数理解析研究所講究録
第 1745 巻 2011 年 102-109

Generalization of Néron models of Green, Griffiths and Kerr

(Joint work with P. Brosnan and G. Pearlstein)

Dedicated to Professor Sampei Usui

Morihiko Saito

RIMS Kyoto University, Kyoto 606-8502 Japan

Abstract. We explain some recent developments in the theory of Néron models for families of Jacobians associated to variations of Hodge structures of weight -1.

1. Classical Néron models

1.1. Let $\mathcal{A}$ be an abelian scheme over a smooth curve $S^* \subset S$. There is a unique group scheme $\mathcal{A}_S$ over S, called the Néron model, and satisfying the following property: For any smooth T over S, we have

$$\mathcal{A}_S(T) = \mathcal{A}(T_{S^*}), \quad \text{i.e.} \quad \mathcal{A}_T(T) = \mathcal{A}_{T_{S^*}}(T_{S^*}).$$

Let $\mathbf{H}$ be the variation of Hodge structure of level 1 and weight -1 corresponding to $\mathcal{A}$. Then we have for $s \in S^*$

$$\mathcal{A}_s = J(H_s) \ (:= H_{s,\mathbf{Z}} \backslash H_{s,\mathbf{C}} / F^0 H_{s,\mathbf{C}}).$$

Note that the right-hand side is isomorphic to

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbf{Z}, H_s),$$

i.e. its element corresponds to the short exact sequence of MHS (see [Ca])

$$0 \to H_s \to H'_s \to \mathbf{Z} \to 0,$$

where MHS denotes the abelian category of mixed $\mathbf{Z}$-Hodge structures [D2].

Assume the monodromy is *unipotent* at $0 \in S \setminus S^*$. By [Sd] we have the limit mixed Hodge structure

$$H_\infty = \big((H_{\infty,\mathbf{C}}; F, W),\ (H_{\infty,\mathbf{Q}}, W),\ H_{\infty,\mathbf{Z}}\big).$$

This is closely related to the Néron model. Indeed, there is a short exact sequence

$$0 \to \mathcal{A}^0_{S,0} \to \mathcal{A}_{S,0} \to G_0 \to 0,$$

where

$$\mathcal{A}^0_{S,0} := H^{\rm inv}_{\infty,\mathbf{Z}}\backslash H_{\infty,\mathbf{C}}/F^0 H_{\infty,\mathbf{C}},$$

and

$$\begin{aligned} G_0 &:= H^1(\Delta^*, \mathbf{H_Z})_{\rm tor} = \mathrm{Coker}(T_{\mathbf{Z}} - id)_{\rm tor} \\ &= (\mathrm{Im}(T_{\mathbf{Q}} - id) \cap H_{\infty,\mathbf{Z}})/\mathrm{Im}(T_{\mathbf{Z}} - id) \\ &= \mathrm{Ker}(T_{\mathbf{Q/Z}} - id)/(\mathrm{Im\ of\ Ker}(T_{\mathbf{Q}} - id)). \end{aligned}$$

For the last isomorphism we use the snake lemma applied to the endomorphism T of the short exact sequence

$$0 \to \mathbf{H_Z} \to \mathbf{H_Q} \to \mathbf{H_{Q/Z}} \to 0.$$

This can be used to get a torsion normal function corresponding to a torsion cohomology class.

1.2. Example. Let $\mathcal{A}$ be a family of elliptic curves with monodromy $T = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$. Then

$$\mathcal{A}^0_{S,0} = \mathbf{Z}\backslash\mathbf{C}^2/\mathbf{C} = \mathbf{C}^*, \quad G_0 = \mathbf{Z}/r\mathbf{Z}, \quad \mathcal{A}_{S,0} = \textstyle\coprod^r \mathbf{C}^*.$$

2. Generalization by Zucker and Clemens

2.1. Generalization by Zucker. Let $\mathbf{H}$ be a variation of Hodge structure of weight -1 on S^*. we have the family of Jacobians

$$J(\mathbf{H}) := \textstyle\coprod_{s\in S^*} J(H_s).$$

Let $\widehat{\mathcal{L}}$ be the Deligne extension of $\mathbf{H}_{\mathcal{O}}$ over S (see [D1]), $\widehat{\mathcal{V}}$ the vector bundle corresponding to $\widehat{\mathcal{L}}/F^0\widehat{\mathcal{L}}$, and $\widehat{\Gamma}$ the image of $j_*\mathbf{H_Z}$ in $\widehat{\mathcal{V}}$ where $j : S^* \hookrightarrow S$.

