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# HAUSDORFF DIMENSION OF SATURATED SETS FOR DIFFEOMORPHISMS WITH DOMINATED SPLITTING

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ABSTRACT. Let $f$ be a diffeomorphism of a manifold having a compact invariant set with dominated splitting. Some lower and upper bounds on the Hausdorff dimension of saturated sets are given in terms of the Lyapunov exponents and the entropy.

Let $M$ be a compact metric space and $f : M \to M$ be a continuous map of $M$. Given a continuous function $\varphi : M \to \mathbb{R}$, we consider the set

$$K_\alpha = \left\{ x \in M \ : \ \lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) = \alpha \right\}$$

for $\alpha \in \mathbb{R}$, which is called the level set of the Birkhoff averages of $\varphi$. The Birkhoff ergodic theorem tells us that when $\mu$ is an ergodic $f$-invariant probability measure, the $\mu$-measure of $K_\alpha$ is equal to 1 for $\alpha = \int \varphi d\mu$, and 0 for any other $\alpha \in \mathbb{R}$. However, the multifractal analysis assures that for several important dynamical systems $f$ and generic $\varphi$, there exist uncountably many values of $\alpha$ such that the 'sizes' of $K_\alpha$ are not small in terms of the dimension and of the entropy (see, e.g., [4], [8], [12], [13]). For example, if $M$ is a repeller of an expanding, $C^{1+\delta}$-conformal mixing map $f$, then the following equation holds:

$$\dim_H(K_\alpha) = \max\left\{ \frac{h_\mu(f)}{\int \log \|D_x f\| d\mu} \ : \ \int \varphi d\mu = \alpha \right\} \tag{0.1}$$

where $h_\mu(f)$ is the measure theoretical entropy of $\mu$, $\|D_x f\|$ is the operator norm of the differential $D_x f$, and $\dim_H$ is the Hausdorff dimension ([8]).

In this paper we deal with the Hausdorff dimension of some saturated sets for diffeomorphisms having a compact invariant set with dominated splitting. Our purpose here is twofold: Firstly, we consider saturated sets instead of the level sets of the Birkhoff averages. Secondly, we get rid of the assumptions of the uniform hyperbolicity (or expansion) and the conformality of $f$. From now on we consider a $C^2$ diffeomorphism $f : M \to M$ of a compact smooth Riemannian manifold $M$.

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To investigate time averages along orbits we introduce the *empirical measure* of order $n$ of $x \in M$, which is defined by

$$\delta_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)},$$

where $\delta_y$ is the Dirac measure at $y \in M$. And we denote as $V(x)$ the limit-point set of the sequence $\{\delta_n(x)\}_{n\in\mathbb{N}}$ in the collection $\mathcal{M}(M)$ of all probability measures on $M$. A subset $D \subset M$ is said to be *saturated* if $x \in M$ satisfies $V(x) = V(y)$ for some $y \in D$, then $x \in D$. We remark that the level set $K_\alpha$ is saturated. This is a simple consequence of the fact that $x \in K_\alpha$ if and only if every $\mu \in V(x)$ satisfies $\int \varphi d\mu = \alpha$. In this paper we shall consider more general saturated sets defined by

$$G(K) = \{x \in M : V(x) \subset K\}$$

for some closed set $K$ in the collection $\mathcal{M}_f(M)$ of all $f$-invariant probability measures on $M$. With this notation we can write

$$K_\alpha = G\left(\left\{\mu \in \mathcal{M}_f(M) : \int \varphi d\mu = \alpha\right\}\right).$$

In the case when $K = \{\mu\}$, we write simply $G_\mu$.

A compact $f$-invariant set $\Lambda$ is said to be an *isolating set* if there is an open neighborhood $U \supset \Lambda$ (called an *isolating block*) such that $\Lambda = \cap_{n\in\mathbb{Z}} f^n(U)$. We say that an isolating set $\Lambda$ admits a *dominated splitting* if there exist a continuous $Df$-invariant splitting $E^{cs} \oplus E^{cu}$ of the tangent bundle of $M$ over $\Lambda$ and a constant $0 < \lambda < 1$ satisfying

$$\|Df|E^{cs}(x)\| \cdot \|(Df|E^{cu}(x))^{-1}\| \leq \lambda,$$

for all $x \in \Lambda$ and $n \geq 1$. Moreover, to avoid complication we impose additional assumptions as follows:

(1) if $x \in U$ and $f(x) \notin U$ then $f^n(x) \notin U$ for every $n \in \mathbb{N}$, and
(2) the dimension of $E^{cs}(x)$ does not depend on $x \in \Lambda$.

Hereafter, the dimension of $E^{cs}$ will be denoted by $d^s$. The domination condition of the splitting is a weaker form of the uniform hyperbolicity, and its statistical properties were intensively studied in several papers (cf. [1], [2], [5], [7], [14]). On the other hand, to the best of our knowledge, the multifractal analysis has not been studied under the domination condition.

