<table>
<thead>
<tr>
<th>Title</th>
<th>Symplectic varieties and Poisson deformations (Hodge theory and algebraic geometry)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Namikawa, Yoshinori</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2011), 1745: 116-122</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2011-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/171033">http://hdl.handle.net/2433/171033</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Symplectic varieties and Poisson deformations

Yoshinori Namikawa

A symplectic variety $X$ is a normal algebraic variety (defined over $\mathbb{C}$) which admits an everywhere non-degenerate d-closed 2-form $\omega$ on the regular locus $X_{\text{reg}}$ of $X$ such that, for any resolution $f : \tilde{X} \to X$ with $f^{-1}(X_{\text{reg}}) \cong X_{\text{reg}}$, the 2-form $\omega$ extends to a regular closed 2-form on $\tilde{X}$. There is a natural Poisson structure $\{ , \}$ on $X$ determined by $\omega$. Then we can introduce the notion of a Poisson deformation of $(X, \{ , \})$. A Poisson deformation is a deformation of the pair of $X$ itself and the Poisson structure on it. When $X$ is not a compact variety, the usual deformation theory does not work in general because the tangent object $T^1_X$ may possibly have infinite dimension, and moreover, infinitesimal or formal deformations do not capture actual deformations of non-compact varieties. On the other hand, Poisson deformations work very well in many important cases where $X$ is not a complete variety. Denote by $PD_X$ the Poisson deformation functor of a symplectic variety. In this lecture, we shall study the Poisson deformation of an affine symplectic variety. The main result is:

**Theorem 1.** Let $X$ be an affine symplectic variety. Then the Poisson deformation functor $PD_X$ is unobstructed.

A Poisson deformation of $X$ is controlled by the Poisson cohomology $HP^2(X)$. When $X$ has only terminal singularities, we have $HP^2(X) \cong H^2((X_{\text{reg}})^{\text{an}}, \mathbb{C})$, where $(X_{\text{reg}})^{\text{an}}$ is the associated complex space with $X_{\text{reg}}$. This description enables us to prove that $PD_X$ is unobstructed. But, in general, there is not such a direct, topological description of $HP^2(X)$. Let us explain our strategy to describe $HP^2(X)$. As remarked, $HP^2(X)$ is identified with $PD_X(\mathbb{C}[\epsilon])$ where $\mathbb{C}[\epsilon]$ is the ring of dual numbers over $\mathbb{C}$. First, note that there is an open locus $U$ of $X$ where $X$ is smooth, or is locally a trivial deformation of a (surface) rational double point at each $p \in U$. Let $\Sigma$ be the singular locus of $U$. Note that $X \setminus U$ has codimension $\geq 4$ in $X$. Moreover, we have $PD_X(\mathbb{C}[\epsilon]) \cong PD_U(\mathbb{C}[\epsilon])$. Put $T^1_{U^{\text{an}}} := \mathbb{EHom}^1(\Omega^1_{U^{\text{an}}}, \mathcal{O}_{U^{\text{an}}})$. As is well-
known, a (local) section of $T_{U^{\text{an}}}^{1}$ corresponds to a 1-st order deformation of $U^{\text{an}}$. Let $\mathcal{H}$ be a locally constant $\mathbb{C}$-modules on $\Sigma$ defined as the subsheaf of $T_{U^{\text{an}}}^{1}$ which consists of the sections coming from Poisson deformations of $U^{\text{an}}$. Now we have an exact sequence:

$$0 \to H^{2}(U^{\text{an}}, \mathbb{C}) \to PD_{U}(\mathbb{C}[\epsilon]) \to H^{0}(\Sigma, \mathcal{H}).$$

Here the first term $H^{2}(U^{\text{an}}, \mathbb{C})$ is the space of locally trivial\(^1\) Poisson deformations of $U$. By the definition of $U$, there exists a minimal resolution $\pi : \tilde{U} \to U$. Let $m$ be the number of irreducible components of the exceptional divisor of $\pi$. A key result is:

**Proposition 2.** The following equality holds:

$$\dim H^{0}(\Sigma, \mathcal{H}) = m.$$

In order to prove Proposition 2, we need to know the monodromy action of $\pi_{1}(\Sigma)$ on $\mathcal{H}$. The idea is to compare two sheaves $R^{2}\pi_{*}^{\text{an}}\mathbb{C}$ and $\mathcal{H}$. Note that, for each point $p \in \Sigma$, the germ $(\tilde{U}, p)$ is isomorphic to the product of an ADE surface singularity $S$ and $(\mathbb{C}^{2n-2}, 0)$. Let $\tilde{S}$ be the minimal resolution of $S$. Then, $(R^{2}\pi_{*}^{\text{an}}\mathbb{C})_{p}$ is isomorphic to $H^{2}(\tilde{S}, \mathbb{C})$. A monodromy of $R^{2}\pi_{*}^{\text{an}}\mathbb{C}$ comes from a graph automorphism of the Dynkin diagram determined by the exceptional $(-2)$-curves on $\tilde{S}$. As is well known, $S$ is described in terms of a simple Lie algebra $\mathfrak{g}$, and $H^{2}(\tilde{S}, \mathbb{C})$ is identified with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$; therefore, one may regard $R^{2}\pi_{*}^{\text{an}}\mathbb{C}$ as a local system of the $\mathbb{C}$-module $\mathfrak{h}$ on $\Sigma$, whose monodromy action coincides with the natural action of a graph automorphism on $\mathfrak{h}$. On the other hand, $\mathcal{H}$ is a local system of $\mathfrak{h}/W$, where $\mathfrak{h}/W$ is the linear space obtained as the quotient of $\mathfrak{h}$ by the Weyl group $W$ of $\mathfrak{g}$. The action of a graph automorphism on $\mathfrak{h}$ descends to an action on $\mathfrak{h}/W$, which gives a monodromy action for $\mathcal{H}$. This description of the monodromy enables us to compute $\dim H^{0}(\Sigma, \mathcal{H})$.

Proposition 2 together with the exact sequence above gives an upper-bound of $\dim PD_{U}(\mathbb{C}[\epsilon])$ in terms of some topological data of $X$ (or $U$). We shall prove Theorem 1 by using this upper-bound. The rough idea is the following. There is a natural map of functors $PD_{\tilde{U}} \to PD_{U}$ induced by the resolution map $\tilde{U} \to U$. The tangent space $PD_{\tilde{U}}(\mathbb{C}[\epsilon])$ to $PD_{\tilde{U}}$ is identified with $H^{2}(\tilde{U}^{\text{an}}, \mathbb{C})$. We have an exact sequence

$$0 \to H^{2}(U^{\text{an}}, \mathbb{C}) \to H^{2}(\tilde{U}^{\text{an}}, \mathbb{C}) \to H^{0}(U^{\text{an}}, R^{2}\pi_{*}^{\text{an}}\mathbb{C}) \to 0,$$

\(^1\)More exactly, this means that the Poisson deformations are locally trivial as usual flat deformations of $U^{\text{an}}$.\)
and $\dim H^0(U^{\text{an}}, R^2\pi^{\text{an}}_*C) = m$. In particular, we have $\dim H^2(\tilde{U}^{\text{an}}, C) = \dim H^2(U^{\text{an}}, C) + m$. But, this implies that $\dim \text{PD}_{\tilde{U}}(C[\epsilon]) \geq \dim \text{PD}_U(C[\epsilon])$. On the other hand, the map $\text{PD}_{\tilde{U}} \rightarrow \text{PD}_U$ has a finite closed fiber; or more exactly, the corresponding map $\text{Spec}R_{\tilde{U}} \rightarrow \text{Spec}R_U$ of pro-representable hulls, has a finite closed fiber. Since $\text{PD}_{\tilde{U}}$ is unobstructed, this implies that $\text{PD}_U$ is unobstructed and $\dim \text{PD}_{\tilde{U}}(C[\epsilon]) = \dim \text{PD}_U(C[\epsilon])$. Finally, we obtain the unobstructedness of $\text{PD}_X$ from that of $\text{PD}_U$.

Theorem 1 is only concerned with the formal deformations of $X$; but, if we impose the following condition\(^(*)\), then the formal universal Poisson deformation of $X$ has an algebraization.