2.2. Definition (Zucker extension) [Zu].

$$J^Z_S(\mathbf{H}) := \widehat{\Gamma}\backslash\widehat{\mathcal{V}} \text{ (fiberwise)}.$$

Assume $\mathbf{H}$ geometric, i.e. $\mathbf{H} = R^{2p-1}f_*\mathbf{Z}_{X^*}(p)$ with $f : X^* \to S^*$. Then we have the following.

2.3. Theorem (El Zein, Zucker) [EZ]. Let ν be a normal function defined by an algebraic cycle σ with $\gamma(\sigma|_{X_s}) = 0$. Then ν extends to a section of $J^Z_S(\mathbf{H})$ over S, if $\gamma(\sigma) = 0$.

Here $\gamma(\sigma)$ denotes the cohomology class as a cycle.

2.4. Generalization by Clemens. Assume

Hypothesis (C) : $N^2 = 0$ and $\mathrm{Gr}^W_0 H_\infty$ has type (0,0).

Then we have the following.

2.5. Theorem (Clemens) [Cl]. *There is $J_S^C(\mathbf{H})$ (Clemens Néron model) such that any normal function ν on S^* defined by an algebraic cycle is extended to a section of $J_S^C(\mathbf{H})$ over S. Moreover, there is a short exact sequence*

$$0 \to J_S^Z(\mathbf{H})_0 \to J_S^C(\mathbf{H})_0 \to G_0 \to 0,$$

with

$$J_S^Z(\mathbf{H})_0 = H_{\infty,\mathbf{Z}}^{\mathrm{inv}} \backslash H_{\infty,\mathbf{C}}/F^0 H_{\infty,\mathbf{C}},$$

$$G_0 = H^1(\Delta^*, \mathbf{H}_{\mathbf{Z}})_{\mathrm{tor}}.$$

In fact, $J_S^C(\mathbf{H})$ is obtained by gluing $J_S^Z(\mathbf{H})$.

3. Improvement using admissible normal functions

3.1. By [Ca] we have a canonical isomorphism

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbf{Z}, \mathbf{H}_s) = J(\mathbf{H}_s)\ (:= H_{s,\mathbf{Z}} \backslash H_{s,\mathbf{C}}/F^0 H_{s,\mathbf{C}}),$$

and a normal function $\nu \in \mathrm{NF}(S^*, \mathbf{H})$ (which is a holomorphic section of $J(\mathbf{H})$) corresponds to a short exact sequence

$$0 \to \mathbf{H} \to \mathbf{H}' \to \mathbf{Z}_{S^*} \to 0,$$

where the Griffiths transversality of ν corresponds to that of $\mathbf{H}'$. So we get

$$\mathrm{NF}(S^*, \mathbf{H}) = \mathrm{Ext}^1_{\mathrm{VMHS}}(\mathbf{Z}_{S^*}, \mathbf{H})\ (\subset J(\mathbf{H})(S^*)).$$

By construction [Ca], this is induced by taking the difference of the two local splittings σ_F and $\sigma_{\mathbf{Z}}$ of the above short exact sequence, where the σ_F is compatible with F and $\sigma_{\mathbf{Z}}$ is defined over $\mathbf{Z}$. Note that the cohomology class $\gamma(\nu) \in H^1(S^*, \mathbf{H})$ is defined by using the cohomology long exact sequence of the above short exact sequence.

Let $S^* \subset S$ be a partial compactification such that $S \setminus S^* \subset S$ is closed analytic.

3.2. Definition (Admissible normal functions with respect to $S^* \subset S$, see [Sa2]).

$$\mathrm{NF}(S^*, \mathbf{H})_S^{\mathrm{ad}} := \mathrm{Ext}^1_{\mathrm{VMHS}(S^*)_S^{\mathrm{ad}}}(\mathbf{Z}_{S^*}, \mathbf{H})\ (\subset J(\mathbf{H})(S^*)),$$

where $\mathrm{VMHS}(S^*)_S^{\mathrm{ad}}$ denotes the category of admissible variation of mixed Hodge structure with respect to $S^* \subset S$.

