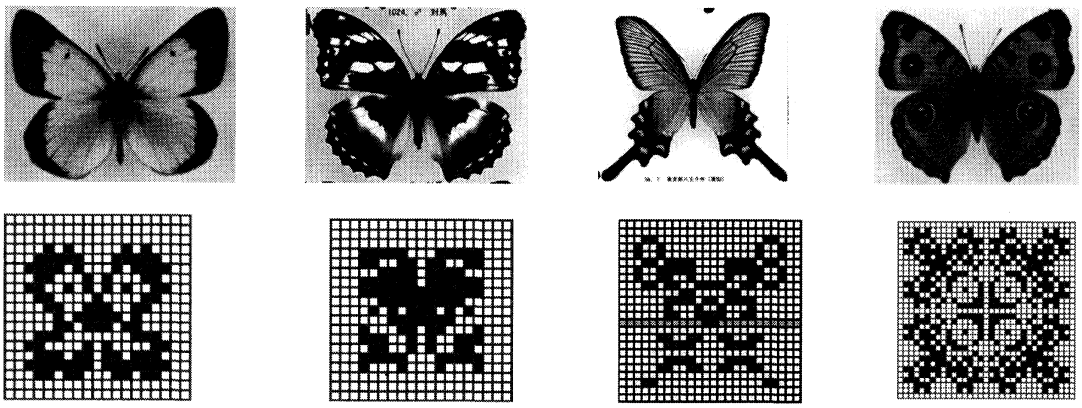


Figure 6. Generation of butterfly design patterns

3. MATHEMATICAL PROBLEMS ON DISCRETE LAPLACIANS

Here we recall some basic notations on the dynamical systems and state problems on mathematical structures ([1], [3], [4])). At first we notice that we consider dynamical systems under the periodic condition. Namely, choosing an integer M, which is called the size, we consider the following periodic functions:

$$F(M) = \{f \in F \mid f(x + mM, y + nM) = f(x, y), (n, m \in Z)\}$$

Choosing a neighborhood, we can define a discrete Laplacian and we can consider its iteration dynamical system under the periodic condition. Hence we can understand that we consider the dynamical system on the torus with the size $M \times M$. The torus is denoted by $T(M)$. We prepare several basic notations:

(1) A dynamical system is called to have a fixed point, if $\exists k \in N$ s.t. $f_n = f_k$ ($\forall n \geq k$)

(2) A dynamical system is called periodic if $\exists n, \exists l$ such that $f_n = f_{n+kl}$ $\forall k \in N$

If n=0, it is called periodic simply and if $n \neq 0$, it is called asymptotically periodic, respectively.

(3) A set $\{Q_1, Q_2, \ldots, Q_k\} \in T(M)$ is called source of a dynamical system, if

$$f_n(Q_j)=1 \quad \forall n \in N, j = 1, 2, \dots, k \ .$$

CONJECTURE ([1], [3], [4])

We can propose conjectures for some cases:

(1) In the case $M = 2^p$ and a single source, we have the following results:

(α) If a neighborhood is even, we see that the dynamical system has a fixed point and its fixed point can be attained after 2^p (or 2^{p-1}) steps for Moore, Hexagonal, and Neumann (resp. Sierpinski) neighborhoods.

(β) If the neighborhood is odd, we see that the dynamical system is periodic, its period depends on neighborhoods.

(2) In the case where M is odd, the dynamical system is periodic in the case of a single source with Neumann neighborhood. We give the table of periods for small values of M (see Table 1).

M	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
Period	1	5	6	13	30	62	29	30	511	126	2046	2045	1021	16384	61
Recurrence point	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 1. Periods for small odd size

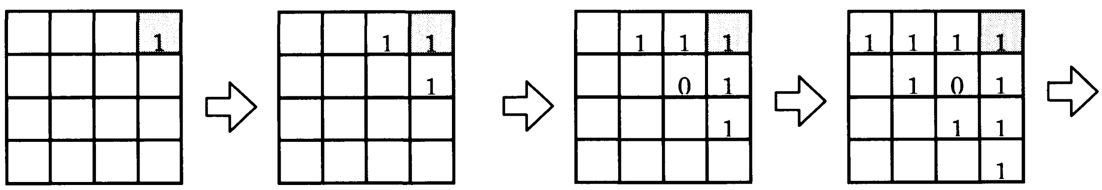
4. FIXED POINT THEOREMS FOR EVEN NEIGHBORHOODS

Theorem1

Let $M = 2^p$ and let the neighborhood be of Sierpinski type (resp. Neumann type). Then, in case of a one point source, the dynamical system reaches at a fixed point after 2^p (resp. 2^{p-1}) steps.