\(^(*)\): $X$ has a $C^*$-action with positive weights with a unique fixed point $0 \in X$. Moreover, $\omega$ is positively weighted for the action.

We shall briefly explain how this condition\(^(*)\) is used in the algebraization. Let $R_X := \lim R_X/(m_X)^{n+1}$ be the pro-representable hull of $\text{PD}_X$. Then the formal universal deformation $\{X_n\}$ of $X$ defines an $m_X$-adic ring $A := \lim \Gamma(X_n, \mathcal{O}_{X_n})$ and let $\hat{A}$ be the completion of $A$ along the maximal ideal of $A$. The rings $R_X$ and $\hat{A}$ both have the natural $C^*$-actions induced from the $C^*$-action on $X$, and there is a $C^*$-equivariant map $R_X \rightarrow \hat{A}$. By taking the $C^*$-subalgebras of $R_X$ and $\hat{A}$ generated by eigen-vectors, we get a map

$$C[x_1, \ldots, x_d] \rightarrow S$$

from a polynomial ring to a $C$-algebra of finite type. We also have a Poisson structure on $S$ over $C[x_1, \ldots, x_d]$ by the second condition of\(^(*)\). As a consequence, there is an affine space $A^d$ whose completion at the origin coincides with $\text{Spec}(R_X)$ in such a way that the formal universal Poisson deformation over $\text{Spec}(R_X)$ is algebraized to a $C^*$-equivariant map

$$\mathcal{X} \rightarrow A^d.$$

According to a result of Birkar-Cascini-Hacon-McKernan, we can take a crepant partial resolution $\pi : Y \rightarrow X$ in such a way that $Y$ has only $\mathbb{Q}$-factorial terminal singularities. This $Y$ is called a $\mathbb{Q}$-factorial terminalization of $X$. In our case, $Y$ is a symplectic variety and the $C^*$-action on $X$ uniquely extends to that on $Y$. Since $Y$ has only terminal singularities, it is relatively easy to show that the Poisson deformation functor $\text{PD}_Y$ is unobstructed. Moreover, the formal universal Poisson deformation of $Y$ has an
algebraization over an affine space $A^d$:
\[
\mathcal{Y} 	o A^d.
\]

There is a $C^\ast$-equivariant commutative diagram
\[
\begin{array}{ccc}
\mathcal{Y} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
A^d & \psi & A^d
\end{array}
\] (1)

We have the following.

**Theorem 3** (a) $\psi$ is a finite Galois covering.
(b) $\mathcal{Y} \to A^d$ is a locally trivial deformation of $Y$.
(c) The induced map $\mathcal{Y}_t \to X_{\psi(t)}$ is an isomorphism for a general point $t \in A^d$.