3.3. Definition. The category of admissible VMHS in the one-dimensional case is defined by the following two conditions of Steenbrink and Zucker [SZ] where $(S, S^*) = (\Delta, \Delta^*)$:

(a) The $\mathrm{Gr}_F^p \mathrm{Gr}_k^W \widehat{\mathcal{L}}$ are free $\mathcal{O}_\Delta$-modules (in the unipotent case).

(b) The relative monodromy filtration exists.

3.4. Remarks. (i) In the non-unipotent case, (a) is not sufficient, and we have to take a ramified covering to reduce to the unipotent monodromy case (or use the V-filtration).

(ii) Condition (b) in the weight -1 case is equivalent to a splitting of the short exact sequence of $\mathbf{Q}$-local systems over Δ^*.

(iii) The generalization to the higher dimensional case is by the curve test, see [Ka].

3.5. Remark. Taking a multivalued lifting $\widetilde{\nu}$ of ν in $\mathcal{V}$, conditions (a), (b) correspond to the conditions given by Green, Griffiths, Kerr [GGK] (in the unipotent case):

(a)′ $\widetilde{\nu}$ has a logarithmic growth.

(b)′ $T\widetilde{\nu} - \widetilde{\nu} \in \mathrm{Im}(T_{\mathbf{Q}} - id)$.

Note that the variation $T\widetilde{\nu} - \widetilde{\nu}$ gives the cohomology class of ν.

Using the theory of admissible normal functions, we can show the following.

3.6. Theorem [Sa2]. *Theorem (2.5) holds for any admissible normal function ν (not necessarily associated to an algebraic cycle) without assuming the hypothesis (C), and $J_S^C(\mathbf{H})$ has a structure of a complex Lie group over S.*

The key point is the following generalization of Theorem (2.3).

3.7. Proposition [Sa2]. *For an admissible normal function ν on S^*, ν extends to a section of $J_S^Z(\mathbf{H})$ if and only if $\gamma_0(\nu) = 0$.*

Here $\gamma_0(\nu)$ is the cohomology class of $\nu|_{\Delta^*}$ where $\Delta \subset S$ is a sufficiently small disk with center 0. Theorem (3.6) is then proved by setting

$$J_S^C(\mathbf{H})|_\Delta := \bigcup\nolimits_{g\in G_0} \left(\nu_g + J_S^Z(\mathbf{H})|_\Delta\right),$$

where $\nu_g \in \mathrm{NF}(\Delta^*, \mathbf{H}_{\Delta^*})_\Delta^{\mathrm{ad}}$ such that $\gamma_0(\nu_g) = g$. This is independent of the choice of ν_g by Proposition (3.7).

3.8. Remark. If $\mathbf{H}$ corresponds to an abelian scheme $\mathcal{A}$, then

$$\mathcal{A}_S^{\mathrm{an}} \xrightarrow{\sim} J_S^C(\mathbf{H}),$$

even in the non-unipotent case, see [Sa2], 4.5.

4. Generalization by Green, Griffiths and Kerr [GGK]

4.1. Problem. In general, $J_S^Z(\mathbf{H})$, $J_S^C(\mathbf{H})$ are not Hausdorff as is shown by an example in [Sa2], 3.5(iv) where H_∞ has type

$$(1, -1), (-1, 1), (0, -2), (-2, 0).$$

4.2. Theorem (Green, Griffith, Kerr) [GGK]. *Except for the last assertion on the structure of a complex Lie group over S, Theorem* (3.6) *also holds for a subspace $J_S^{\mathrm{GGK}}(\mathbf{H})$ of $J_S^{\mathcal{C}}(\mathbf{H})$ which is obtained by replacing $J_S^{\mathcal{Z}}(\mathbf{H})_0$ with*

$$J_S^{\mathrm{GGK}}(\mathbf{H})_0^0 := J(H_\infty^{\mathrm{inv}})\,(= H_{\infty,\mathbf{Z}}^{\mathrm{inv}}\backslash H_{\infty,\mathbf{C}}^{\mathrm{inv}}/F^0H_{\infty,\mathbf{C}}^{\mathrm{inv}}),$$

where $H_\infty^{\mathrm{inv}} := \operatorname{Ker} N \subset H_\infty$ with $N = \log T$.

Here the monodromy is assumed *unipotent.* Note that $J_S^{\mathrm{GGK}}(\mathbf{H})$ is not an analytic space in the usual sense, see [GGK]. We have moreover the following.

4.3. Theorem [Sa3]. *As a topological space endowed with the quotient topology, $J_S^{\mathrm{GGK}}(\mathbf{H})$ is Hausdorff* (*assuming* $\dim S = 1$).

