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Application of normal form theory to a chemotaxis system in one dimension

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1 Introduction

Reduction principles are helpful to study local bifurcation structures and solutions to a nonlinear equation around the degenerate points. These methods bring us a simpler problem which dominates the local dynamics of the original problem. Moreover, the normal form can be derived with symmetries which are inherited from the original problem.

Let us consider the following problem in real Banach space $\mathcal{B}$:

$$u_t = \mathcal{L}u + \mathcal{N}(u),\ (t,x) \in (0,\infty) \times I, \tag{1.1}$$

$$u_x(t,0) = u_x(t,L) = 0, t \in (0,\infty). \tag{1.2}$$

Here, $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, and I denotes an interval $(0,L) \subset \mathbb{R}$, $u(t,x) \in \mathbb{R}^n$ is an unknown function. We assume that the linear operator $\mathcal{L}$ is a generator of an analytic semigroup, the nonlinear map $\mathcal{N}$ is smooth enough, and $\mathcal{N}(0) = \mathcal{N}'(0) = 0$ holds. We also assume that $\mathcal{L}$ has a finite number of eigenvalues with zero real parts, and the other eigenvalues are bound away from the imaginary axis. Let us define actions γ_θ and κ on $\mathcal{B}$:

$$\gamma_\theta u(x) = u(x+\theta),\ \kappa u(x) = u(-x).$$

We suppose that $\mathcal{F}(u) := \mathcal{L}u + \mathcal{N}(u)$ itself is invariant under the actions γ_θ and κ:

$$\gamma_\theta \mathcal{F}(u) \equiv \mathcal{F}(\gamma_\theta u),^\forall \theta \in \mathbb{R},\ \kappa\mathcal{F}(u) \equiv \mathcal{F}(\kappa u).$$

In this case, a smooth solution $u(t,x)$ of (1.1)-(1.2) can be extended to the solution of (1.1) with periodic boundary conditions on $(0,2L)$. Indeed, suppose $u(t,x)$ is a solution of (1.1)-(1.2), and define a function $\hat{u}(t,x)$ as follows:

$$\hat{u}(t,x) = \begin{cases} u(t,x) & x \in [0,L], \\ u(t,2L-x) & x \in (L,2L]. \end{cases}$$

It is easy to see that function $\hat{u}(t,x)$ is a solution of (1.1) imposed periodic boundary conditions with period $2L$. On the contrary, if a $2L$-periodic function $u(t,x)$ satisfying $u(t,x) = u(t,-x)$ solves (1.1), then it solves (1.1)-(1.2). Thus, the problem (1.1)-(1.2) is equivalent to the problem (1.1) on $(0,2L)$ imposed periodic boundary conditions with the restriction $u(x) = u(-x)$. Let us consider the Fourier expansion of u:

$$u(t,x) = \sum_{j\in\mathbb{Z}} u_j(t) e^{ij\pi x/L}. \tag{1.3}$$

Here i denotes $\sqrt{-1}$. Since $u(t,x)$ is real valued, the Fourier coefficients must satisfy: $u_j = \bar{u}_j, j \in \mathbb{Z}$. Here, $\bar{\cdot}$ denotes the complex conjugate. Using Fourier expansion (1.3), partial differential equation (1.1) can be transformed to a system of ordinary differential equations for the Fourier amplitudes in infinite dimension. Then, applying the center manifold theory, it can be reduced to a system of ordinary differential equations for finite number of critical modes (for instance, see [5, 8, 19, 21]). Moreover, it is invariant under the O(2) action. That is, if z_{j_k}, $k = 1,\ldots,N$ are critical modes of (1.1) in Fourier space, and system

$$\dot{z}_{j_k} = f_{j_k}(z_{j_1},\ldots,z_{j_N}) \tag{1.4}$$

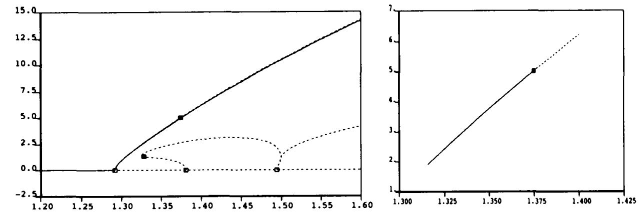

Figure 1: The bifurcation diagram of chemotaxis system(1.7). Coefficients are $a = 1/4, d = 16, f = 1, g = p = 1/16$ and $L = 28.56$. The horizontal and vertical axes correspond to b and $||(u-1, v-f/g)||_{L^2}$, respectively. Right figure is closeup around a Hopf-bifurcation point on a branch of 2-mode stationary solution.

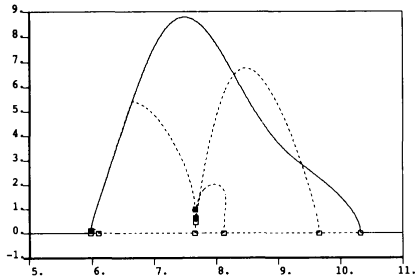

Figure 2: The bifurcation diagram of chemotaxis system (1.7). Coefficients are $a = 1/4, d = 16. f = 1, g = p = 1/16$. L and b are parameterized as $L = \pi/(\sqrt{6}/24 + \cos\theta)$, $b = 65/48 + 2\sqrt{2}\sin\theta/25$. The horizontal and vertical axes correspond to θ and $||(u-1, v-f/g)||_{L^2}$, respectively.

is a reduced system of (1.1)-(1.2) on the center manifolds, then reduced system (1.4) can be transformed into equivalent system which is O(2) invariant, or (1.4) itself is invariant with respect to O(2) action:

$$e^{i\theta\ell_{j_k}} f_{j_k}(z_{j_1}, z_{j_2}, \dots, z_{j_N}) \equiv f_{j_k}(e^{i\theta j_1} z_{j_1}, e^{i\theta j_2} z_{j_2}, \dots, e^{i\theta j_N} z_{j_N}), \tag{1.5}$$

$$\bar{f}_{j_k}(z_{j_1}, z_{j_2}, \dots, z_{j_N}) \equiv f_{j_k}(\bar{z}_{j_1}, \dots, \bar{z}_{j_N}). \tag{1.6}$$

Moreover, to study the dynamics of problem (1.1)-(1.2) on the center manifold, it is sufficient to consider the system (1.4) on the real subspace. Let us see the simple case when $N = 1$. In this case, the normal form on the real subspace restriction of O(2) symmetry is given by the following: $\dot{z} = (\mu + Cz^2)z$, $z(t) \in \mathbb{R}$. Hence, if $C \neq 0$ then we can see that the pure mode stationary solutions bifurcate at $\mu = 0$ through pitchfork bifurcation (see also [7]).