The Galois group of $\psi$ is described as follows. Let $\Sigma$ be the singular locus of $X$. There is a closed subset $\Sigma_0 \subset \Sigma$ such that $X$ is locally isomorphic to $(S,0) \times (C^{2n-2},0)$ at every point $p \in \Sigma - \Sigma_0$ where $S$ is an ADE surface singularity. We have $\text{Codim}_X \Sigma_0 \geq 4$. Let $B$ be the set of connected components of $\Sigma - \Sigma_0$. Let $B \in \mathcal{B}$. Pick a point $b \in B$ and take a transversal slice $S_B \subset Y$ of $B$ passing through $b$. In other words, $X$ is locally isomorphic to $S_B \times (B,b)$ around $b$. $S_B$ is a surface with an ADE singularity. Put $\tilde{S}_B := \pi^{-1}(S_B)$. Then $\tilde{S}_B$ is a minimal resolution of $S_B$. Put $T_B := S_B \times (B,b)$ and $\tilde{T}_B := \pi^{-1}(T_B)$. Note that $\tilde{T}_B = \tilde{S}_B \times (B,b)$. Let $C_i (1 \leq i \leq r)$ be the $(-2)$-curves contained in $\tilde{S}_B$ and let $[C_i] \in H^2(\tilde{S}_B, \mathbb{R})$ be their classes in the 2-nd cohomology group. Then
\[
\Phi := \{\Sigma a_i[C_i]; a_i \in \mathbb{Z}, (\Sigma a_i[C_i])^2 = -2\}
\]
is a root system of the same type as that of the ADE-singularity $S_B$. Let $W$ be the Weyl group of $\Phi$. Let $\{E_i(B)\}_{1 \leq i \leq \bar{r}}$ be the set of irreducible exceptional divisors of $\pi$ lying over $B$, and let $e_i(B) \in H^2(X, \mathbb{Z})$ be their classes. Clearly, $\bar{r} \leq r$. If $\bar{r} = r$, then we define $W_B := W$. If $\bar{r} < r$, the Dynkin diagram of $\Phi$ has a non-trivial graph automorphism. When $\Phi$ is of type $A_r$ with $r > 1$, $\bar{r} = [r + 1/2]$ and the Dynkin diagram has a graph automorphism $\tau$ of order 2. When $\Phi$ is of type $D_r$ with $r \geq 5$, $\bar{r} = r - 1$ and the Dynkin diagram has a graph automorphism $\tau$ of order 2. When $\Phi$ is of type $D_4$, the Dynkin diagram has two different graph automorphisms of order 2 and 3. There are
two possibilities of $\bar{\tau}$; $\bar{\tau} = 2$ or $\bar{\tau} = 3$. In the first case, let $\tau$ be the graph automorphism of order 3. In the latter case, let $\tau$ be the graph automorphism of order 2. Finally, when $\Phi$ is of type $E_6$, $\bar{\tau} = 4$ and the Dynkin diagram has a graph automorphism $\tau$ of order 2. In all these cases, we define

$$W_B := \{w \in W; \tau w \tau^{-1} = w\}.$$ 

The Galois group of $\psi$ coincides with $W_B$.

As an application of Theorem 3, we have

**Corollary 4:** Let $(X, \omega)$ be an affine symplectic variety with the property (*). Then the following are equivalent.

1. $X$ has a crepant projective resolution.
2. $X$ has a smoothing by a Poisson deformation.

**Example 5** (i) Let $O \subset g$ be a nilpotent orbit of a complex simple Lie algebra. Let $\tilde{O}$ be the normalization of the closure $\bar{O}$ of $O$ in $g$. Then $\tilde{O}$ is an affine symplectic variety with the Kostant-Kirillov 2-form $\omega$ on $O$. Let $G$ be a complex algebraic group with $\text{Lie}(G) = g$. By [Fu], $\tilde{O}$ has a crepant projective resolution if and only if $O$ is a Richardson orbit (cf. [C-M]) and there is a parabolic subgroup $P$ of $G$ such that its Springer map $T^*(G/P) \to \tilde{O}$ is birational. In this case, every crepant resolution of $\tilde{O}$ is actually obtained as a Springer map for some $P$. If $\tilde{O}$ has a crepant resolution, $\tilde{O}$ has a smoothing by a Poisson deformation. The smoothing of $\tilde{O}$ is isomorphic to the affine variety $G/L$, where $L$ is the Levi subgroup of $P$. Conversely, if $\tilde{O}$ has a smoothing by a Poisson deformation, then the smoothing always has this form.

(ii) In general, $\tilde{O}$ has no crepant resolutions. But, by [Na 4], at least when $g$ is a classical simple Lie algebra, every $\mathbb{Q}$-factorial terminalization of $\tilde{O}$ is given by a generalized Springer map. More explicitly, there is a parabolic subalgebra $\mathfrak{p}$ with Levi decomposition $\mathfrak{p} = \mathfrak{n} \oplus \mathfrak{l}$ and a nilpotent orbit $O'$ in $\mathfrak{l}$ so that the generalized Springer map $G \times^P (n + \mathfrak{o}') \to \tilde{O}$ is a crepant, birational map, and the normalization of $G \times^P (n + \mathfrak{o}')$ is a $\mathbb{Q}$-factorial terminalization of $\tilde{O}$. By a Poisson deformation, $\tilde{O}$ deforms to the normalization of $G \times^L \tilde{O}'$. Here $G \times^L \tilde{O}'$ is a fiber bundle over $G/L$ with a typical fiber $\tilde{O}'$, and its normalization can be written as $G \times^L \tilde{O}'$ with the normalization $\tilde{O}'$ of $O'$.