Proof of the assertion for Sierpinski neighborhood.

We give a proof of Theorem I in the case p=2. We put the source at the corner which is denoted by yellow color. Making iterations, we have the assertion. By this short observation, we know that our assertion may hold for any p.



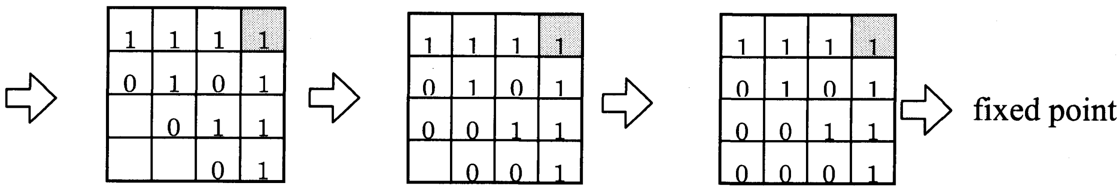


Figure 7. Proof for Sierpinski dynamical system (p=4)

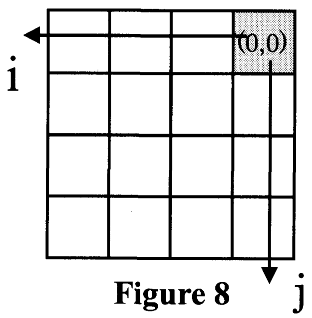


Figure 8

We introduce a coordinate system such that the origin (0,0) is located at the upper right corner of the rectangle as shown in Figure 8. We put $N_n = \{(i,j): i + j = n, i,j \geq 0\}$, $M_n = \bigcup_{k=0}^{n} N_k$. We can prove the following proposition which proves the assertion of Theorem I in the case of general 2^p.

PROPOSITION For the dynamical system $\{f_n\}$ with the source at the origin, we have:

(1) $f_n(i,j) = f_n(j,i)$ on N_n,

(2) $f_n(n,0) = f_n(0,n) = 1 \quad (0 \leq n \leq M-1)$

(3) $f_n(i,j) = f_n(i-1,j) + f_n(i,j-1)$ on N_n $(mod\ 2)$

(4) The image of the Laplacian is invariant on M_n: $f_{n+1|M_n} = f_n$.

Remark 1.

By this proposition we see the following facts:

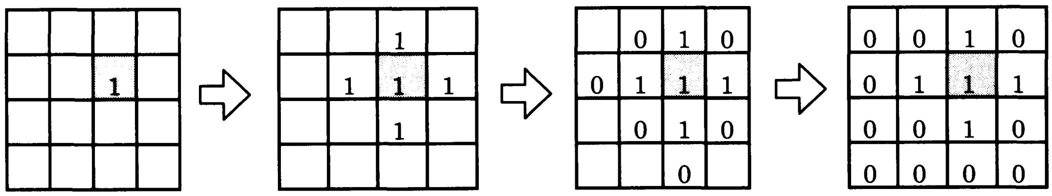
(i) We see that the Pascal triangle mod 2 appears in the upper triangle part.

(ii) At step 2^p, every element in the diagonal is 1.

(iii) Then the lower triangle is filled by 0 (see Figure 7).

Proof of the assertion for Neumann neighborhood.

At first we give a proof of Theorem I in the case p=2 (see Figures 9,10). We put the source at the position which is denoted by yellow color. Iterating in the following manner we recognize the idea of the proof.



⇨ Fixed point

Figure 9. Proof for Neumann dynamical system (p=2)

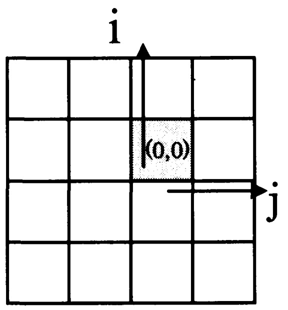


Figure 10.

We introduce a coordinate system such that the origin (0,0) is at the center as in the Figure 10. We put $N_n = \{(i,j): |i+j| = n\}$ and $M_n = \cup_{k=0}^{n} N_k$. We can prove the following proposition which proves the assertion of Theorem I for the case of general 2^p:

PROPOSITION

Let $\{f_n\}$ be a dynamical system with a source at the origin. Then the following properties hold for an integer n of the form $n = 2^q (0 \leq q \leq p-1)$:

(1) The Laplacian Δ maps the support of M_n to that of M_{n+1}.