We can consider the normal form for the several critical modes. Armbruster, Guckenheimer and Holmes [1] studied the normal form with 1 : 2 resonance. They proved the existence of limit cycles and heteroclinic orbits in the normal form (see also [9]). They applied the results in [1] to study the Kuramoto-Sivashinsky dynamics on the center-unstable manifold ([2]). In this paper, we apply the normal form theory to the chemotaxis-diffusion-growth system:

$$\begin{cases} u_t = au_{xx} - b(uv_x)_x + c(u), & (t,x) \in (0,\infty) \times I, \\ v_t = dv_{xx} + fu - gv, & (t,x) \in (0,\infty) \times I, \\ u_x(t,0) = u_x(t,L) = v_x(t,0) = v_x(t,L) = 0, & t \in (0,\infty). \end{cases} \tag{1.7}$$

The chemotaxis system (1.7) is introduced by Mimura and Tsujikawa [13] as a model for chemotactic aggregation of biological individuals, e.g. bacteria (see also [3, 4]). Here, all coefficients in (1.7) are positive, and we consider the chemotaxis system with logistic source: $c(u) := pu(1-u)$, $p > 0$.

Kurata, et al. [10] studied the nonlinear solutions and bifurcation structures of (1.7) numerically. They obtained oscillatory solutions, and Hopf-bifurcation points on branches of pure mode stationary solutions. Here, we recall the bifurcation diagrams of (1.7) by AUTO ([6]) (see fig. 1 and fig. 2). We note that the black squares in these diagrams correspond to Hopf-bifurcation points. In fig. 2 (especially, see the right), we can see a Hopf-bifurcation point on a branch of 2-mode stationary solution. If we only consider the two modal interaction, we can not analyze this bifurcation phenomenon. Therefore, to analyze the dynamics around the pure mode stationary solutions, we need to consider three or more modal interaction. We can find the studies to several modal interaction in nonlinear partial differential equations. Lorenz [12] and Saltzman [18] derived the second order truncated system of ordinary differential equations corresponding to the dynamics of several Fourier modes from equations governing convection in a liquid. Their studies clarified the existence of oscillatory dynamics and attractors. Smith, Moehlis and Holmes [20] studied second order normal form with $0:1:2$ resonance in $\mathbb{C}^2 \times \mathbb{R}$. They proved the existence of modulated traveling waves and heteroclinic cycles in the normal form. However, in (1.7), since the pure mode stationary solutions bifurcate through pitchfork bifurcation, it is necessary to derive the third order normal form to study the bifurcation structures (Hopf-bifurcation) around it. Therefore, in the next section, we derive the third order normal form with $1:2:3$ resonance. In section 3, we apply the normal form theory to chemotaxis system (1.7).

2 Normal form

In this section, we consider the general normal form to problem (1.1)-(1.2) with $1:2:3$ resonance. That is, we consider the case when $N=3$, $j_1=1, j_2=2$ and $j_3=3$ in (1.4). Using (1.5) and (1.6), we get the third order truncated normal form on the real subspace restriction of O(2) symmetry:

$$\begin{cases} \dot{z}_1 = (a_1 z_1 + a_2 z_3) z_2 + (\mu_1 + a_3 z_1^2 + a_4 z_2^2 + a_5 z_3^2 + a_6 z_1 z_3) z_1 + a_7 z_2^2 z_3, \\ \dot{z}_2 = (b_1 z_1 + b_2 z_3) z_1 + (\mu_2 + b_3 z_1^2 + b_4 z_2^2 + b_5 z_3^2 + b_6 z_1 z_3) z_2, \\ \dot{z}_3 = c_1 z_1 z_2 + (\mu_3 + c_2 z_1^2 + c_3 z_2^2 + c_4 z_3^2) z_3 + c_5 z_1^3 + c_6 z_1 z_2^2. \end{cases} \tag{2.1}$$

We study the stability of the equilibriums of normal form 2.1. Especially, we are interested in the case when linearized eigenvalues of equilibriums include pure imaginary numbers.

Firstly, let us consider the second order truncated system:

$$\begin{cases} \dot{z}_1 = \mu_1 z_1 + (a_1 z_1 + a_2 z_3) z_2, \\ \dot{z}_2 = \mu_2 z_2 + (b_1 z_1 + b_2 z_3) z_1, \\ \dot{z}_3 = \mu_3 z_3 + c_1 z_1 z_2. \end{cases} \tag{2.2}$$

By solving the stationary problem (2.2) with $z_3 = \rho z_1$ for (z_1, z_2, z_3, μ_1), we have

$$\begin{cases} z_1 = \pm z_1^* := \pm\sqrt{\frac{\mu_2\mu_3\rho}{c_1(b_1+\rho b_2)}}, \\ z_2 = z_2^* := -\frac{\mu_3\rho}{c_1}, \\ z_3 = \pm z_3^* := \pm \rho z_1^* \end{cases} \tag{2.3}$$

and the bifurcation point is

$$\mu_1 = \mu_1^* := \frac{\mu_3\rho(a_1 + \rho a_2)}{c_1}.$$

Here, ρ is a real parameter. Then, we have linearized matrix $\mathcal{M}_1$ around equilibriums (2.3) as follows:

$$\mathcal{M}_1 = \begin{pmatrix} \mu_1^* + a_1 z_2^* & \pm(a_1 z_1^* + a_2 z_3^*) & a_2 z_2^* \\ \pm(2b_1 z_1^* + b_2 z_3^*) & \mu_2 & \pm b_2 z_1^* \\ c_1 z_2^* & \pm c_1 z_1^* & \mu_3 \end{pmatrix}.$$

We get the characteristic polynomial of the matrix $\mathcal{M}_1$: $\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3$, where $s_1 = -\operatorname{tr}\mathcal{M}_1$, $s_3 = \det\mathcal{M}_1$ and

$$s_2 = \frac{\mu_1\mu_3}{c_1}\left\{c_1 + a_2\rho^2 - \frac{\rho}{(b_1+b_2\rho)}(b_2c_1 + 2a_1b_1 + a_2b_2\rho^2 + a_1b_2\rho + 2a_2b_1\rho)\right\}.$$