We can apply Theorem 3 to the Poisson deformations of an affine symplectic variety related to a nilpotent orbit in a complex simple Lie algebra.
Let \( g \) be a complex simple Lie algebra and let \( G \) be the adjoint group. For a parabolic subgroup \( P \) of \( G \), denote by \( T^*(G/P) \) the cotangent bundle of \( G/P \). The image of the Springer map \( s : T^*(G/P) \to g \) is the closure \( \bar{O} \) of a nilpotent (adjoint) orbit \( O \) in \( g \). Then the normalization \( \bar{O} \) of \( O \) is an affine symplectic variety with the Kostant-Kirillov 2-form. If \( s \) is birational onto its image, then the Stein factorization \( T^*(G/P) \to \bar{O} \to \tilde{O} \) of \( s \) gives a crepant resolution of \( \bar{O} \). In this situation, we have the following commutative diagram

\[
\begin{align*}
G \times^P r(p) & \longrightarrow \bar{G} \cdot r(p) \\
\downarrow & \downarrow \\
\mathfrak{e}(p) & \longrightarrow \mathfrak{e}(p)/W'
\end{align*}
\]

where \( r(p) \) is the solvable radical of \( p \), \( \bar{G} \cdot r(p) \) is the normalization of the adjoint \( G \)-orbit of \( r(p) \) and \( \mathfrak{e}(p) \) is the centralizer of the Levi part \( l \) of \( p \). Moreover, \( W' := N_W(L)/W(L) \), where \( L \) is the Levi subgroup of \( P \) and \( W(L) \) is the Weyl group of \( L \).

**Theorem 6.** The diagram above coincides with the \( C^* \)-equivariant commutative diagram of the universal Poisson deformations of \( T^*(G/P) \) and \( \bar{O} \).

Note that \( W' \) has been extensively studied by Howlett and others. Another important example is a transversal slice of \( g \). In the commutative diagram above, put \( p = b \) the Borel subalgebra. Then we have:

\[
\begin{align*}
G \times^B b & \longrightarrow \mathcal{V}_B \\
\downarrow & \downarrow \\
h & \longrightarrow h/W.
\end{align*}
\]

Let \( x \in g \) be a nilpotent element of \( g \) and let \( O \) be the adjoint orbit containing \( x \). Let \( \mathcal{V} \subset g \) be a transversal slice for \( O \) passing through \( x \). Put \( \mathcal{V}_B := \pi_B^{-1}(\mathcal{V}) \). Denote by \( \mathcal{V} \) (resp. \( \tilde{V}_B \)) the central fiber of \( \mathcal{V} \to h/W \) (resp. \( G \times^B b \to h \)). Note that \( \tilde{V}_B \) is isomorphic to the cotangent bundle \( T^*(G/B) \) of \( G/B \), and \( \tilde{V}_B \to V \) is a crepant resolution.
Theorem 7 The commutative diagram

\[
\begin{array}{c}
\tilde{\mathcal{V}}_B \\
\downarrow \phi \downarrow \\
h \\
\end{array}
\begin{array}{c}
\mathcal{V} \\
h/W \\
\end{array}
\]

(4)

is the $C^*$-equivariant commutative diagram of the universal Poisson deformations of $\tilde{\mathcal{V}}_B$ and $\mathcal{V}$ if $\mathfrak{g}$ is simply laced.

When $\mathfrak{g}$ is not simply-laced, Theorem 7 is no more true. In fact, Slodowy pointed out that the transversal slice $\mathcal{V}$ for a subregular nilpotent orbit of non-simply-laced $\mathfrak{g}$ does not give the universal deformation. However, we have a criterion of the universality. Let

\[ \rho : A(O) \to GL(H^2(\pi_{B,0}^{-1}(x), \mathbb{Q})) \]

be the monodromy representation of the component group $A(O)$ of $O$.

Theorem 8. Let $\mathfrak{g}$ be a complex simple Lie algebra which is not necessarily simply-laced. Then the above commutative diagram is universal if and only if $\rho$ is trivial.

Department of Mathematics, Kyoto University, namikawa@math.kyoto-u.ac.jp