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Symplectic varieties and Poisson deformations

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A symplectic variety $X$ is a normal algebraic variety (defined over $\mathbb{C}$) which admits an everywhere non-degenerate $d$-closed 2-form $\omega$ on the regular locus $X_{\text{reg}}$ of $X$ such that, for any resolution $f : \hat{X} \to X$ with $f^{-1}(X_{\text{reg}}) \cong X_{\text{reg}}$, the 2-form $\omega$ extends to a regular closed 2-form on $\hat{X}$. There is a natural Poisson structure $\{ , \}$ on $X$ determined by $\omega$. Then we can introduce the notion of a Poisson deformation of $(X, \{ , \})$. A Poisson deformation is a deformation of the pair of $X$ itself and the Poisson structure on it. When $X$ is not a compact variety, the usual deformation theory does not work in general because the tangent object $T^1_X$ may possibly have infinite dimension, and moreover, infinitesimal or formal deformations do not capture actual deformations of non-compact varieties. On the other hand, Poisson deformations work very well in many important cases where $X$ is not a complete variety. Denote by $PD_X$ the Poisson deformation functor of a symplectic variety. In this lecture, we shall study the Poisson deformation of an affine symplectic variety. The main result is:

**Theorem 1.** Let $X$ be an affine symplectic variety. Then the Poisson deformation functor $PD_X$ is unobstructed.

A Poisson deformation of $X$ is controlled by the Poisson cohomology $HP^2(X)$. When $X$ has only terminal singularities, we have $HP^2(X) \cong H^2((X_{\text{reg}})^{\text{an}}, \mathbb{C})$, where $(X_{\text{reg}})^{\text{an}}$ is the associated complex space with $X_{\text{reg}}$. This description enables us to prove that $PD_X$ is unobstructed. But, in general, there is not such a direct, topological description of $HP^2(X)$. Let us explain our strategy to describe $HP^2(X)$. As remarked, $HP^2(X)$ is identified with $PD_X(C[\epsilon])$ where $C[\epsilon]$ is the ring of dual numbers over $\mathbb{C}$. First, note that there is an open locus $U$ of $X$ where $X$ is smooth, or is locally a trivial deformation of a (surface) rational double point at each $p \in U$. Let $\Sigma$ be the singular locus of $U$. Note that $X \setminus U$ has codimension $\geq 4$ in $X$. Moreover, we have $PD_X(C[\epsilon]) \cong PD_U(C[\epsilon])$. Put $T^1_{U^{\text{an}}} := \mathbb{E}xt^1(\Omega^1_{U^{\text{an}}}, \mathcal{O}_{U^{\text{an}}})$. As is well-
known, a (local) section of $T^{1}_{U^{an}}$ corresponds to a 1-st order deformation of $U^{an}$. Let $\mathcal{H}$ be a locally constant $\mathbb{C}$-modules on $\Sigma$ defined as the subsheaf of $T^{1}_{U^{an}}$ which consists of the sections coming from Poisson deformations of $U^{an}$. Now we have an exact sequence:

$$0 \to H^{2}(U^{an}, \mathbb{C}) \to PD_{U}(\mathbb{C}[\epsilon]) \to H^{0}(\Sigma, \mathcal{H}).$$

Here the first term $H^{2}(U^{an}, \mathbb{C})$ is the space of locally trivial Poisson deformations of $U$. By the definition of $U$, there exists a minimal resolution $\pi : \tilde{U} \to U$. Let $m$ be the number of irreducible components of the exceptional divisor of $\pi$. A key result is:

**Proposition 2.** The following equality holds:

$$\dim H^{0}(\Sigma, \mathcal{H}) = m.$$

In order to prove Proposition 2, we need to know the monodromy action of $\pi_{1}(\Sigma)$ on $\mathcal{H}$. The idea is to compare two sheaves $R^{2}\pi^{an}_{*}\mathbb{C}$ and $\mathcal{H}$. Note that, for each point $p \in \Sigma$, the germ $(\tilde{U}, p)$ is isomorphic to the product of an ADE surface singularity $S$ and $(\mathbb{C}^{2n-2}, 0)$. Let $\tilde{S}$ be the minimal resolution of $S$. Then, $(R^{2}\pi^{an}_{*}\mathbb{C})_{p}$ is isomorphic to $H^{2}(\tilde{S}, \mathbb{C})$. A monodromy of $R^{2}\pi^{an}_{*}\mathbb{C}$ comes from a graph automorphism of the Dynkin diagram determined by the exceptional (-2)-curves on $\tilde{S}$. As is well known, $S$ is described in terms of a simple Lie algebra $\mathfrak{g}$, and $H^{2}(\tilde{S}, \mathbb{C})$ is identified with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$; therefore, one may regard $R^{2}\pi^{an}_{*}\mathbb{C}$ as a local system of the $\mathbb{C}$-module $\mathfrak{h}$ (on $\Sigma$), whose monodromy action coincides with the natural action of a graph automorphism on $\mathfrak{h}$. On the other hand, $\mathcal{H}$ is a local system of $\mathfrak{h}/W$, where $\mathfrak{h}/W$ is the linear space obtained as the quotient of $\mathfrak{h}$ by the Weyl group $W$ of $\mathfrak{g}$. The action of a graph automorphism on $\mathfrak{h}$ descends to an action on $\mathfrak{h}/W$, which gives a monodromy action for $\mathcal{H}$. This description of the monodromy enables us to compute $\dim H^{0}(\Sigma, \mathcal{H})$.

Proposition 2 together with the exact sequence above gives an upper-bound of $\dim PD_{U}(\mathbb{C}[\epsilon])$ in terms of some topological data of $X$ (or $U$). We shall prove Theorem 1 by using this upper-bound. The rough idea is the following. There is a natural map of functors $PD_{\tilde{U}} \to PD_{U}$ induced by the resolution map $\tilde{U} \to U$. The tangent space $PD_{\tilde{U}}(\mathbb{C}[\epsilon])$ to $PD_{\tilde{U}}$ is identified with $H^{2}(\tilde{U}^{an}, \mathbb{C})$. We have an exact sequence

$$0 \to H^{2}(U^{an}, \mathbb{C}) \to H^{2}(\tilde{U}^{an}, \mathbb{C}) \to H^{0}(U^{an}, R^{2}\pi^{an}_{*}\mathbb{C}) \to 0,$$

1More exactly, this means that the Poisson deformations are locally trivial as usual flat deformations of $U^{an}$.
and $\dim H^0(U^{an}, R^2\pi^{an}C) = m$. In particular, we have $\dim H^2(\tilde{\mathcal{U}}^{an}, C) = \dim H^2(U^{an}, C) + m$. But, this implies that $\dim \text{PD}_{\tilde{\mathcal{U}}} (C[\epsilon]) \geq \dim \text{PD}_U (C[\epsilon])$. On the other hand, the map $\text{PD}_{\tilde{\mathcal{U}}} \to \text{PD}_U$ has a finite closed fiber; or more exactly, the corresponding map $\text{Spec} R_{\tilde{\mathcal{U}}} \to \text{Spec} R_U$ of pro-representable hulls, has a finite closed fiber. Since $\text{PD}_{\tilde{\mathcal{U}}}$ is unobstructed, this implies that $\text{PD}_U$ is unobstructed and $\dim \text{PD}_{\tilde{\mathcal{U}}} (C[\epsilon]) = \dim \text{PD}_U (C[\epsilon])$. Finally, we obtain the unobstructedness of $\text{PD}_X$ from that of $\text{PD}_U$.

Theorem 1 is only concerned with the formal deformations of $X$; but, if we impose the following condition (*), then the formal universal Poisson deformation of $X$ has an algebraization.

(*) $X$ has a $C^\ast$-action with positive weights with a unique fixed point $0 \in X$. Moreover, $\omega$ is positively weighted for the action.