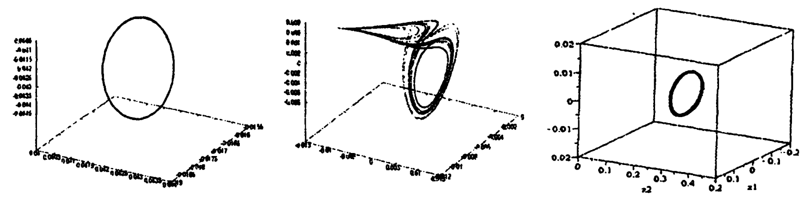

Figure 3: Numerical results to the system (2.2) and (2.1). [Left]: Coefficients are $\mu_2 = -0.099, a_1 = -2, a_2 = -6, b_1 = -3, b_2 = -2, c_1 = 4$ and $\mu_1 = \mu_1^*$, $\mu_3 = -0.068968$ in (2.2). [Middle] : $\mu_1 = 0.01, \mu_2 = -0.01, \mu_3 = -0.05, a_1 = -2, a_2 = -6, b_1 = -3, b_2 = -2, c_1 = 4$ in (2.2). [Right] : Periodic orbit around $(0, z_{2*}, 0)$ in (2.1). Coefficients are $\mu_2 = 0.04, a_1 = 2, a_2 = -4, a_3 = -2, a_4 = a_5 = a_6 = -1, a_7 = 1$, $b_1 = -3, b_2 = -2, b_3 = -1, b_4 = -7, b_5 = b_6 = -1$, $c_1 = 4, c_2 = -1, c_3 = 1, c_4 = -3, c_5 = 1, c_6 = 1$, and $\mu_1 = (-a_1 - a_4 z_{2*}) z_{2*} + 0.01, \mu_3 = c_3 z_{2*} + 0.01$.

By the simple calculation, if

$$s_2 > 0 \text{ and } s_3 - s_1 s_2 = 0 \tag{2.4}$$

holds, then the eigenvalues of matrix $\mathcal{M}_1$ are $\pm i\sqrt{s_2}$ and $-s_1$. Thus, (2.4) and $\mu_2\mu_3\rho c_1(b_1 + \rho b_2) > 0$ are necessary conditions to Hopf-bifurcation in (2.2) (see [11] for Hopf-bifurcation theorem).

Secondly, let us study the linearized stability of the other equilibriums of (2.1): $(z_1, z_2, z_3) = (0, \pm\sqrt{-\mu_2/b_4}, 0)$. The linearized matrix is

$$\mathcal{M}_2 = \begin{pmatrix} \mu_1 \pm a_1 z_{2*} + a_4 z_{2*}^2 & 0 & \pm a_2 z_{2*} + a_7 z_{2*}^2 \\ 0 & -2\mu_2 & 0 \\ \pm c_1 z_{2*} + c_6 z_{2*}^2 & 0 & \mu_3 + c_3 z_{2*}^2 \end{pmatrix}.$$

Here, z_{2*} denotes $\sqrt{-\mu_2/b_4}$. It can be observed that the necessary conditions to Hopf-bifurcation are $\mu_1 = (\mp a_1 - a_4 z_{2*}) z_{2*}, \mu_3 = -c_3 z_{2*}^2$, $(c_1 \pm c_6 z_{2*})(a_2 \pm a_7 z_{2*}) < 0$ and $\mu_2 b_4 < 0$. In this case, the eigenvalues of $\mathcal{M}_2$ are $-2\mu_2$ and $\pm i\omega z_{2*}$, where $\omega^2 = -(c_1 \pm c_6 z_{2*})(a_2 \pm a_7 z_{2*})$. Similarly, the linearized eigenvalues around equilibriums $(0, 0, \pm z_{3*}) := (0, 0, \pm\sqrt{-\mu_3/c_4})$ are $-2\mu_3$ and $\pm z_{3*}\sqrt{a_2 b_2}$ at $(\mu_1, \mu_2) = (-a_5 z_{3*}^2, -b_5 z_{3*}^2)$. Thus, if $a_2 b_2 < 0$, then pure imaginary eigenvalues appear.

We show the numerical results of (2.2) and (2.1) in fig. 3. We can see that the stable limit cycles appear with suitable choice of coefficients. The left and middle figures correspond to the time periodic orbit around (z_1^*, z_2^*, z_3^*) and Lorenz attractor, respectively. The right figure correspond to the periodic orbit around a pure mode solution $(0, z_{2*}, 0)$ in (2.1).

3 Bifurcation analysis to a chemotaxis system

In this section, we study the dynamics of chemotaxis system (1.7) using normal form. The equations of (1.7) have constant stationary states $(u, v) \equiv (0, 0)$ and $(1, f/g)$. It is easy to see that the trivial solution $(0, 0)$ is unstable against the spatially uniform perturbation with any choice of parameters. Changing the variables $(u_*, v_*) = (u - 1, v - f/g)$, we have

$$\begin{cases} (u_*)_t = a(u_*)_{xx} - b(v_*)_{xx} - b\{u_*(v_*)_x\}_x - pu_*(1 + u_*), & (t, x) \in (0, \infty) \times I, \\ (v_*)_t = d(v_*)_{xx} + fu_* - gv_*, \quad (t, x) \in (0, \infty) \times I, \\ (u_*)_x(t, 0) = (u_*)_x)(t, L) = (v_*)_x(t, 0) = (v_*)_x(t, L) = 0, \; t \in (0, \infty). \end{cases} \tag{3.1}$$

We consider the bifurcation problem of the equations (3.1) around the trivial solution $(u_*, v_*) \equiv (0, 0)$. To consider the system (3.1) in Fourier space, we introduce the similar settings used in [15, 16] as follows. The functions $u_*(t, x)$ and $v_*(t, x)$ can be expanded in the Fourier series:

$$u_*(t, x) = u_0 + \sum_{\ell \in \mathbb{N}} u_\ell(t) \cos(\ell \alpha x), \; v_*(t, x) = v_0 + \sum_{\ell \in \mathbb{N}} v_\ell(t) \cos(\ell \alpha x) \tag{3.2}$$

in a function space

$$X := \left\{ \mathbf{u} \in [H^2(I)]^2 : \mathbf{u}_x(0) = \mathbf{u}_x(L) = 0, ||\mathbf{u}||_X^2 = \sum_{\ell \in \mathbb{N}_0} (1+\ell^2)^2 |\mathbf{u}_\ell|^2 < \infty \right\}.$$

Here, $\mathbf{u}$, α and $\mathbb{N}_0$ denote (u_*, v_*), π/L and $\mathbb{N} \cup \{0\}$ respectively. In addition, $\mathbf{u}_\ell$ denotes (u_ℓ, v_ℓ). In this situation, the linearized operator of (3.1) is a generator of an analytic semigroup (for instance, see [8]). We consider the space of Fourier coefficients:

$$Y := \left\{ \hat{\mathbf{u}} = \{\mathbf{u}_\ell\}_{\ell \in \mathbb{N}_0} : ||\hat{\mathbf{u}}||_Y^2 = \sum_{\ell \in \mathbb{N}_0} (1+\ell^2)^2 |\mathbf{u}_\ell|^2 < \infty \right\},$$

which is equivalent to X by the map $\mathcal{R} : X \to Y$, where

$$\mathcal{R}(\mathbf{u}) = \left\{ \frac{2}{L} \int_0^L \mathbf{u} \, \cos(\ell \alpha x) \, dx \right\}_{\ell \in \mathbb{N}_0}.$$