4.4 Remark. These assertions can be extended to the case $\dim S > 1$ if $D := S \setminus S^*$ is *smooth.*

4.5. Corollary. *The closure of the zero locus of an admissible norma function is analytic if D is smooth.*

4.6. Remark. This is independently proved by P. Brosnan and G. Pearlstein using another method, and they recently give a proof in the general case [BP], see also [KNU], [Sl]. The generalization of (4.3) seems to be closely related to [CKS].

5. Generalization by Brosnan, Pearlstein and Saito [BPS]

5.1. Assume S^* smooth, but S may be singular. Consider the inclusions $j : S^* \hookrightarrow S$ and $i_s : \{s\} \hookrightarrow S$ for $s \in S$. Set

$$H_s := H^0 i_s^*(\mathbf{R}j_*\mathbf{H})\,(= \lim_{U \ni 0} H^0(U \cap S^*, \mathbf{H})),$$
$$J(H_s) := \operatorname{Ext}^1(\mathbf{Z}, H_s)\,(= H_{s,\mathbf{Z}}\backslash H_{s,\mathbf{C}}/F^0H_{s,\mathbf{C}}).$$

5.2. Identity component of the BPS-Néron model [BPS]. Define

$$J_S^{\mathrm{BPS}}(\mathbf{H})^0 := \textstyle\coprod_{s\in S} J(H_s) \text{ (set-theoretically)}.$$

Then $J_S^{\mathrm{BPS}}(\mathbf{H})^0$ is a topological space (using a resolution of S), and

$$J_S^{\mathrm{BPS}}(\mathbf{H})_{S_\alpha}^0 := J_S^{\mathrm{BPS}}(\mathbf{H})^0|_{S_\alpha}$$

is a Lie group over S_α for any stratum S_α of a Whitney stratification of (S, D). Moreover, ν defines a continuous section of $J_S^{\mathrm{BPS}}(\mathbf{H})^0$ if $\gamma_s(\nu) = 0$ ($\forall\, s \in D$). Indeed, ν corresponds to a short exact sequence

$$0 \to \mathbf{H} \to \mathbf{H}' \to \mathbf{Z}_{S^*} \to 0,$$

and·this induces a long exact sequence

$$0 \to H^0 i_s^* \mathbf{R}j_* \mathbf{H} \to H^0 i_s^* \mathbf{R}j_* \mathbf{H}' \to \mathbf{Z} \to H^1 i_s^* \mathbf{R}j_* \mathbf{H}.$$

So we have $\nu_s \in J(H_s)$ if $\gamma_s(\nu) = 0$.

For the reduction to the normal crossing case, set

$$\mathrm{NF}_{(s)}(S^*, \mathbf{H})_S^{\mathrm{ad}} := \{\nu \in \mathrm{NF}(S^*, \mathbf{H})_S^{\mathrm{ad}} \mid \gamma_s(\nu) = 0\}.$$

Then we have the following.

5.3. Proposition. *For $\pi : S' \to S$ proper with $S'^* := \pi^{-1}(S^*)$ smooth, we have the commutative diagram*

$$\begin{array}{ccc} \mathrm{NF}_{(s)}(S^*, \mathbf{H})_S^{\mathrm{ad}} & \to & \mathrm{NF}_{(s')}(S'^*, \pi^* \mathbf{H})_{S'}^{\mathrm{ad}} \\ \downarrow & & \downarrow \\ J(H_s) & \to & J(H_{s'}) \end{array}$$

5.4. Definition. Set $G_s := \{\text{images of admissible } \nu\} \subset H^1 i_s^* \mathbf{R}j_* \mathbf{H}$, and

$$G := \textstyle\coprod_{s \in S} G_s.$$

Then we can prove the following.

5.5. Theorem [BPS]. *There exists $J_S^{\mathrm{BPS}}(\mathbf{H})$ over S together with an exact sequence*

$$0 \to J_S^{\mathrm{BPS}}(\mathbf{H})^0 \to J_S^{\mathrm{BPS}}(\mathbf{H}) \to G \to 0,$$

and any admissible ν on S^* is extendable to a continuous section of $J_S^{\mathrm{BPS}}(\mathbf{H})$ over S.

This is shown by generalizing the gluing argument in the proof of Theorem (3.6) in the one-dimensional case.