We shall briefly explain how this condition (*) is used in the algebraization. Let $R_X := \lim R_X/(m_X)^{n+1}$ be the pro-representable hull of $\text{PD}_X$. Then the formal universal deformation $\{X_n\}$ of $X$ defines an $m_X$-adic ring $A := \lim \Gamma(X_n, \mathcal{O}_{X_n})$ and let $\hat{A}$ be the completion of $A$ along the maximal ideal of $A$. The rings $R_X$ and $\hat{A}$ both have the natural $C^\ast$-actions induced from the $C^\ast$-action on $X$, and there is a $C^\ast$-equivariant map $R_X \to \hat{A}$. By taking the $C^\ast$-subalgebras of $R_X$ and $\hat{A}$ generated by eigen-vectors, we get a map

$$C[x_1, \ldots, x_d] \to S$$

from a polynomial ring to a $C$-algebra of finite type. We also have a Poisson structure on $S$ over $C[x_1, \ldots, x_d]$ by the second condition of (*). As a consequence, there is an affine space $A^d$ whose completion at the origin coincides with $\text{Spec}(R_X)$ in such a way that the formal universal Poisson deformation over $\text{Spec}(R_X)$ is algebraized to a $C^\ast$-equivariant map

$$\mathcal{X} \to A^d.$$ 

According to a result of Birkar-Cascini-Hacon-McKernan, we can take a crepant partial resolution $\pi : Y \to X$ in such a way that $Y$ has only $\mathbb{Q}$-factorial terminal singularities. This $Y$ is called a $\mathbb{Q}$-factorial terminalization of $X$. In our case, $Y$ is a symplectic variety and the $C^\ast$-action on $X$ uniquely extends to that on $Y$. Since $Y$ has only terminal singularities, it is relatively easy to show that the Poisson deformation functor $\text{PD}_Y$ is unobstructed. Moreover, the formal universal Poisson deformation of $Y$ has an
algebraization over an affine space $A^d$:

$$\mathcal{Y} \rightarrow A^d.$$ 

There is a $C^*$-equivariant commutative diagram

\[
\begin{array}{ccc}
\mathcal{Y} & \rightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
A^d & \xrightarrow{\psi} & A^d
\end{array}
\] (1)

We have the following.

**Theorem 3**

(a) $\psi$ is a finite Galois covering.
(b) $\mathcal{Y} \rightarrow A^d$ is a locally trivial deformation of $Y$.
(c) The induced map $\mathcal{Y}_t \rightarrow \mathcal{X}_{\psi(t)}$ is an isomorphism for a general point $t \in A^d$.

The Galois group of $\psi$ is described as follows. Let $\Sigma$ be the singular locus of $X$. There is a closed subset $\Sigma_0 \subset \Sigma$ such that $X$ is locally isomorphic to $(S,0) \times (C^{2n-2},0)$ at every point $p \in \Sigma - \Sigma_0$ where $S$ is an ADE surface singularity. We have $\text{Codim}_X \Sigma_0 \geq 4$. Let $B$ be the set of connected components of $\Sigma - \Sigma_0$. Let $B \in \mathcal{B}$. Pick a point $b \in B$ and take a transversal slice $S_B \subset Y$ of $B$ passing through $b$. In other words, $X$ is locally isomorphic to $S_B \times (B, b)$ around $b$. $S_B$ is a surface with an ADE singularity. Put $\tilde{S}_B := \pi^{-1}(S_B)$. Then $\tilde{S}_B$ is a minimal resolution of $S_B$. Put $T_B := S_B \times (B, b)$ and $\tilde{T}_B := \pi^{-1}(T_B)$. Note that $\tilde{T}_B = \tilde{S}_B \times (B, b)$. Let $C_i (1 \leq i \leq r)$ be the $(-2)$-curves contained in $\tilde{S}_B$ and let $[C_i] \in H^2(\tilde{S}_B, \mathbb{R})$ be their classes in the 2-nd cohomology group. Then