(2) The Laplacian preserves the function f_n on M_n ,i.e., $f_{n+1}|_{M_n}=f_n$

(3) $f_n(i,j) = 1: i+j = \pm n$ and $f_n(i,j)=0$ the outside of M_n,

(4) $f_{n+1}(\pm(n+1,0) = 1, f_{n+1}(0,\pm(n+1)) = 1$ on N_{n+1}.

Remark 2.

The condition (2) in the proposition is called monotonic increasing condition. We can prove the same assertion under this condition.

Remark 3.

In [5], the concept of the linearization of the discrete Laplacian and the iteration dynamical system is considered. The comparison theorems on the fixed point and periodicity between these operators and the original discrete Laplacian are given and interesting similarities can be observed.

Remark 4.

In [8], the concept of characteristic polynomials which are introduced by Wolfram in the case of 1-dimensional lattice can be transported to the plane lattice and it is proved that the period of Neumann neighborhood is identical with that of Moore neighborhood.

5. PERIODICITY THEOREM FOR ODD NEIGHBORHOODS

We can prove the following theorem:

THEOREM II

In the case $M = 2^p$, with a neighborhood of Peano type or Roof type and a one point source, the dynamical system is periodic and its period is 2^p (resp. 2^{p-1}).

Proof. We give the proof for the Peano neighborhood. The proofs for the other cases are similar. We give an idea of the proof in the case p=2 (see Figure 11). Putting the source at the corner which is denoted by yellow color, and making iterations, we have the assertion. .

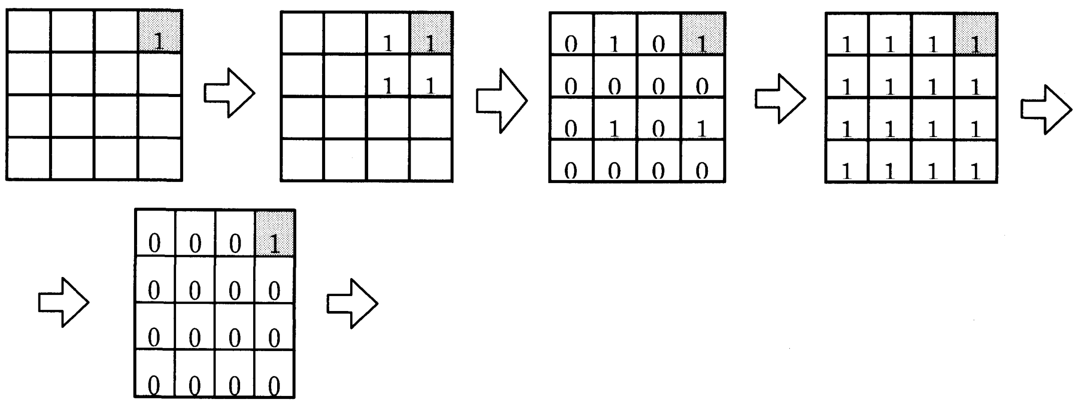


Figure 11. Proof for Peano neighborhood(p=2)

We choose a local coordinate system as in the case of Sierpinski neighborhood. Then we can prove the assertion of Theorem II by the following proposition:

PROPOSITION

We consider the square domain with the origin at the right upper corner of the size $2^q(=m)$ and call it $T(m)$. We consider a dynamical system $\{f_n\}$ with a source at the origin. Then we can prove the following assertions for an integer $m = 2^q, (0 \leq q \leq p-1$:

(1) The Laplacian Δ maps the support of M_m to that of M_{m+1}

(2) $f_m(m = 2^k)$ is harmonic on $T(m)$,i.e., $\Delta f_{m-1}|_{T(m)} = 0$

Remark 5. The condition (2) in the proposition is called harmonic monotonic increasing condition. We can prove the same assertion under this condition.

6. BINOMIAL AND TRINOMIAL SEQUENCES OF ITERATION DYNAMICAL SYSTEMS OF DISCRETE LAPLACIANS

For an iteration dynamical system we have configurations which constitute distributions of 0 and 1. We will obtain binomial and trinomial sequences for the dynamical systems. In fact, we have the usual mod 2 binomial sequence from the Sierpinski dynamical system. We list up some interesting sequences without proofs. The exact mathematical treatments will be given in another paper. At first we introduce the following three "triangles" of integers and associated mod 2 reductions which are called their "sequences":

(1) Wolfram triangle and Wolfram sequence.