5.6. Remarks. (i) A. Young [Yo] constructed a generalization of Néron model for families of Abelian varieties defined on the complement of a divisor with normal crossings where he assumes that the local monodromies are unipotent and the identity component is similar to the classical construction [Na].

(ii) C. Schnell [Sl] has given a definition of a Néron model whose identity component $J_S^{\mathrm{Sch}}(\mathbf{H})^0$ is Hausdorff by using the Hodge filtration of Hodge modules [Sa1] where the partial compactification S is smooth although $S \setminus S^*$ is not necessarily a divisor with normal crossings (see [SS] for the one-dimensional case).

(iii) In the case of abelian schemes over curves, we get something different from the classical Néron model if the monodromy is *non-unipotent.* Indeed, for a family of elliptic curves with non-unipotent monodromy, we have

$$J(H_0) = pt,$$

(since $H_0 := H^0 i_0^* \mathbf{H} = 0$), and there is a '*blow-down*' map

$$\mathcal{A}_S^{\mathrm{an}} = J_S^C(\mathbf{H}) \to J_S^{\mathrm{BPS}}(\mathbf{H}),$$

such that the image of $\mathcal{A}_{S,0}^{\mathrm{an}} = J_S^C(\mathbf{H})_0 = \mathbf{C}$ is $J(H_0) = pt$.

(iv) It is very difficult to determine G_0 even in the normal crossing case. Indeed, there is a cohomology class map

$$\mathrm{NF}((\Delta^*)^n, \mathbf{H}_{\mathbf{Q}})^{\mathrm{ad}}_{\Delta^n} \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbf{Q}, \mathcal{H}^1(\mathrm{IC}_{(\Delta^*)^n}\mathbf{H}_{\mathbf{Q}})_0),$$

which is surjective if H is a nilpotent orbit, but it can be non-surjective in general [Sa4]. However, this does not seem to contradicts the strategy of Green, Griffith for solving the Hodge conjecture since it seems to occur only in rather artificial occasions, e.g. when an unnecessary blowing-up is made.

(v) As is remarked by C. Schnell [Sl], the topology of the Néron models which graph any admissible normal functions with non-vanishing cohomology classes can be rather complicated even in the abelian scheme case [Yo] as is shown by the example below if $\dim S \geq 2$.

5.7. Example. Let $S = \Delta^2$, $S^* = (\Delta^*)^2$, and $\rho : S \to \Delta$ be the morphism defined by $\rho(t_1, t_2) = t_1 t_2$. Let $\mathbf{H}^1$ be the nilpotent orbit of weight -1 having an integral basis e_0, e_1 such that $Ne_0 = e_1$ and $F^0 H_{\infty,\mathbf{C}} = \mathbf{C} e_0$. Set $\mathbf{H} = \rho^* \mathbf{H}^1$. Then

$$\begin{aligned}
&G_0 \cong \mathbf{Z} \text{ (non-canonically)}, \quad G_s = 0 \ (s \neq 0),\\
&J_S^{\mathrm{BPS}}(\mathbf{H})^0 = \mathbf{C}^* \times \Delta^2 / \Gamma' \quad \text{with}\\
&\Gamma' = \{(x, t_1, t_2) \in \mathbf{C}^* \times (\Delta^*)^2 \mid x = (t_1 t_2)^k \ (k \in \mathbf{Z})\}.
\end{aligned}$$

For $(p, \alpha) \in \mathbf{Z} \times \mathbf{C}^*$, there is an admissible normal function $\nu_{p,\alpha}$ with respect to $S^* \subset S$ defined by

$$x = \alpha t_1^p = \alpha t_1^{k+p} t_2^k \bmod \Gamma' \text{ for any } k \in \mathbf{Z}.$$

Its cohomology class $\gamma_0(\nu_{p,\alpha}) \in G_0 \cong \mathbf{Z}$ is equal to p up to a sign, and the closure of its graph over S^* contains the 0-component $J_S^{\mathrm{BPS}}(\mathbf{H})_0^0 = \mathbf{C}^*$ of $J_S^{\mathrm{BPS}}(\mathbf{H})_0$ if $|p| \geq 2$ (restricting over a curve defined by $t_2 = \beta t_1^{|p|-1}$ with $\beta \in \mathbf{C}^*$). However, the extended section $\overline{\nu}_{p,\alpha}$ over S passes through the $\gamma_0(\nu_{p,\alpha})$-component of $J_S^{\mathrm{BPS}}(\mathbf{H})_0$ by construction. (Here the value $\overline{\nu}_{p,\alpha}(0)$ at the origin depends also on α, and this induces an isomorphism $\mathbf{Z} \times \mathbf{C}^* \simeq J_S^{\mathrm{BPS}}(\mathbf{H})_0$. In particular, for any admissible normal function ν, there is a unique $(p, \alpha) \in \mathbf{Z} \times \mathbf{C}^*$ such that $\overline{\nu}(0) = \overline{\nu}_{p,\alpha}(0)$. Then the above assertion on the closure of the graph holds for any admissible function ν with $|\gamma_0(\nu)| \geq 2$.)

