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On the new family of wavelet interpolating to the Shannon wavelet

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1 Introduction

There are various types of orthogonal wavelet with order parameters, e.g., Battle-Lemarié wavelet, Daubechies wavelet, Strömberg wavelet (see [1], [2], [7] and [9]). In particular, [10] showed that the Battle-Lemarié wavelet of order n converges to the Shannon wavelet as n tends to infinity. Let us denote the low pass filter, the scaling function and the wavelet of the Battle-Lemarié by $m_n^{BL}(\xi)$, $\varphi_n^{BL}(x)$ and $\psi_n^{BL}(x)$ respectively. This family of the Battle-Lemarié wavelet interpolates from the (non smooth) Haar wavelet which has the best localization in time to the (smooth) Shannon wavelet which has the best localization in frequency. For some applications, the order parameter n enables us to control the smoothness and the proportion between the time window and the frequency window.

We shall first explain the constructions of the Haar wavelet and the Meyer wavelet. In this paper, the low pass filter of the Haar wavelet $m_1^{BL}(\xi)$ is denoted also by $m_1^H(\xi)$ and given by

$$m_1^H(\xi)\Big(\equiv m_1^{BL}(\xi)\Big)=e^{-i\xi/2}\cos\frac{\xi}{2}.$$

$m_1^H(\xi)$ is 2π-periodic due to the multiplying by $e^{-i\xi/2}$. We immediately see that $m_1^H(0)=1$ and $|m_1^H(\xi)|^2+|m_1^H(\xi+\pi)|^2=1$. Then we also get

$$\begin{aligned}\hat{\varphi}_1^H(\xi) &= \prod_{j=1}^{\infty}m_1^H(2^{-j}\xi)=\prod_{j=1}^{\infty}e^{-i\xi/2^{j+1}}\prod_{j=1}^{\infty}\cos\frac{\xi}{2^{j+1}}=e^{-i\xi/2}\frac{\sin\xi/2}{\xi/2},\\ \hat{\psi}_1^H(\xi) &= e^{i\xi/2}\overline{m_1^H\Big(\frac{\xi}{2}+\pi\Big)}\hat{\varphi}_1^H\Big(\frac{\xi}{2}\Big)=ie^{-i\xi/2}\frac{\sin^2\xi/4}{\xi/4}.\end{aligned}$$

The Fourier inverse transform directly gives the Haar scaling function $\varphi_1^H(x)$ and wavelet $\psi_1^H(x)$.

Now, let us put

$$\nu_n(\xi) = \begin{cases} 0 & \text{for } -1 \leq \xi < 0, \\ p_n(\xi) & \text{for } \quad 0 \leq \xi \leq 1, \\ 1 & \text{for } \quad \xi > 1, \end{cases} \tag{1}$$

where $p_n(\xi)$ is the n-th order polynomial satisfying $p_n(\xi) + p_n(1-\xi) \equiv 1$ and $p_n(0) = 0$. For instance,

$$p_0(\xi) = \xi, \quad p_1(\xi) = \xi^2(3 - 2\xi), \quad p_2(\xi) = \xi^3(10 - 15\xi + 6\xi^2).$$

Then, by neglecting $e^{-i\xi/2}$ and replacing $\frac{\xi}{2}$ by $\frac{\pi}{2}\nu_n\left(\frac{3}{2\pi}|\xi| - 1\right)$ in the argument of the cosine of $m_1^H(\xi)$, one will get

$$m_n^M(\xi) = \cos\left(\frac{\pi}{2}\nu_n\left(\frac{3}{2\pi}|\xi| - 1\right)\right).$$

Hence, we can construct the Meyer scaling function $\varphi_n^M(x)$ and wavelet $\psi_n^M(x)$ by

$$\hat{\varphi}_n^M(\xi) = \prod_{j=1}^{\infty} m_n^M(2^{-j}\xi) \quad \text{and} \quad \hat{\psi}_n^M(\xi) = e^{i\xi/2}\overline{m_n^M\left(\frac{\xi}{2} + \pi\right)}\hat{\varphi}_n^M\left(\frac{\xi}{2}\right),$$

which belong to C^n due to the irregularity of (1) at the points $\xi = 0, 1$. This causes polynomial decay of the Meyer wavelet.