$$\Phi := \{\Sigma a_i[C_i]; a_i \in \mathbb{Z}, (\Sigma a_i[C_i])^2 = -2\}$$

is a root system of the same type as that of the ADE-singularity $S_B$. Let $W$ be the Weyl group of $\Phi$. Let $\{E_i(B)\}_{1 \leq i \leq \overline{r}}$ be the set of irreducible exceptional divisors of $\pi$ lying over $B$, and let $e_i(B) \in H^2(X, \mathbb{Z})$ be their classes. Clearly, $\overline{r} \leq r$. If $\overline{r} = r$, then we define $W_B := W$. If $\overline{r} < r$, the Dynkin diagram of $\Phi$ has a non-trivial graph automorphism. When $\Phi$ is of type $A_r$ with $r > 1$, $\overline{r} = [r + 1/2]$ and the Dynkin diagram has a graph automorphism $\tau$ of order 2. When $\Phi$ is of type $D_r$ with $r \geq 5$, $\overline{r} = r - 1$ and the Dynkin diagram has a graph automorphism $\tau$ of order 2. When $\Phi$ is of type $D_4$, the Dynkin diagram has two different graph automorphisms of order 2 and 3. There are
two possibilities of \( \bar{r} ; \bar{r} = 2 \) or \( \bar{r} = 3 \). In the first case, let \( \tau \) be the graph automorphism of order 3. In the latter case, let \( \tau \) be the graph automorphism of order 2. Finally, when \( \Phi \) is of type \( E_6 \), \( \bar{r} = 4 \) and the Dynkin diagram has a graph automorphism \( \tau \) of order 2. In all these cases, we define

\[
W_B := \{ w \in W ; \tau w \tau^{-1} = w \}.
\]

The Galois group of \( \psi \) coincides with \( W_B \).

As an application of Theorem 3, we have

**Corollary 4:** Let \((X, \omega)\) be an affine symplectic variety with the property (*). Then the following are equivalent.

1. \( X \) has a crepant projective resolution.
2. \( X \) has a smoothing by a Poisson deformation.

**Example 5** (i) Let \( O \subset \mathfrak{g} \) be a nilpotent orbit of a complex simple Lie algebra. Let \( \tilde{O} \) be the normalization of the closure \( \tilde{O} \) of \( O \) in \( \mathfrak{g} \). Then \( \tilde{O} \) is an affine symplectic variety with the Kostant-Kirillov 2-form \( \omega \) on \( O \). Let \( G \) be a complex algebraic group with \( \text{Lie}(G) = \mathfrak{g} \). By [Fu], \( \tilde{O} \) has a crepant projective resolution if and only if \( O \) is a Richardson orbit (cf. [C-M]) and there is a parabolic subgroup \( P \) of \( G \) such that its Springer map \( T^*(G/P) \to \tilde{O} \) is birational. In this case, every crepant resolution of \( \tilde{O} \) is actually obtained as a Springer map for some \( P \). If \( \tilde{O} \) has a crepant resolution, \( \tilde{O} \) has a smoothing by a Poisson deformation. The smoothing of \( \tilde{O} \) is isomorphic to the affine variety \( G/L \), where \( L \) is the Levi subgroup of \( P \). Conversely, if \( \tilde{O} \) has a smoothing by a Poisson deformation, then the smoothing always has this form.

(ii) In general, \( \tilde{O} \) has no crepant resolutions. But, by [Na 4], at least when \( \mathfrak{g} \) is a classical simple Lie algebra, every \( \mathbb{Q} \)-factorial terminalization of \( \tilde{O} \) is given by a generalized Springer map. More explicitly, there is a parabolic subalgebra \( \mathfrak{p} \) with Levi decomposition \( \mathfrak{p} = \mathfrak{n} \oplus \mathfrak{l} \) and a nilpotent orbit \( O' \) in \( \mathfrak{l} \) so that the generalized Springer map \( G \times^P (n + O') \to \tilde{O} \) is a crepant, birational map, and the normalization of \( G \times^P (n + O') \) is a \( \mathbb{Q} \)-factorial terminalization of \( \tilde{O} \). By a Poisson deformation, \( \tilde{O} \) deforms to the normalization of \( G \times^L \tilde{O}' \). Here \( G \times^L \tilde{O}' \) is a fiber bundle over \( G/L \) with a typical fiber \( \tilde{O}' \), and its normalization can be written as \( G \times^L \tilde{O}' \) with the normalization \( \tilde{O}' \) of \( \tilde{O}' \).