Wolfram triangle

```
1 0 0
1 1 0
1 1 1 0
1 2 1 1 0
1 2 3 1 1 0
1 4 3 4 1 1 0
1 4 8 4 5 1 1 0
1 9 8 13 5 6 1 1 0
```

$\Longrightarrow$

Wolfram sequence

```
1 0
1 1 0
1 1 1 0
1 0 1 1 0
1 0 1 1 1 0
1 0 1 0 1 1 0
1 0 0 0 1 1 1 0
1 1 0 1 1 0 1 1 0
```

(2) Pascal triangle and binomial sequence.

Pascal triangle

```
          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5  10  10   5   1
```

$\Longrightarrow$

Binominal sequence

```
          1
        1   1
      1   0   1
    1   1   1   1
  1   0   0   0   1
1   1   0   0   0   1
```

(3) Trinomial triangle and trinomial sequence.

Trinomial sequence

1
1 1 1
1 2 3 2 1
1 3 6 7 6 3 1
1 4 10 16 10 4 1
1 5 15 30 36 30 15 1

Moore trinomial sequence

1
1 1 1
1 0 1 0 1
1 1 0 1 0 1 1
1 0 0 0 1 0 0 0 1
1 1 0 0 1 1 1 0 0 1 1
1 0 0 1 1 0 1 0 1 1 0 0 1

(1) Wolfram #-90 dynamical system.

We begin with the configurations obtained for the case of 1-dimensional dynamics. We notice that the dynamical system is identical with the Wolfram #90 dynamical system. Then we have the configuration which is given in the Figure 12 (left side). Hence we have the Wolfram sequence.

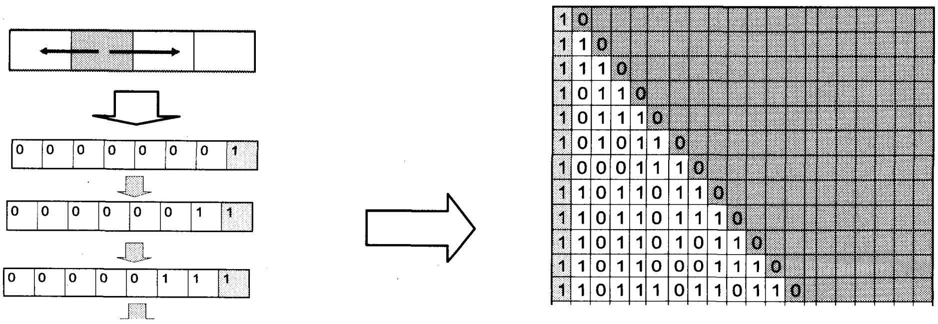


Figure 12. Binomial sequence of Wolfram dynamical system

(2) The Sierpinski dynamical system.

Next we will be concerned with binomial sequences of the Sierpinski dynamical system. By this scheme we see that we can obtain the binominal sequence in the diagonal direction.

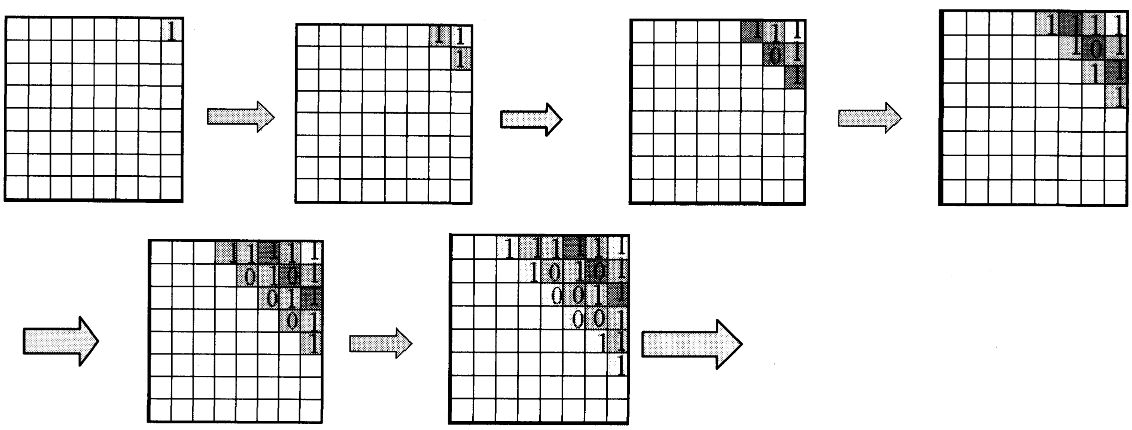


Figure 13. Binomial sequence of Sierpinski dynamical system

(3) Neumann dynamical system.

