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Kyoto University
A period map for cubic surfaces

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1 Construction of the period map

In this report, we construct a period map for cubic surfaces, and we prove the injectivity of the period map. Let \( X = X_F \subset \mathbb{P}^3 \) be a nonsingular cubic surface defined by \( F \in \mathbb{C}[x_0, \ldots, x_3] \). We remark that the cubic surface \( X \) has no holomorphic 2-form, therefore we cannot have a nontrivial Hodge structure by the period integral on \( X \) itself. We will consider another variety associated with \( X \). Let \( B = B_F \) be the zeros of the Hessian of the cubic polynomial \( F \):

\[
B_F = \{ p \in X \mid \det \left( \frac{\partial^2 F}{\partial x_i \partial x_j}(p) \right)_{0 \leq i,j \leq 3} = 0 \},
\]

and let \( \phi : Y = Y_F \to X \) be the double cover branched along \( B \). We remark that \( B \) has at most node as its singularity, and \( Y \) is the canonical resolution of the finite double cover. Here we want to classify the isomorphism class of \( X \) by using period integral on \( Y \).

The double cover \( Y \) is a minimal surface of general type with the geometric genus \( p_g(Y) = 4 \), the irregularity \( q(Y) = 0 \) and the square of the canonical divisor \( K_Y^2 = 6 \). Then the second Betti number is \( h^2(Y, \mathcal{Z}) = 52 \). The Néron-Severi group \( \text{NS}(Y) \) is contained in \( H^2(Y, \mathcal{Z}) \), and the Picard number depends on the equation \( F \). We can prove that the Picard number of \( Y \) for the generic equation is 28. We denote by \( H^2_{\text{inv}}(Y, \mathcal{Z}) \) the subgroup of rank 28 in \( H^2(Y, \mathcal{Z}) \) which corresponds to the Néron-Severi group of the generic equation, and we denote by \( H^2_{\text{var}}(Y, \mathcal{Z}) \) the subgroup of rank 24 in \( H^2(Y, \mathcal{Z}) \) orthogonal to \( H^2_{\text{inv}}(Y, \mathcal{Z}) \) by the symmetric form

\[
\langle , \rangle_Y : H^2(Y, \mathcal{Z}) \times H^2(Y, \mathcal{Z}) \to H^4(Y, \mathcal{Z}) \simeq \mathcal{Z}.
\]

We study the Hodge structure defined on \( H^2_{\text{var}}(Y, \mathcal{Z}) \).

Let \( (H, \langle , \rangle) \) be a lattice which is isomorphic to \( (H^2_{\text{var}}(Y, \mathcal{Z}), \langle , \rangle_Y) \). We have an Hermitian form on \( H_C = \mathbb{C} \otimes_{\mathbb{Z}} H \) by

\[
\langle , \rangle : H_C \times H_C \to \mathbb{C}; (\alpha, \beta) \mapsto \langle \alpha, \beta \rangle,
\]
where $\overline{\beta}$ denotes the complex conjugate of $\beta \in H_C$. We define the classifying space of the polarized Hodge structure by

$$D = \{ W \in \text{Grass}(4, H_C) \mid W \subset W^\perp, \langle \cdot, \cdot \rangle_W > 0 \},$$

and we call elements of $D$ polarized Hodge structure on $H$. We define the polarized Hodge structure on $H^2_{\text{var}}(Y, Z)$ by the image of the injective homomorphism

$$H^0(Y, \Omega_Y^2) \longrightarrow \text{Hom}\left( \frac{H^2(Y, Z)}{H^2_{\text{inv}}(Y, Z)}, \mathbb{C} \right) \simeq H^2_{\text{var}}(Y, \mathbb{C}); \eta \mapsto \left[ \alpha \mapsto \int_{\alpha} \eta \right].$$

Let $C \subset \text{Grass}(1, H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(3)))$ be the space of smooth cubic surfaces. We fix a base point $[F_0] \in C$ and an isomorphism $(H^2_{\text{var}}(Y_{F_0}, Z), \langle \cdot, \cdot \rangle_{Y_{F_0}}) \simeq (H, \langle \cdot, \cdot \rangle)$. Then the monodromy group $\Gamma$ is defined as the image of the monodromy representation

$$\pi_1(C, [F_0]) \longrightarrow \text{Aut}(H, \langle \cdot, \cdot \rangle),$$

and we have a period map

$$C \longrightarrow \Gamma \backslash D; [F] \mapsto [H^0(Y_F, \Omega_Y^2) \subset H^2_{\text{var}}(Y_F, \mathbb{C}) \simeq H^2_{\text{var}}(Y_{F_0}, \mathbb{C}) \simeq H_C],$$

where the isomorphism $H^2_{\text{var}}(Y_F, \mathbb{C}) \simeq H^2_{\text{var}}(Y_{F_0}, \mathbb{C})$ is defined by a path from $[F_0]$ to $[F]$ in $C$. This map gives the period map $\Psi : \mathcal{M} \rightarrow \Gamma \backslash D$, where $\mathcal{M} = C / \text{PGL}(4)$ is the moduli space of nonsingular cubic surfaces.