Remark 1.1 One can find a Gevrey function p_∞ such that ν_∞ is also a Gevrey function. Then, the Meyer wavelet would have arbitrary polynomial decay. But, there does not exist an analytic function p_∞ such that ν_∞ is also an analytic function. Therefore in this construction with (1), it is impossible to have exponential decay.

2 New family of Haar type

Following [4] and [5], we shall construct a new family of the wavelet interpolating to the Shannon wavelet. Replacing $\frac{\xi}{2}$ by $\frac{\pi}{2}\sin^2\frac{\xi}{2}$ in the argument of the cosine of $m_1^H(\xi)$, we define

$$m_2^H(\xi) = \cos\left(\frac{\pi}{2}\sin^2\frac{\xi}{2}\right).$$

Here we remark that $m_2^H(\xi)$ is 2π-periodic and satisfies $m_2^H(0) = 1$ and $|m_2^H(\xi)|^2 + |m_2^H(\xi+\pi)|^2 = 1$, since

$$m_2^H(\xi+\pi) = \cos\left(\frac{\pi}{2}\cos^2\frac{\xi}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{2}\sin^2\frac{\xi}{2}\right) = \sin\left(\frac{\pi}{2}\sin^2\frac{\xi}{2}\right).$$

To construct a new wavelet family, let us consider $\Theta_n(\xi)$ given recursively by

$$\Theta_1(\xi) = \frac{\xi}{2} \quad\text{and}\quad \Theta_n(\xi) = \frac{\pi}{2}\sin^2\Theta_{n-1}(\xi) \quad\text{for}\quad n \geq 2. \tag{2}$$

Then we also define the 2π-periodic function

$$m_n^H(\xi) = \cos\Theta_n(\xi) \quad\text{for}\quad n \geq 2.$$

$m_n^H(\xi)$ satisfies $m_n^H(0) = 1$. Noting that $m_n^H(\xi+\pi) = \sin\Theta_n(\xi)$ still holds, we can obtain $|m_n^H(\xi)|^2 + |m_n^H(\xi+\pi)|^2 = 1$. We shall define $\varphi_n^H(x)$ and $\psi_n^H(x)$ by

$$\hat{\varphi}_n^H(\xi) = \prod_{j=1}^{\infty} m_n^H(2^{-j}\xi) \quad\text{and}\quad \hat{\psi}_n^H(\xi) = e^{i\xi/2}\overline{m_n^H\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}_n^H\left(\frac{\xi}{2}\right).$$

Letting $\varphi_n^{SH}(x)$ and $\psi_n^{SH}(x)$ be the Shannon scaling function and wavelet, we get the following:

Theorem 2.1 *For $2 \leq q \leq \infty$, we have*

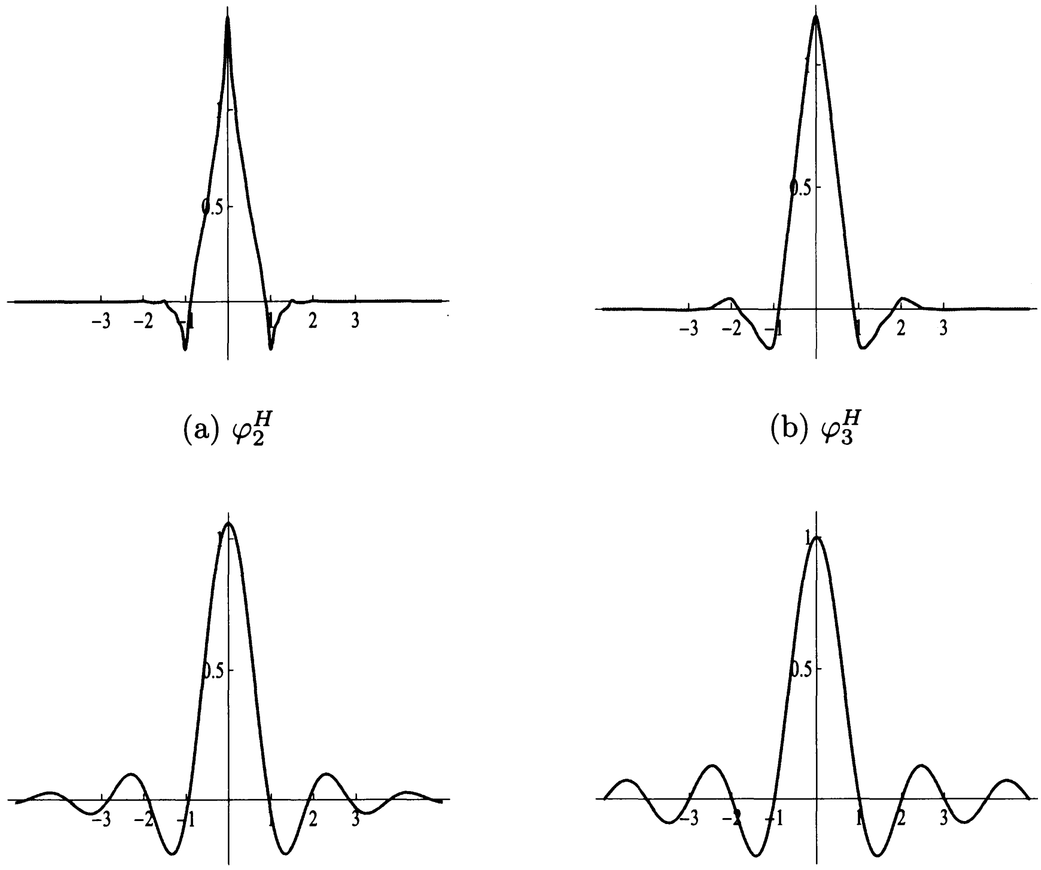
$$\lim_{n\to\infty} \left\|\varphi_n^H - \varphi_\infty^{SH}\right\|_{L^q} = 0 \quad\text{and}\quad \lim_{n\to\infty} \left\|\psi_n^H - \psi_\infty^{SH}\right\|_{L^q} = 0.$$

In [10] one can see the corresponding results for the Battle-Lemarié scaling function and wavelet. Moreover, we also know the decays and regularities of φ_n^H and ψ_n^H $(n \geq 2)$ as follows:

Theorem 2.2 *Let $n \geq 2$. The scaling function φ_n^H and wavelet ψ_n^H have exponential decays and belong to $C^{\alpha_n}(\mathbf{R}_x)$ for some $\alpha_n > 0$ increasing in the parameter n.*

In conclusion, we observe that the scaling function $\varphi(x)$ defined by $\hat{\varphi}(\xi) = \prod_{j=1}^{\infty}\cos\left(\frac{\pi}{2}\sin^2\left(\frac{\pi}{2}\sin^2\frac{\xi}{2^{j+1}}\right)\right)$ is differentiable in x and also satisfies $\Delta_\varphi\Delta_{\hat{\varphi}} = 0.669$ which is near the limit $1/2$ by the uncertainty principle, where Δ_f is defined by

$$\Delta_f := \left\{\frac{\int_{-\infty}^{\infty}(x-x_0)^2|f(x)|^2dx}{\int_{-\infty}^{\infty}|f(x)|^2dx}\right\}^{1/2} \quad\text{with}\quad x_0 := \frac{\int_{-\infty}^{\infty}x|f(x)|^2dx}{\int_{-\infty}^{\infty}|f(x)|^2dx}.$$



(a) φ_2^H (b) φ_3^H

(c) φ_5^H (d) φ_∞^{SH}

Figure 1: graphs of φ_2^H, φ_3^H, φ_5^H and φ_∞^{SH}

Table 1: Regularities of φ_n^H and ψ_n^H

n	2	3	4	5	6
α_n	0.386	1.133	2.616	5.580	11.508

Table 2: Time-bandwidth products of the scaling function and wavelet

n	2	3	4	5	6
$\Delta_{\varphi_n^H}\Delta_{\hat{\varphi}_n^H}$	0.926	**0.669**	0.772	0.947	1.177
$\Delta_{\psi_n^H}\Delta_{\hat{\psi}_n^H}$	2.603	2.136	2.500	3.069	5.393