For a given $m \in \mathbb{N}_0$, we define the projection $\mathcal{P}_m : Y \to Y$ as follows:

$$\mathcal{P}_m(\{\mathbf{u}_\ell\}_{\ell \in \mathbb{N}_0}) = \delta_\ell^m \mathbf{u}_\ell,$$

where

$$\delta_\ell^m = \begin{cases} 1 \ (\ell = m), \\ 0 \ (\ell \neq m). \end{cases}$$

Using (3.2), we obtain the equivalent dynamics on Y to the system (1.7) as an infinite dimensional system of ordinary differential equations:

$$^t\dot{\mathbf{u}}_m = M_m \, {}^t\mathbf{u}_m - \begin{pmatrix} \mathcal{N}_m \\ 0 \end{pmatrix}, m \in \mathbb{N}_0. \tag{3.3}$$

Here,

$$M_m = \begin{pmatrix} -A_m & bm^2\alpha^2 \\ f & -B_m \end{pmatrix},$$
$$A_m = am^2\alpha^2 + p, B_m = dm^2\alpha^2 + g$$

and

$$\begin{aligned} \mathcal{N}_m &= \frac{b\alpha^2}{2} \left(\sum_{\substack{|m_1 - m_2| = m \\ m_1, m_2 \in \mathbb{N}_0}} (m_1 - m_2) m_2 u_{m_1} v_{m_2} - \sum_{\substack{m_1 + m_2 = m \\ m_1, m_2 \in \mathbb{N}_0}} (m_1 + m_2) m_2 u_{m_1} v_{m_2} \right) \\ &\quad + \frac{p}{2} \left(\sum_{\substack{|m_1 - m_2| = m \\ m_1, m_2 \in \mathbb{N}_0}} u_{m_1} u_{m_2} + \sum_{\substack{m_1 + m_2 = m \\ m_1, m_2 \in \mathbb{N}_0}} u_{m_1} u_{m_2} \right). \end{aligned}$$

It is obvious that the matrix M_m has a 0 eigenvalue if and only if $\det M_m = 0$, i.e., $b = A_m B_m / (f \alpha^2 m^2)$. For a given positive constant ε_0, let $\mathcal{S} \subset \mathbb{N}_0$ be a set of natural numbers which satisfies $\det M_m = O(\varepsilon_0)$. It is convenient to introduce new variables $\tilde{\mathbf{u}}_m := (\tilde{u}_m, \tilde{v}_m), m \in \mathcal{S}$ as follows:

$$^t\tilde{\mathbf{u}}_m = \Phi_m^{-1} \, {}^t\mathbf{u}_m, \tag{3.4}$$

$$\Phi_m = \begin{pmatrix} B_m & -A_m \\ f & f \end{pmatrix}. \tag{3.5}$$

Then, the infinite dimensional system (3.3) is represented as follows:

$$\begin{cases} {}^t\dot{\tilde{\mathbf{u}}}_m = \begin{pmatrix} \mu_m & 0 \\ 0 & -A_m - B_m + O(\varepsilon_0) \end{pmatrix} {}^t\tilde{\mathbf{u}}_m - \Phi_m^{-1} \begin{pmatrix} \mathcal{N}_m \\ 0 \end{pmatrix}, m \in \mathcal{S}, \\ {}^t\dot{\mathbf{u}}_m = M_m \, {}^t\mathbf{u}_m - \begin{pmatrix} \mathcal{N}_m \\ 0 \end{pmatrix}, m \in \mathbb{N}_0 \setminus \mathcal{S}. \end{cases} \tag{3.6}$$

Here,

$$\mu_m := \frac{1}{2}\{ - A_m - B_m + \sqrt{(A_m - B_m)^2 + 4fb\alpha^2 m^2}\}.$$

Let j and k be natural numbers. We can see that if

$$\alpha = \alpha^{j,k} := \frac{1}{\sqrt{jk}}\left(\frac{cg}{ad}\right)^{\frac{1}{4}}, \; b = b^{j,k} := \left.\frac{A_j B_j}{f\alpha^2 j^2}\right|_{\alpha=\alpha^{j,k}}$$

then $\mu_j = \mu_k = 0$ holds.

Before we study the three modal interaction, let us see the 1 : 2 modal interaction in the system (1.7). We note again that the normal form with 1 : 2 resonance is studied in [1, 9]. Using the explicit form of coefficients of the reduced system, we can check the conditions for Hopf-bifurcation and existence of heteroclinic orbits (see [1]) to the chemotaxis system (1.7). Set $(\alpha, b) = (\alpha^{1,2}, b^{1,2})$, and let $\mathcal{W}^c_{loc}$ be a center manifold in a neighborhood of $\mathbb{R}^2 \times Y$. Then the following holds.

Theorem 1 *The dynamics of (3.6) on $\mathcal{W}^c_{loc}$ is topologically equivalent to the dynamics of the following system:*

$$\begin{cases} \dot{z}_1 = e_{10} z_1 z_2 + (\mu_1 + e_{11} z_1^2 + e_{12} z_2^2) z_1, \\ \dot{z}_2 = e_{20} z_1^2 + (\mu_2 + e_{21} z_1^2 + e_{22} z_2^2) z_2, \end{cases} \tag{3.7}$$

where $z_j(t) = \tilde{u}_j(t)$, $j = 1, 2$. Moreover, the coefficients e_{jk}, $j = 1, 2$, $k = 0, 1, 2$ are given in Appendix B.

We note that Theorem 1 can be proved by the same arguments shown in [2, 15, 16]. For reader's convenience, we give the proof in Appendix A.

Let us focus attention on Hopf-bifurcation phenomena around mixed mode equilibrium: $(z_1, z_2) = (z_{1_*}, z_{2_*})$, $z_{j_*} \neq 0$ of (3.7). Let M be a linearized matrix around (z_1^*, z_2^*). Then, matrix M has pure imaginary eigenvalues if and only if $\det M > 0$ and $\operatorname{tr} M = 0$. By solving the stationary problem of (3.7) with $z_1 = \rho z_2$ and $\operatorname{tr} M = 0$ for (z_1, z_2, μ_1, μ_2), we have $z_2 = z_{2*} := \frac{\rho^2 e_{20}}{2(e_{11}+e_{22})}$, $z_1 = z_{1*} := \rho z_{2*}$, and the bifurcation point is

$$\mu_1 = \mu_{1*} := -\frac{\rho^2 e_{20}\{2e_{10}(\rho^2 e_{11}+e_{22})+\rho^2 e_{20}(\rho^2 e_{11}+e_{12})\}}{4(\rho^2 e_{11}+e_{22})^2},$$
$$\mu_2 = \mu_{2*} := -\frac{\rho^4 e_{20}^2(2\rho^2 e_{11}+3e_{22}+\rho^2 e_{21})}{4(\rho^2 e_{11}+e_{22})^2}.$$