We will deal now with binomial sequences of the Neumann dynamical system. Then, we can observe Wolfram sequences in the horizontal and vertical directions (Figure 14, upper sequence). In an analogous manner we can expect to obtain the binominal sequence in the diagonal direction (Figure 14, lower sequence):

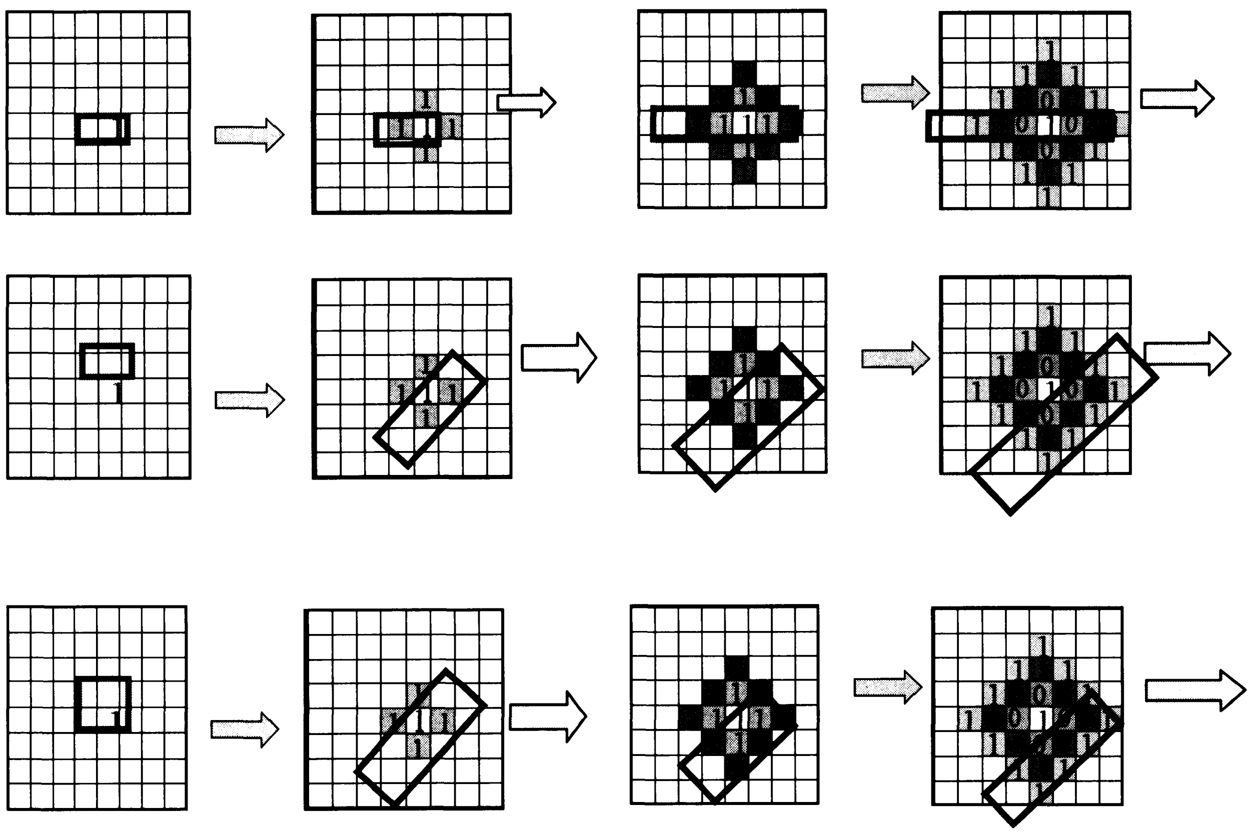
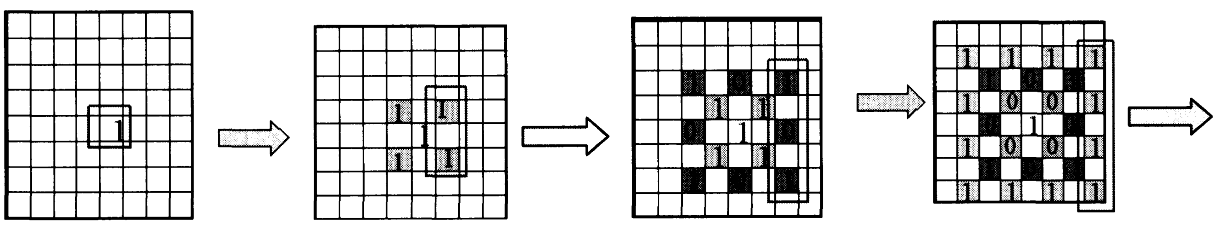
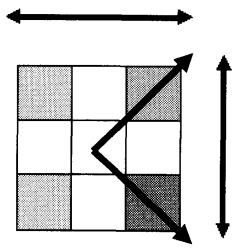