Table 3: Time-bandwidth products of the scaling functions of Battle-Lemarié, Meyer and Daubechies

n	1	2	3	4	5
$\Delta_{\varphi_n^{BL}}\Delta_{\hat{\varphi}_n^{BL}}$	∞	0.686	0.741	0.837	0.928
$\Delta_{\varphi_n^{M}}\Delta_{\hat{\varphi}_n^{M}}$	0.810	0.875	0.949	1.012	1.065
$\Delta_{\varphi_n^{D}}\Delta_{\hat{\varphi}_n^{D}}$	∞	1.057	0.828	0.849	0.984

3 New family of Strömberg type

In the previous section, starting from the Haar wavelet and using the recursion (2) we have constructed a new family of the wavelet. A similar procedure could be done also for other orthonormal wavelets. Now the following question arises:

"*Does* **non-symmetric** *family of Strömberg type converge to the* **symmetric** *Shannon wavelet in our construction?*"

So, in this section we shall consider the wavelet starting from the Strömberg wavelet.

According to the method in [3], we first derive the low pass filter of the Strömberg wavelet. Let us put the scaling function

$$\varphi_1^{ST}(x) = \sum_{k\in\mathbf{Z}} a_k N_2(x-k),$$

where $N_2(x)$ is the B-spline defined by

$$N_2(x) = \begin{cases} x & \text{for } 0 \le x \le 1, \\ 2-x & \text{for } 1 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

By Fourier transform, we get

$$\hat{N}_2(\xi) = e^{-i\xi}\frac{\sin^2(\xi/2)}{(\xi/2)^2}.$$

Hence it follows that

$$\hat{\varphi}_1^{ST}(\xi) = \sum_{k\in\mathbf{Z}} a_k e^{-ik\xi}\left(e^{-i\xi}\frac{\sin^2(\xi/2)}{(\xi/2)^2}\right). \tag{3}$$

The orthonormality of the basis $\{\varphi(x-k) : k \in \mathbf{Z}\}$ is equivalent to $\sum_{k\in\mathbf{Z}} |\hat{\varphi}(\xi+2k\pi)|^2 = 1$. Substituting (3) into this identity yields

$$\Big|\sum_{k\in\mathbf{Z}} a_k e^{-ik\xi}\Big|^2 \Big(2\sin\frac{\xi}{2}\Big)^4 \sum_{k\in\mathbf{Z}} \frac{1}{(\xi+2k\pi)^4} = 1.$$

Therefore, we obtain

$$\Big|\sum_{k\in\mathbf{Z}} a_k e^{-ik\xi}\Big| = \Big(1 - \frac{2}{3}\sin^2\frac{\xi}{2}\Big)^{-\frac{1}{2}},$$

here we used the fact that $\sum_{k\in\mathbf{Z}} 1/(\xi+2k\pi)^4 = \big(2\sin\frac{\xi}{2}\big)^{-4}\big(1-\frac{2}{3}\sin^2\frac{\xi}{2}\big)$ (see [6]). Thus, $\sum_{k\in\mathbf{Z}} a_k e^{-ik\xi}$ can be written as

$$\sum_{k\in\mathbf{Z}} a_k e^{-ik\xi} = e^{i\theta(\xi)}\Big(1 - \frac{2}{3}\sin^2\frac{\xi}{2}\Big)^{-\frac{1}{2}},$$

with a real function $\theta(\xi)$. To find a suitable $\theta(\xi)$, we use the following lemma:

Lemma 3.1 *There exist a real function $\theta_0(\xi)$ and positive constants γ_1 and γ_2 such that*

$$e^{i\theta_0(\xi)}\Big(1 - \frac{2}{3}\sin^2\frac{\xi}{2}\Big)^{-\frac{1}{2}} = \frac{\sqrt{3}}{\gamma_1+\gamma_2 e^{i\xi}}.$$