Thus, the condition: $\det M > 0$ at $(\mu_{1*}, \mu_{2*}, z_{1*}, z_{2*})$ is a necessary condition for Hopf-bifurcation around (z_{1*}, z_{2*}). Let us choose the coefficients in (1.7) as follows:

$$a = 1/16, d = 1, f = 1, g = 32, p = 2.$$

Then, using explicit form of e_{jk} shown in Appendix B, we have

$$e_{10} = -\tfrac{3072}{17},\; e_{11} = -\tfrac{10991616}{4913},\; e_{12} = -\tfrac{1046707104}{24565},$$

$$e_{20} = \tfrac{768}{17},\; e_{21} = -\tfrac{468314112}{24565},\; e_{22} = -\tfrac{811008}{17}.$$

Choosing $\rho = 1$, the Hopf-bifurcation point is

$$\mu_{1*} = -\frac{4628915091}{63798558760} \approx -0.0725, \; \mu_{2*} = \frac{271863864}{7974819845} \approx 0.03409.$$

The left of fig. 4 shows the limit cycle corresponding to the standing wave solution to (1.7).

Let us consider the three modal interaction. For given natural numbers j ,k, and ℓ, $(\ell \neq j, k)$, we can compute

$$2\mu_\ell|_{(\alpha,b)=(\alpha^{j,k},b^{j,k})} = -(K_2\sqrt{gp}+g+p)+\sqrt{(K_2\sqrt{gp}+g+p)^2-K_1 gp}\ ,$$

where

$$K_1 = (j^2-\ell^2)\Big(1-\frac{\ell^2}{k^2}\Big)\frac{1}{j^2},\ K_2 = \frac{\ell^2(a+d)}{jk\sqrt{ad}}.$$

Therefore, if we take the parameters $(\alpha, b) = (\alpha^{1,3}, b^{1,3})$ and $g, p \backsim \varepsilon_1 > 0$, it follows that

$$\mu_1 = \mu_3 = 0\,, 0 < \mu_2 = O(\varepsilon_1) \text{ and } \mu_m < 0\,, m \in \mathbb{N}_0 \setminus \{1,2,3\}.$$

Thus, by taking g and p small, we can apply the center-unstable manifold theorem to analyze the three modal interaction in the chemotaxis system (1.7) (for instance, see [5, 8]). Let $\mathcal{W}^{cu}_{loc}$ be a center-unstable manifold of (3.6) in a neighborhood of $\mathbb{R}^3 \times Y$. Set $(\alpha, b) = (\alpha^{1,3}, b^{1,3})$, and taking g and p small, we have the following theorem.

Theorem 2 *For given positive constants a, d and f, there exist constants $a_j\,, j = 1,\ldots,7, b_j\,, j = 1,\ldots,7$ and $c_j\,, j = 1,\ldots,6$ such that the dynamics of (3.6) on $\mathcal{W}^{cu}_{loc}$ is topologically equivalent to the dynamics of the system (2.1) by replacing $z_j(t)$ with $\tilde{u}_j(t)$, $j = 1, 2, 3$.*

Proof is given in Appendix A, and explicit forms of the coefficients of normal form are shown in Appendix B. Let us study the stability of nontrivial stationary solutions to (1.7) using the normal form. We take the coefficients as follows:

$$a = 1/4,\ d = 16,\ f = 1,\ g = p = 1/16. \tag{3.8}$$

Then, we have $(\alpha^{1,3}, b^{1,3}) = (275/192, \sqrt{6}/24)$, and the coefficients of second order system (2.2) are

$$\mu_2 = \mu_2^* := \tfrac{-77+\sqrt{6169}}{192} \approx 0.008,\ a_1 = -\tfrac{22055}{520704},\ a_2 = -\tfrac{15375}{57856},$$

$$b_1 = \tfrac{143}{64512},\ b_2 = -\tfrac{475}{10752},\ c_1 = \tfrac{9515}{972288}.$$

Choosing $\rho = -1$, we obtain $z_1^* \approx 0.1656, z_2^* \approx -0.1584$ and $z_3^* = -z_1^*$. Moreover, solving $s_3 - s_1 s_2 = 0$ for μ_3, we get

$$\mu_3 = \frac{8047537176063 - 104513469819\sqrt{6169}}{104029331289901} \approx -0.0016.$$

In this setting, it follows that

$$s_2 = \frac{62870415469646796 - 800301205350108\sqrt{6169}}{311359788550673693} \approx 0.3974 \times 10^{-4} > 0.$$

Consequently, the necessary condition to Hopf-bifurcation holds around the triple mixed mode solutions $(\pm z_1^*, z_2^*, \pm z_3^*)$.

In the same case, the coefficients of third order terms are given by the following:

$$a_3 = \tfrac{217653469}{30005047296},\ a_4 = -\tfrac{227113319143975}{7104343523733504},\ a_5 = -\tfrac{163625}{173568},$$

$$a_6 = \tfrac{1322893}{34728064},\ a_7 = -\tfrac{91622502625}{159572865024},$$

$$b_3 = \tfrac{181109972065}{440092076691456},\ b_4 = -\tfrac{6580325}{51093504},\ b_5 = -\tfrac{6956901071875}{14139065892864},$$

$$b_6 = -\tfrac{9156412925}{46130208768},$$

$$c_2 = -\tfrac{330935}{6806016},\ c_3 = -\tfrac{172891631284375}{1220458460479488},\ c_4 = -\tfrac{8715625}{41484288},$$

$$c_5 = -\tfrac{17943211}{4863465600},\ c_6 = -\tfrac{12108977485}{446945236992}.$$

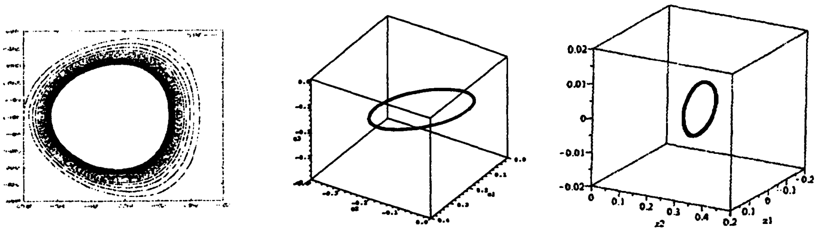

Figure 4: [Left]: Time periodic orbit in (3.7). The horizontal and vertical axes correspond to z_1 and z_2, respectively. [MIddle]: The unstable time periodic orbit in the reduced system (2nd order truncated). Parameters are shown in (3.8). Figure shows the orbit $(z_1(t), z_2(t), z_3(t))$, $t \in [-3000, -1000]$. [Right]: The time periodic orbit around a pure mode solution in third order reduced system. $t \in [5000, 6000]$.