Figure 14 Binomial sequence of Neumann dynamical system

(4) The binomial sequence of Diagonal Neumann dynamical system.

The next example is concerned with binomial sequences of the diagonal Neumann dynamical system.



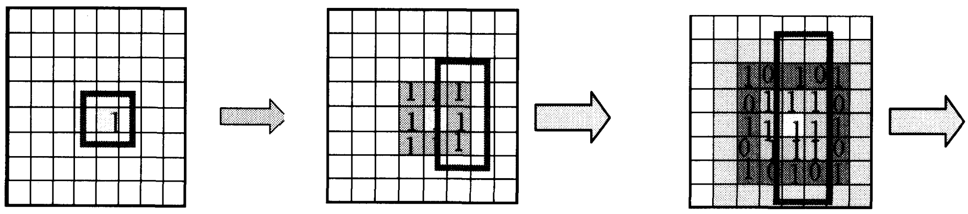


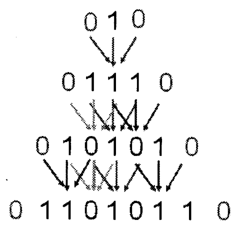
In the vertical and the horizontal directions, we obtain the usual binomial sequences. In the diagonal direction, Wolfram sequence appears

Figure 15. Binomial sequence of diagonal dynamical system

(5)The Moore dynamical system.

The last example leads to the binomial sequences of the dynamical system of the Moore neighborhood.





In this case we obtain the sequences of new type which might be called trinominal sequence. The generation rule is given in scheme on the left side.

Figure 16. Trinomial sequence of Moore dynamical system

Remark 6.

The trinomial sequence appears in the time evolution of the dynamical system of Wolfram #150 dynamical system. This is communicated by Dr. Kawaharaguchi (Hokkaido University) privately. We will hunt the sequences in the table of Wolfram dynamical systems and make the collections in more details.

7. SEQUENCE ANALYSIS ON ITERATION DYNAMICAL SYSTEMS OF DISCRETE LAPLACIANS

In this section we suggest a new method for the analysis of dynamical systems by use of binomial and trinomial sequences. We introduce a concept of "sequence analysis" of the dynamical system. Here we mean the analysis by that we describe the original dynamical systems choosing sequences appearing in the dynamical systems. Here we propose several problems concerning sequence analysis.

Comparison problem.

We consider the comparison of two dynamical systems. One of the interesting questions is for instance, whether the dynamical systems are equivalent. For example, we can ask the following: Choosing some sequences of the both systems and comparing them, can we derive the equivalence of the systems? Here we compare the Wolfram dynamical system with the Neumann system. We notice the following theorem due to X. Li([8]):

Theorem (Comparison Theorem [8]).

The access speeds to the fixed points of Neumann dynamical system is identical with that of Wolfram dynamical system.

This theorem can be treated by use of the comparison of both associated binomial sequences. In a similar manner, we may expect to prove a similar theorem between Neumann dynamical system and diagonal Neumann dynamical system. We may expect to prove the following assertion:

ASSERTION.

The fixed point theorem holds for the Neumann neighborhood , if and only it holds for the diagonal Neumann neighborhood.

Strategy to the proof of the assertion:.

The result may follow from the fact that the access speeds to the fixed points of the both Laplacians of diagonal Neumann neighborhoods and Neumann neighborhood are identical after rotation by the angle $\pi/4$.

Generation problem.

We have obtained only three kinds of sequences, the Pascal sequence, the Wolfram sequence and the trinomial sequence. Can we find other sequences for more complicated neighborhoods, for example Hexagonal neighborhoods? In general we may propose the following problem:

PROBLEM.

Can we characterize the dynamical system in terms of the sequences associated to the dynamical systems?

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