In the present paper we consider a new class of invariant measures instead of hyperbolic measures. Let $\mathcal{M}_f(\Lambda)$ be the set of all $f$-invariant probability measures on an isolating set $\Lambda$ with dominated splitting. For $\mu \in \mathcal{M}_f(\Lambda)$, by the Kingman sub-additive ergodic theorem [10], the following limits exist for $\mu$-almost every $x \in M$:

$$\chi_1(x) = \lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n|E^{cu}\|,$$
$$\chi_c(x) = \lim_{n\to\infty} \frac{1}{n} \log \|D_{f^n x} f^{-n}|E^{cu}\|^{-1},$$

$$\chi_s(x) = \lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n | E^{cs}\|.$$

Using these characteristics we define a subset $\mathcal{H}_f(\Lambda)$ of $\mathcal{M}_f(\Lambda)$ as follows:

$$\mathcal{H}_f(\Lambda) = \{\mu \in \mathcal{M}_f(\Lambda) \ : \ \chi_c(\mu) > 0 > \chi_s(\mu)\}.$$

Here we set

$$\chi_\sigma(\mu) = \int_M \chi_\sigma(x) d\mu(x) \ (\sigma = 1, c, s).$$

Then we have the following:

**Theorem 0.1.** *$\mathcal{H}_f(\Lambda)$ is open in $\mathcal{M}_f(\Lambda)$.*

To state our main theorem, we use the notion of topological entropy of non-compact sets which was defined by Bowen ([6]). Recently Pfister and Sullivan [12] showed that

$$\sup\{h_{\mathrm{top}}(f, G_\mu) : \mu \in K\} \le h_{\mathrm{top}}(f, G(K)) \le \sup\{h_\mu(f) : \mu \in K\},$$

where $h_{\mathrm{top}}(f, Z)$ is the topological entropy of $Z \subset M$. Our main theorem gives lower and upper bounds on the Hausdorff dimension of $G(K)$ as follows:

**Theorem 0.2.** *Let $f\colon M \to M$ be a $C^2$ diffeomorphism exhibiting an isolating set $\Lambda$ with a dominated splitting which satisfies the conditions* (1) *and* (2). *If $K$ is a closed subset contained in $\mathcal{H}_f(\Lambda)$ and satisfies that $G_\mu \neq \emptyset$ for some $\mu \in K$, then we have*

$$d^s + \sup_{\mu\in K} \left\{ \frac{h_{\mathrm{top}}(f, G_\mu)}{\chi_1(\mu)} \right\} \le \dim_H G(K) \le d^s + \sup_{\mu \in K} \left\{ \frac{h_\mu(f)}{\chi_c(\mu)} \right\}.$$

In the case when $K = \{\mu\}$, we can obtain an upper bound by using the topological entropy of $G_\mu$.

**Theorem 0.3.** *Let $f$ and $\Lambda$ be as in Theorem 0.2. For $\mu \in \mathcal{H}_f(\Lambda)$ with $G_\mu \neq \emptyset$, we have*

$$d^s + \frac{h_{\mathrm{top}}(f, G_\mu)}{\chi_1(\mu)} \le \dim_H G_\mu \le d^s + \frac{h_{\mathrm{top}}(f, G_\mu)}{\chi_c(\mu)}.$$

By the result of [9] we can give a sufficient condition for the equality to hold.

**Theorem 0.4.** *Let $f$ and $\Lambda$ be as in Theorem 0.2. If $K$ is a closed subset of $\mathcal{H}_f(\Lambda)$ such that for every $\mu \in K$*

(a) $\chi_1(\mu) = \chi_c(\mu)$ *and*

(b) *$\mu$ is hyperbolic and satisfies the almost transversality condition,*

*then we have*

$$\dim_H G(K) = d^s + \sup_{\mu\in K} \left\{ \frac{h_\mu(f)}{\chi_1(\mu)} \right\}.$$

Here the hyperbolicity and the almost transversality of invariant measures are defined as follows: A point $x \in \Lambda$ is said to be *Lyapunov regular* if there exist real numbers $\chi_1(x) > \chi_2(x) > \cdots > \chi_{r(x)}(x)$ and a $D_x f$-invariant decomposition $T_x M = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_{r(x)}(x)$ such that for each $i = 1, 2, \ldots, r(x)$

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \chi_i(x) \qquad (v \in E_i(x) \setminus \{0\})$$

exists, and

$$\lim_{n \to \pm\infty} \frac{1}{n} \log |\det(D_x f^n)| = \sum_{i=1}^{r(x)} \chi_i(x) \dim E_i(x).$$

By the multiplicative ergodic theorem ([11]) $\Gamma$ has full $\mu$-measure. The numbers $\chi_i(x)$ are called the *Lyapunov exponents* of $f$ at the point $x$. We call the measure $\mu$ *hyperbolic* if none of the Lyapunov exponents for $\mu$ vanish and there exist Lyapunov exponents with different signs for $\mu$-almost everywhere.

For $x \in \Gamma$, we define the *unstable* and *stable manifolds* at $x$ as

$$\mathcal{W}^u(x) = \left\{ y \in M \colon \limsup_{n \to \infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < 0 \right\},$$
$$\mathcal{W}^s(x) = \left\{ y \in M \colon \limsup_{n \to \infty} \frac{1}{n} \log d(f^{n}(x), f^{n}(y)) < 0 \right\},$$

where $d$ is the distance on $M$ induced by the Riemannian metric. Then $\mathcal{W}^u(x)$ and $\mathcal{W}^s(x)$ are injectively immersed manifolds satisfying

$$T_x \mathcal{W}^u(x) = \bigoplus_{\chi_i(x) > 0} E_i(x) \quad \text{and} \quad T_x \mathcal{W}^s(x) = \bigoplus_{\chi_i(x) < 0} E_i(x)$$

(see [3]). We say that $\mu$ satisfies the *almost transversality condition* if for $\mu \otimes \mu$-almost every pair $(x, y) \in M \times M$ there exist integers $p, q \in \mathbb{Z}$ and a point $z \in \mathcal{W}^u(f^p(x)) \cap \mathcal{W}^s(f^q(y))$ such that

$$T_z \mathcal{W}^u(f^p(x)) \oplus T_z \mathcal{W}^s(f^q(y)) = T_z M.$$

Recently, in [9] we gave some lower bound on the Hausdorff dimension of $G_\mu$.

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