Since $a_2 b_2 > 0$ holds, Hopf-bifurcation around the pure mode equilibrium $(0, 0, \pm z_{3*})$ does not occur. On the other hand, we can see that the necessary condition to the Hopf-bifurcation around pure mode solution $(z_1, z_2, z_3) = (0, \sqrt{-\mu_2^*/b_4}, 0) \approx (0, 0.2498, 0)$ holds:

$$(c_1 + c_6 z_{2*})(a_2 + a_7 z_{2*}) \approx -0.1235 \times 10^{-2} < 0,$$

and bifurcation point is

$$\mu_1 = (a_1 - a_4 z_{2*}) z_{2*} \approx 0.011, \mu_3 = -c_3 z_{2*}^2 \approx 0.0088$$

(see Figure 4). We note that the equilibrium $(0, -\sqrt{-\mu_2^*/b_4}, 0) \approx (0, -0.2498, 0)$ also has pure imaginary eigenvalues at a point

$$\mu_1 = (-a_1 - a_4 z_{2*}) z_{2*} \approx -0.010, \mu_3 = -c_3 z_{2*}^2 \approx 0.0088.$$

However, in this case, μ_4 is positive. That is, in this parameter value, it is away from the region that the reduced system is dominant. In fact, Hopf-bifurcation point lies on only one of a branch of 2-mode stationary solution (see Figure 1).

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Appendix A: Proof of Theorems

We give a proof of Theorem 1 and Theorem 2. We recall the dynamics of (1.7) on Fourier space Y (3.6):

$$\begin{cases} {}^t\dot{\tilde{\mathbf{u}}}_m = \begin{pmatrix} \mu_m & 0 \\ 0 & -A_m - B_m + O(\varepsilon_0) \end{pmatrix} {}^t\tilde{\mathbf{u}}_m + \Phi_m^{-1} \begin{pmatrix} \mathcal{N}_m \\ 0 \end{pmatrix}, m \in \mathcal{S}, \\ {}^t\dot{\mathbf{u}}_m = M_m\, {}^t\mathbf{u}_m - \begin{pmatrix} \mathcal{N}_m \\ 0 \end{pmatrix}, m \in \mathbb{N}_0 \setminus \mathcal{S}. \end{cases}$$

To prove theorem 1, we define $\mathcal{S} := \{1, 2\}$. It should be noted that if we define $\mathcal{S} := \{1, 2.3\}$, then theorem 2 can be proved similarly by considering the dynamics on the center unstable manifold $\mathcal{W}_{loc}^{cu}$. Let us define the projection $\mathcal{P}$ to critical modes by

$$\mathcal{P} = \sum_{m \in \mathcal{S}} \mathbf{e}_{\mathbf{x}} \cdot \Phi_m^{-1} \mathcal{P}_m,$$

and define the projection $\mathcal{Q} := I - \mathcal{P}$. Here, $\mathbf{e_x}$ denotes $(1,0)$, and Φ_m is defined in Section 2. We consider the extended system (3.6) with the trivial equations $\dot{\mu}_m = 0, m \in \mathcal{S}$. It follows that the center space of this extended flow is spanned by $(\mu, \tilde{u})$, where $\mu := (\mu_1, \mu_2)$ and $\tilde{u} := (\tilde{u}_1, \tilde{u}_2)$. Therefore, the center manifold theory tells us that there exists a neighborhood $\mathcal{U}$ of $\mathbb{R}^2 \times Y \equiv \mathbb{R}^4 \times \mathcal{Q}Y$ with radius s:

$$\mathcal{U} := \{(\mu, \tilde{u}) : |\mu| + |\tilde{u}| + ||\hat{\mathbf{u}}||_Y < s\} \tag{3.9}$$

such that there exists a smooth invariant manifold $\mathcal{W}^c_{loc}$ of (3.1) contained in $\mathcal{U}$. Here, $\hat{\mathbf{u}}$ denotes $Q(\breve{\mathbf{u}})$ for $\breve{\mathbf{u}} \in Y$. Moreover, there exists a smooth map $\mathcal{H} : \mathbb{R}^4 \to \mathcal{Q}Y$ satisfying

$$\frac{\partial \mathcal{H}}{\partial J}(0) = 0, \text{ for } J = \mu_m, \tilde{u}_m, \ m \in \mathcal{S}$$

by which $\mathcal{W}^c_{loc}$ is represented as $\mathcal{W}^c_{loc} = \{(\mu, \tilde{u}, \hat{\mathbf{u}}) : \hat{\mathbf{u}} = \mathcal{H}(\mu, \tilde{u})\}$. Furthermore, we define

$$(h^u_m(\mu, \tilde{u}), h^v_m(\mu, \tilde{u})) = \mathcal{P}_m(\mathcal{H}(\mu, \tilde{u})), \quad \text{for } m \in \mathbb{N}_0 \setminus \mathcal{S}, \tag{3.10}$$
$$\tilde{h}^v_m(\mu, \tilde{u}) = \mathbf{e_y} \cdot \Phi_m^{-1} \mathcal{P}_m(\breve{\mathbf{u}}), \text{ for } m \in \mathcal{S}. \tag{3.11}$$

It is necessary to calculate the quadratic approximation of center manifolds to obtain the cubic normal form. We present the following lemma.

Lemma 1 *Let $m_c \in \mathcal{S}$ and $m_* \in \mathbb{N}_0 \setminus \mathcal{S}$. The quadratic approximation of the map $h^u_{m_*}$, $h^v_{m_*}$ and $\tilde{h}^v_{m_c}$ (which are characterized in (3.10) and (3.11)) are given by the graph of the functions:*

$$u_{m_*} = h^u_{m_*}(\tilde{u}_1, \tilde{u}_2), \ v_{m_*} = h^v_{m_*}(\tilde{u}_1, \tilde{u}_2), \ \tilde{v}_{m_c} = \tilde{h}^v_{m_c}(\tilde{u}_1, \tilde{u}_2),$$

and each which are approximated as follows:

$$\begin{pmatrix} h^u_{m_*}(\tilde{u}_1, \tilde{u}_2) \\ h^v_{m_*}(\tilde{u}_1, \tilde{u}_2) \end{pmatrix} = M^{-1}_{m_*} \begin{pmatrix} \mathcal{N}_{m_*} \\ 0 \end{pmatrix}, \tag{3.12}$$
$$\tilde{h}^v_{m_c}(\tilde{u}_1, \tilde{u}_2) = \frac{\mathcal{N}_{m_*}}{(A_{m_c} + B_{m_c})^2}. \tag{3.13}$$