The above argument implies that $J_S^{\mathrm{BPS}}(\mathbf{H})$ cannot be Hausdorff as a topological space, and this would be the same for the Néron models in [Sl], [Yo] which should coincide with $J_S^{\mathrm{BPS}}(\mathbf{H})$ for this simple example (even if the identity component $J_S^{\mathrm{Sch}}(\mathbf{H})^0$ in [Sl] is Hausdorff in general).

References

[BP] P. Brosnan and G. Pearlstein, On the algebraicity of the zero locus of an admissible normal function, preprint (arXiv:0910.0628)

[BPS] P. Brosnan, G. Pearlstein and M. Saito, A generalization of the Néron models of Green, Griffiths and Kerr, preprint (arXiv:0809.5185)

[Ca] J. Carlson, Extensions of mixed Hodge structures, in Journées de Géométrie Algébrique d'Angers 1979, Sijthoff-Noordhoff Alphen a/d Rijn, 1980, pp. 107–127.

[CKS] E. Cattani, A. Kaplan and W. Schmid, Degeneration of Hodge structures, Ann. of Math. 123 (1986), 457–535.

[Cl] H. Clemens, The Néron model for families of intermediate Jacobians acquiring "algebraic" singularities, Publ. Math. IHES 58 (1983), 5–18.

[D1] P. Deligne, Equations différentiellesà points singuliers réguliers, Lect. Notes in Math. vol. 163, Springer, Berlin, 1970.

[D2] P. Deligne, Théorie de Hodge II, Publ. Math. IHES 40 (1971), 5–58.

[EZ] F. El Zein and S. Zucker, Extendability of normal functions associated to algebraic cycles, in Topics in transcendental algebraic geometry, Ann. Math. Stud., 106, Princeton Univ. Press, Princeton, N.J., 1984, pp. 269–288.

[GGK] M. Green, P. Griffiths and M. Kerr, Néron models and limits of Abel-Jacobi mappings (preprint).

[Ka] M. Kashiwara, A study of variation of mixed Hodge structure, Publ. RIMS, Kyoto Univ. 22 (1986), 991–1024.

[KNU] K. Kato, C. Nakayama and S. Usui, Moduli of log mixed Hodge structures (arXiv:0910.4454).

[Na] Y. Namikawa, A new compactification of the Siegel space and degeneration of Abelian varieties, I and II, Math. Ann. 221 (1976), 97–141 and 201–241.

[Sa1] M. Saito, Mixed Hodge modules, Publ. RIMS, Kyoto Univ. 26 (1990), 221–333.

[Sa2] M. Saito, Admissible normal functions, J. Algebraic Geom. 5 (1996), 235–276.

[Sa3] M. Saito, Hausdorff property of the Néron models of Green, Griffiths and Kerr (arXiv:0803.2771).

[Sa4] M. Saito, Cohomology classes of admissible normal functions, preprint (arXiv:0904.1593).

[SS] M. Saito and C. Schnell, A variant of Néron models over curves, preprint (arXiv:0909.4276).

[Sd] W. Schmid, Variation of Hodge structure: The singularities of the period mapping, Inv. Math. 22 (1973), 211–319.

[Sl] C. Schnell, Complex analytic Néron models for arbitrary families of intermediate Jacobians, preprint (arXiv:0910.0662).

[SZ] J.H.M. Steenbrink and S. Zucker, Variation of mixed Hodge structure, I, Inv. Math. 80 (1985), 489–542.

[Yo] A. Young, Complex analytic Néron models for degenerating Abelian varieties over higher dimensional parameter spaces, Ph. D. Thesis, Princeton University, 2008.

[Zu] S. Zucker, Generalized intermediate Jacobians and the theorem on normal functions, Inv. Math. 33 (1976),185–222.

Nov. 19, 2009, v.1