We can apply Theorem 3 to the Poisson deformations of an affine symplectic variety related to a nilpotent orbit in a complex simple Lie algebra.
Let $\mathfrak{g}$ be a complex simple Lie algebra and let $G$ be the adjoint group. For a parabolic subgroup $P$ of $G$, denote by $T^*(G/P)$ the cotangent bundle of $G/P$. The image of the Springer map $s : T^*(G/P) \to \mathfrak{g}$ is the closure $\bar{O}$ of a nilpotent (adjoint) orbit $O$ in $\mathfrak{g}$. Then the normalization $\bar{O}$ of $\bar{O}$ is an affine symplectic variety with the Kostant-Kirillov 2-form. If $s$ is birational onto its image, then the Stein factorization $T^*(G/P) \to \bar{O} \to \tilde{O}$ of $s$ gives a crepant resolution of $\bar{O}$. In this situation, we have the following commutative diagram

$$
\begin{array}{ccc}
G \times^P r(\mathfrak{p}) & \longrightarrow & \tilde{G} \cdot r(\mathfrak{p}) \\
\downarrow & & \downarrow \\
\mathfrak{k}(\mathfrak{p}) & \longrightarrow & \mathfrak{k}(\mathfrak{p})/W'
\end{array}
$$

(2)

where $r(\mathfrak{p})$ is the solvable radical of $\mathfrak{p}$, $\tilde{G} \cdot r(\mathfrak{p})$ is the normalization of the adjoint $G$-orbit of $r(\mathfrak{p})$ and $\mathfrak{k}(\mathfrak{p})$ is the centralizer of the Levi part $\mathfrak{l}$ of $\mathfrak{p}$. Moreover, $W' := N_W(L)/W(L)$, where $L$ is the Levi subgroup of $P$ and $W(L)$ is the Weyl group of $L$.

**Theorem 6.** The diagram above coincides with the $C^*$-equivariant commutative diagram of the universal Poisson deformations of $T^*(G/P)$ and $\bar{O}$.

Note that $W'$ has been extensively studied by Howlett and others. Another important example is a transversal slice of $\mathfrak{g}$. In the commutative diagram above, put $\mathfrak{p} = \mathfrak{b}$ the Borel subalgebra. Then we have:

$$
\begin{array}{ccc}
G \times^B \mathfrak{b} & \longrightarrow & \mathfrak{g} \\
\downarrow & \phi \downarrow & \\
\mathfrak{h} & \longrightarrow & \mathfrak{h}/W.
\end{array}
$$

(3)

Let $x \in \mathfrak{g}$ be a nilpotent element of $\mathfrak{g}$ and let $O$ be the adjoint orbit containing $x$. Let $\mathcal{V} \subset \mathfrak{g}$ be a transversal slice for $O$ passing through $x$. Put $\mathcal{V}_B := \pi_B^{-1}(\mathcal{V})$. Denote by $\mathcal{V}$ (resp. $\tilde{\mathcal{V}}_B$) the central fiber of $\mathcal{V} \to \mathfrak{h}/W$ (resp. $G \times^B \mathfrak{b} \to \mathfrak{h}$). Note that $\tilde{\mathcal{V}}_B$ is somorphic to the cotangent bundle $T^*(G/B)$ of $G/B$, and $\tilde{\mathcal{V}}_B \to V$ is a crepant resolution.
Theorem 7  The commutative diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{V}}_B & \rightarrow & \mathcal{V} \\
\downarrow & \varphi_{\mathcal{V}} & \downarrow \\
h & \rightarrow & h/W
\end{array}
\]  (4)

is the $\mathbb{C}^*$-equivariant commutative diagram of the universal Poisson deformations of $\tilde{V}_B$ and $V$ if $\mathfrak{g}$ is simply laced.

When $\mathfrak{g}$ is not simply-laced, Theorem 7 is no more true. In fact, Slodowy pointed out that the transversal slice $\mathcal{V}$ for a subregular nilpotent orbit of non-simply-laced $\mathfrak{g}$ does not give the universal deformation. However, we have a criterion of the universality. Let

\[
\rho : A(O) \rightarrow GL(H^2(\pi_{B,0}^{-1}(x), \mathbb{Q}))
\]

be the monodromy representation of the component group $A(O)$ of $O$.

Theorem 8. Let $\mathfrak{g}$ be a complete simple Lie algebra which is not necessarily simply-laced. Then the above commutative diagram is universal if and only if $\rho$ is trivial.

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