Proof. The center manifold theory states that for given pair of integers $m_* \notin \mathcal{S}$, u_{m_*} and u_{m_*} are characterized by the map

$$\begin{pmatrix} u_{m_*} \\ v_{m_*} \end{pmatrix} = \begin{pmatrix} h^u_{m_*}(\tilde{u}_1, \tilde{u}_2) \\ h^v_{m_*}(\tilde{u}_1, \tilde{u}_2) \end{pmatrix}.$$

Differentiating with respect to t, we have

$$\begin{pmatrix} \sum_{m_c \in \mathcal{S}} \frac{\partial h^u_{m_*}}{\partial \tilde{u}_{m_c}} \dot{\tilde{u}}_{m_c} \\ \sum_{m_c \in \mathcal{S}} \frac{\partial h^v_{m_*}}{\partial \tilde{u}_{m_c}} \dot{\tilde{u}}_{m_c} \end{pmatrix} = M_{m_*} \begin{pmatrix} h^u_{m_*} \\ h^v_{m_*} \end{pmatrix} + \begin{pmatrix} \mathcal{N}_{m_*} \\ 0 \end{pmatrix}. \tag{3.14}$$

By the center manifold theory, for sufficiently small $\delta > 0$, if $|\tilde{u}_{m_c}| < O(\delta)$, then it holds that $|h^u_{m_*}| < O(\delta^2)$ and $|h^v_{m_*}| < O(\delta^2)$. Furthermore, we already know that $\dot{\tilde{u}}_{m_c} = \mu_{m_c}\tilde{u}_{m_c} + O(\delta^2)$, and we can take μ_{m_c} small. Therefore, the left hand side of (3.14) is $O(\delta^3)$ by taking $\mu_{m_c} \simeq \delta^2$. Finally, since the matrix M_{m_*} is regular, we obtain the quadratic approximations of $h^u_{m_*}$ and $h^v_{m_*}$ as shown in (3.12). The approximation (3.13) can be obtained similarly.

□

Proof of Theorems. Let us consider the equations for $\tilde{u}_m$, $m \in \mathcal{S}$ with the equation $\dot{\mu}_m = 0$. Using the approximations (3.12) and (3.13), we can compute the nonlinear terms up to $O(\delta^3)$ as follows.

$$\begin{aligned}
\dot{\tilde{u}}_m &= \mu_m \tilde{u}_m - \frac{b\alpha^2}{2(A_m+B_m)} \\
&\quad \times \left(\sum_{\substack{|m_1-m_2|=m \\ m_1,m_2\in S}} (m_1-m_2)m_2 u_{m_1} v_{m_2} - \sum_{\substack{m_1+m_2=m \\ m_1,m_2\in S}} (m_1+m_2)m_2 u_{m_1} v_{m_2} \right) \\
&\quad - \frac{p}{2(A_m+B_m)} \left(\sum_{\substack{|m_1-m_2|=m \\ m_1,m_2\in S}} u_{m_1} u_{m_2} + \sum_{\substack{m_1+m_2=m \\ m_1,m_2\in S}} u_{m_1} u_{m_2} \right) \\
&\quad - \frac{b\alpha^2}{2(A_m+B_m)} \\
&\quad \times \left(\sum_{\substack{|m_1-m_2|=m \\ m_1\in S, m_2\in \mathbb{N}_0\setminus S}} (m_1-m_2)m_2 u_{m_1} h^v_{m_2} - \sum_{\substack{|m_1-m_2|=m \\ m_1\in S, m_2\in \mathbb{N}_0\setminus S}} (m_1+m_2)m_2 u_{m_1} h^v_{m_2} \right) \\
&\quad - \frac{p}{2(A_m+B_m)} \left(\sum_{\substack{|m_1-m_2|=m \\ m_1\in S, m_2\in \mathbb{N}_0\setminus S}} u_{m_1} h^u_{m_2} + \sum_{\substack{|m_1-m_2|=m \\ m_1\in S, m_2\in \mathbb{N}_0\setminus S}} u_{m_1} h^u_{m_2} \right) \\
&\quad - \frac{b\alpha^2}{2(A_m+B_m)} \\
&\quad \times \left(\sum_{\substack{|m_1-m_2|=m \\ m_1\in \mathbb{N}_0\setminus S, m_2\in S}} (m_1-m_2)m_2 h^u_{m_1} v_{m_2} - \sum_{\substack{|m_1-m_2|=m \\ m_1\in \mathbb{N}_0\setminus S, m_2\in S}} (m_1+m_2)m_2 h^u_{m_1} v_{m_2} \right) \\
&\quad - \frac{p}{2(A_m+B_m)} \left(\sum_{\substack{|m_1-m_2|=m \\ m_1\in \mathbb{N}_0\setminus S, m_2\in S}} h^u_{m_1} u_{m_2} + \sum_{\substack{|m_1-m_2|=m \\ m_1\in \mathbb{N}_0\setminus S, m_2\in S}} h^u_{m_1} u_{m_2} \right).
\end{aligned}$$

This yields the cubic normal form (3.7).

□

Appendix B: Coefficients of reduced systems

We show the explicit form of coefficients of reduced system. The coefficients of the system (3.7) are given by the following:

$$\begin{aligned}
e_{10} &= \frac{-1}{A_1+B_1}\left\{\frac{1}{2}bf\alpha^2(-2B_1+B_2)+pB_2B_1\right\}, \\
e_{20} &= \frac{B_1}{A_2+B_2}\left(\alpha^2 bf - \frac{p}{2}B_1\right), \\
e_{11} &= \frac{-e_{20}}{(A_1+B_1)^2}\left\{\frac{1}{2}bf\alpha^2(2B_1+A_2)+pA_2B_1\right\} - \frac{b\alpha^2 f - pB_1}{A_1+B_1}B_1^2, \\
e_{12} &= \frac{-e_{10}}{(A_1+B_1)^2}\left\{\frac{1}{2}bf\alpha^2(2A_1+B_2)-pA_1B_1\right\} \\
&\quad - \frac{G_1}{A_1+B_1}\left\{\frac{1}{2}bf\alpha^2(-3B_2+2B_3)+pB_2B_3\right\} - \frac{b\alpha^2 f - pB_1}{A_1+B_1}B_2^2, \\
e_{21} &= \frac{-e_{10}}{(A_2+B_2)(A_1+B_1)}\left\{bf\alpha^2(B_1-A_1)+pA_1B_1\right\} \\
&\quad - \frac{G_1}{A_2+B_2}\left\{bf\alpha^2(B_3-3B_1)+pB_1B_3\right\} - \frac{4b\alpha^2 f - pB_2}{A_2+B_2}B_1^2, \\
e_{22} &= \frac{-G_2}{(A_2+B_2)}\left\{2bf\alpha^2(B_4-2B_2)+pB_2B_4\right\} - \frac{4b\alpha^2 f - pB_2}{A_2+B_2}B_2^2,
\end{aligned}$$

where

$$
\begin{aligned}
G_1 &= \frac{1}{A_3B_3 - 9bf\alpha^2}\Big\{\frac{3}{2}bf\alpha^2(B_2+2B_1) - pB_1B_2\Big\},\\
G_2 &= \frac{B_2}{A_4B_4 - 16bf\alpha^2}\Big(4bf\alpha^2 - \frac{p}{2}B_2^2\Big).
\end{aligned}
$$

Here, We omitted the $\cdot^{1,2}$ for the sake of simplicity (i.e., $(\alpha, b) = (\alpha^{1,2}, b^{1,2})$).