Now we shall take $\theta(\xi) = \theta_0(\xi)$ given in Lemma 3.1, i.e.,

$$\sum_{k\in\mathbf{Z}} a_k e^{-ik\xi} = \frac{\sqrt{3}}{\gamma_1+\gamma_2 e^{i\xi}},$$

consequently, it follows that

$$\hat{\varphi}_1^{ST}(\xi) = \hat{N}_2(\xi)\frac{\sqrt{3}}{\gamma_1+\gamma_2 e^{i\xi}} = e^{-i\xi}\frac{\sin^2(\xi/2)}{(\xi/2)^2}\frac{\sqrt{3}}{\gamma_1+\gamma_2 e^{i\xi}}.$$

Thus, the low pass filter of the Strömberg wavelet is obtained by

$$m_1^{ST}(\xi) \equiv \frac{\hat{\varphi}_1^{ST}(2\xi)}{\hat{\varphi}_1^{ST}(\xi)} = e^{-i\xi}\cos^2\frac{\xi}{2}\frac{\gamma_1+\gamma_2 e^{i\xi}}{\gamma_1+\gamma_2 e^{2i\xi}},$$

where $\gamma_1 = \frac{\sqrt{3}-1}{2}$ and $\gamma_2 = \frac{\sqrt{3}+1}{2}$.

Similarly as the previous section, we shall replace $\frac{\xi}{2}$ by $\Theta_n(\xi)$ and define

$$m_n^{ST}(\xi) = e^{-2i\Theta_n(\xi)} \cos^2 \Theta_n(\xi) \frac{\gamma_1 + \gamma_2 e^{2i\Theta_n(\xi)}}{\gamma_1 + \gamma_2 e^{4i\Theta_n(\xi)}}.$$

More generally, we can prove the following:

Proposition 3.2 *Let ψ be a MRA wavelet. Suppose that a low pass filter m associated with the scaling function φ has real Fourier coefficients. Assume that $|\varphi(x)|, |\psi(x)| \leq C(1+|x|)^{-1-\varepsilon}$, then m_n defined by*

$$m_n(\xi) = m\Big(2\Theta_n(\xi)\Big)$$

satisfies $m_n(0) = 1$ and $|m_n(\xi)|^2 + |m_n(\xi+\pi)|^2 = 1$.

Hence, we can construct new families by change of starting wavelets (e.g. Franklin wavelet, Daubechies wavelet, etc.). So, the scaling function $\varphi_n^{ST}(x)$ and wavelet $\psi_n^{ST}(x)$ are defined by

$$\hat{\varphi}_n^{ST}(\xi) = \prod_{j=1}^{\infty} m_n^{ST}(2^{-j}\xi) \quad \text{and} \quad \hat{\psi}_n^{ST}(\xi) = e^{i\xi/2}\overline{m_n^{ST}\Big(\frac{\xi}{2}+\pi\Big)}\hat{\varphi}_n^{ST}\Big(\frac{\xi}{2}\Big).$$