We also show the coefficients of the reduced system in Theorem 2. We will also omit $\cdot^{1,3}$ for α and b. The coefficients are given as follows:

$$
\begin{aligned}
a_1 &= e_{10},\\
a_2 &= \frac{-1}{A_1+B_1}\Big\{-\frac{1}{2}bf\alpha^2(3B_2-2B_3) + pB_1B_3\Big\},\\
a_3 &= R_1H_2 - \frac{b\alpha^2 f - pB_1}{A_1+B_1}B_1^2,\\
a_4 &= R_1H_1^{1,2} - \frac{H_3}{A_1+B_1}\Big\{-\frac{1}{2}bf\alpha^2(3B_2+2A_3) - pA_3B_2\Big\} - \frac{b\alpha^2 f - pB_1}{A_1+B_1}B_2^2,\\
a_5 &= R_2H_4^{1,3} - \frac{b\alpha^2 f - pB_1}{A_1+B_1}B_3^2,\\
a_6 &= \frac{-H_2}{A_1+B_1}\Big\{\frac{1}{2}bf\alpha^2(3A_2+2B_3) - pA_2B_3\Big\},\\
a_7 &= R_1H_1^{2,3} + R_2H_4^{2,2},
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= e_{20},\\
b_2 &= \frac{-1}{A_2+B_2}\Big\{bf\alpha^2(-3B_1+B_3) + pB_1B_3\Big\},\\
b_3 &= R_3H_1^{1,2} + \frac{H_3}{A_2+B_2}\{bf\alpha^2(3B_1+A_3) + pB_1A_3\} - \frac{4b\alpha^2 f - pB_2}{A_2+B_2}B_1^2,\\
b_4 &= R_5H_4^{2,2} - \frac{4b\alpha^2 f - pB_2}{A_2+B_2}B_2^2,\\
b_5 &= R_4H_1^{2,3} - \frac{H_5}{A_2+B_2}\Big\{\frac{1}{2}bf\alpha^2(-5B_3+3B_5) + pB_3B_5\Big\} - \frac{4b\alpha^2 f - pB_2}{A_2+B_2}B_3^2,\\
b_6 &= R_3H_1^{2,3} + R_4H_1^{1,2} + R_5H_4^{1,3},
\end{aligned}
$$

$$
\begin{aligned}
c_1 &= \frac{1}{A_3+B_3}\Big\{\frac{1}{2}bf\alpha^2(6B_1+3B_2) - pB_1B_2\Big\},\\
c_2 &= R_7H_4^{1,3} - \frac{9b\alpha^2 f - pB_3}{A_3+B_3}B_1^2,\\
c_3 &= R_6H_1^{2,3} - \frac{H_5}{A_3+B_3}\Big\{\frac{1}{2}bf\alpha^2(-15B_2+6B_5) + pB_2B_5\Big\} - \frac{9b\alpha^2 f - pB_3}{A_3+B_3}B_2^2,\\
c_4 &= \frac{-H_6}{A_3+B_3}\Big\{\frac{1}{2}bf\alpha^2(-18B_3+9B_6) + pB_3B_6\Big\} - \frac{9b\alpha^2 f - pB_3}{A_3+B_3}B_3^2,\\
c_5 &= \frac{H_2}{A_3+B_3}\Big\{\frac{1}{2}bf\alpha^2(6B_1+3B_2) - pB_1B_2\Big\},\\
c_6 &= R_6H_1^{1,2} + R_7H_4^{2,2},
\end{aligned}
$$

where

$$
\begin{aligned}
H_1^{j,k} &= \frac{1}{(A_1+B_1)^2}\Big\{\frac{1}{2}bf\alpha^2(-kB_j+jB_k)+pB_jB_k\Big\},\\
H_2 &= \frac{B_1}{(A_2+B_2)^2}\Big(-bf\alpha^2+\frac{p}{2}B_1\Big),\\
H_3 &= \frac{-1}{(A_3+B_3)^2}\Big\{\frac{1}{2}bf\alpha^2(6B_1+3B_2)-pB_1B_2\Big\},\\
H_4^{1,3} &= \frac{1}{\det M_4}\Big\{bf\alpha^2(6B_1+2B_3)-pB_1B_3\Big\},\\
H_4^{2,2} &= \frac{B_2}{\det M_4}\Big\{4bf\alpha^2-\frac{p}{2}B_2\Big\},\\
H_5 &= \frac{1}{\det M_5}\Big\{\frac{1}{2}bf\alpha^2(15B_2+10B_3)-pB_3B_5\Big\},\\
H_6 &= \frac{B_3}{\det M_6}\Big(9bf\alpha^2-\frac{p}{2}B_3\Big),
\end{aligned}
$$

$$
\begin{aligned}
R_1 &= \frac{-1}{A_1+B_1}\Big\{\frac{1}{2}bf\alpha^2(2A_1+B_2)-pA_1B_2\Big\},\\
R_2 &= \frac{-1}{A_1+B_1}\Big\{\frac{1}{2}bf\alpha^2(-4B_3+3B_4)+PB_3B_4\Big\},\\
R_3 &= \frac{1}{A_2+B_2}\Big\{bf\alpha^2(B_1-A_1)+pA_1B_1\},\\
R_4 &= \frac{-1}{A_2+B_2}\Big\{bf\alpha^2(3A_1+B_3)-pA_1B_3\Big\},\\
R_5 &= \frac{-1}{A_2+B_2}\Big\{bf\alpha^2(-4B_2+2B_4)+pB_2B_4\Big\},\\
R_6 &= \frac{1}{A_3+B_3}\Big\{\frac{1}{2}bf\alpha^2(-6A_1+3B_2)+pA_1B_2\Big\},\\
R_7 &= \frac{-1}{A_3+B_3}\Big\{\frac{1}{2}bf\alpha^2(12B_1-3B_4)+pB_1B_4\Big\}.
\end{aligned}
$$

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