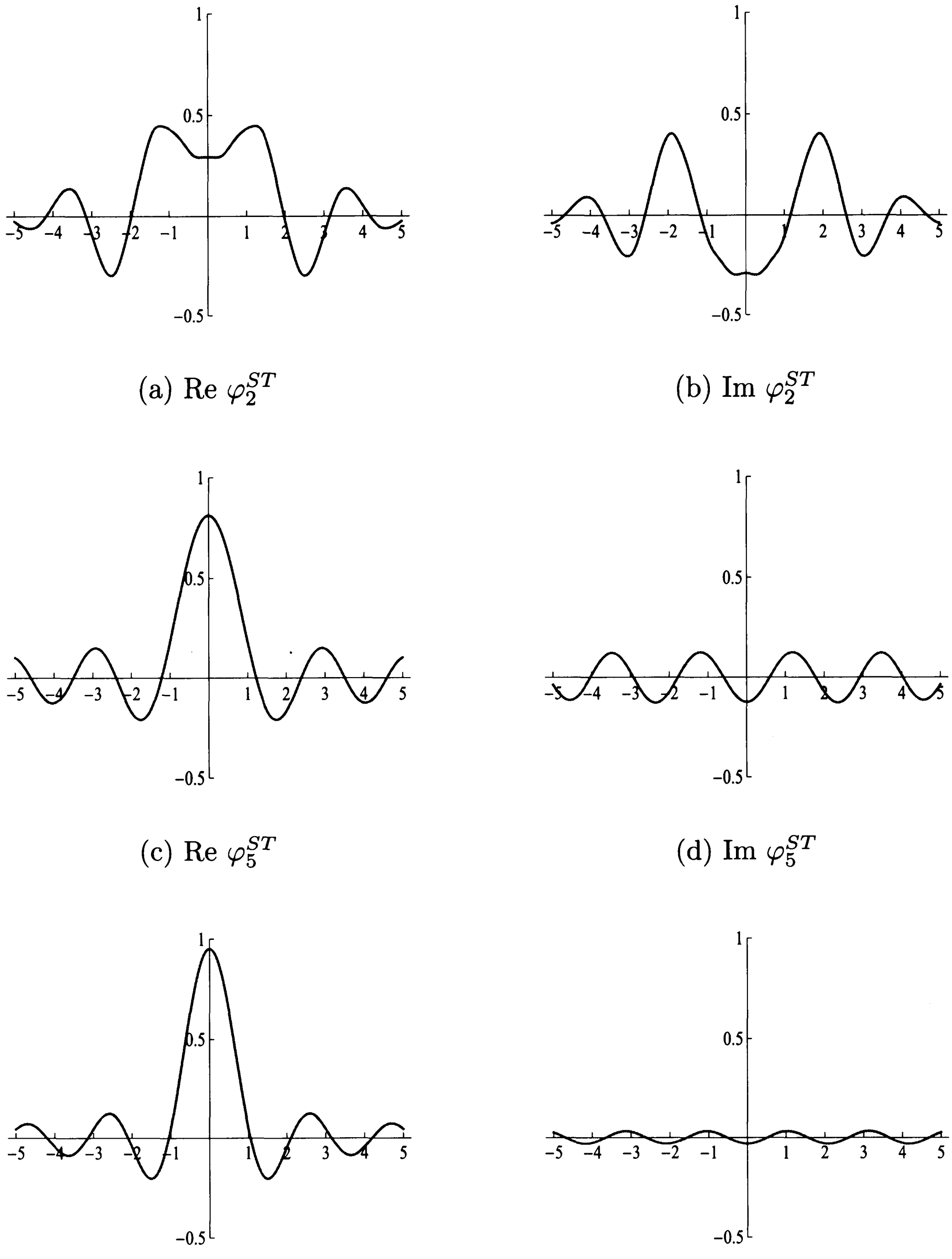
As for this family from the Strömberg wavelet, we also get the followings:

Theorem 3.3 *For $2 \leq q \leq \infty$, we have*

$$\lim_{n\to\infty} \left\|\varphi_n^{ST} - \varphi_\infty^{SH}\right\|_{L^q} = 0 \quad \text{and} \quad \lim_{n\to\infty} \left\|\psi_n^{ST} - \psi_\infty^{SH}\right\|_{L^q} = 0.$$

Theorem 3.4 *Let $n \geq 2$. The scaling function φ_n^{ST} and wavelet ψ_n^{ST} have exponential decays and belong to $C^{\beta_n}(\mathbf{R}_x)$ for some $\beta_n > 0$ increasing in the parameter n.*

In conclusion, we find that **non-symmetric** family of Strömberg type with our construction also converges to the **symmetric** Shannon wavelet. Moreover, we observe that the regularity of Strömberg type is better than Haar type.



(a) Re φ_2^{ST}

(b) Im φ_2^{ST}

(c) Re φ_5^{ST}

(d) Im φ_5^{ST}

(e) Re φ_8^{ST}

(f) Im φ_8^{ST}

Figure 2: graphs of φ_2^{ST}, φ_5^{ST}, and φ_8^{ST}

Table 4: Regularities of φ_n^{ST} and ψ_n^{ST}

n	2	3	4	5	6
β_n	1.175	2.569	5.358	10.934	22.086

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