**Theorem 1.1.** The period map $\Psi$ is injective.

Indeed, this theorem depends on the injectivity of another period map constructed by Allcock-Carlson-Toledo [1]. In the next section, we review the work of Allcock-Carlson-Toledo.

## 2 The period map by Allcock-Carlson-Toledo

Let $X = X_F \subset \mathbb{P}^3$ be a nonsingular cubic surface defined by $F(x_0, \ldots, x_3)$, and let $V = V_F \subset \mathbb{P}^4$ be the cubic 3-fold defined by the equation $F(x_0, \ldots, x_3) = x_4^3$. Then the natural projection

$$\rho : V \longrightarrow \mathbb{P}^3; [x_0 : \cdots : x_3 : x_4] \longmapsto [x_0 : \cdots : x_3]$$

is the triple Galois cover branched along $X$. Let $\sigma$ be a generator of the Galois group. Since $H^3(V, \mathbb{Z})$ has no invariant vector by the Galois action, we consider $H^3(V, \mathbb{Z})$ as a $\mathbb{Z}[\omega]$-module of rank 5 by $\omega \alpha = \sigma^*(\alpha)$ for $\alpha \in H^3(V, \mathbb{Z})$, where $\omega \in \mathbb{C}$ denotes a primitive 3-rd root of unity. By using the alternating form

$$\langle \cdot, \cdot \rangle_V : H^3(V, \mathbb{Z}) \times H^3(V, \mathbb{Z}) \longrightarrow H^6(V, \mathbb{Z}) \simeq \mathbb{Z},$$

we consider the period map $\Psi : \mathcal{M} \rightarrow \Gamma \backslash D$.
we define a Hermitian form on $H^3(V, \mathbb{Z})$ by
\[
    h_V : H^3(V, \mathbb{Z}) \times H^3(V, \mathbb{Z}) \to \mathbb{Z}[\omega]; \quad (\alpha, \beta) \mapsto \langle \alpha, \omega \beta \rangle_V - \omega \langle \alpha, \beta \rangle_V.
\]
Then we have a natural isomorphism of Hermitian space $H^3(V, \mathbb{C})_\omega \simeq \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H^3(V, \mathbb{Z})$, where
\[
    H^3(V, \mathbb{C})_\omega = \{ \alpha \in H^3(V, \mathbb{C}) \mid \sigma^*(\alpha) = \omega \alpha \}
\]
is the eigenspace of $\omega$ in $H^3(V, \mathbb{C})$ by the action $\sigma^*$, and the Hermitian form on $H^3(V, \mathbb{C})_\omega$ is defined by
\[
    H^3(V, \mathbb{C})_\omega \times H^3(V, \mathbb{C})_\omega \to \mathbb{C}; \quad (\alpha, \beta) \mapsto (\omega^2 - \omega) \langle \alpha, \overline{\beta} \rangle_V.
\]
Let $(H', h)$ be a Hermitian $\mathbb{Z}[\omega]$-lattice which is isomorphic to $(H^3(V, \mathbb{Z}), h_V)$. The period domain of Allcock-Carlson-Toledo is defined by
\[
    D' = \{ E \in \text{Grass}(4, \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H') \mid h|_E > 0 \},
\]
which is isomorphic to the 4-dimensional complex ball
\[
    \Delta = \{ (a_1, \ldots, a_4) \in \mathbb{C}^4 \mid |a_1| + \cdots + |a_4| < 1 \}.
\]
We fix an isomorphism $(H^3(V_{F_0}, \mathbb{Z}), h_{V_{F_0}}) \simeq (H', h)$. Then an element of $D'$ is defined by
\[
    H^{2,1}(V_{F})_\omega \subset H^3(V_F, \mathbb{C})_\omega \simeq \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H^3(V_F, \mathbb{Z}) \simeq \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H^3(\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{C}),
\]
where the isomorphism $H^3(V_F, \mathbb{Z}) \simeq H^3(V_{F_0}, \mathbb{Z})$ is defined by a path from $[F_0]$ to $[F]$ in $C$. This gives a period map
\[
    \Psi' : \mathcal{M} \to \Gamma' \backslash D'; \quad [F] \mapsto [H^{2,1}(V)_\omega \subset \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H'],
\]
where $\Gamma'$ is the monodromy group.

**Theorem 2.1** (Hartling [4], Allcock-Carlson-Toledo [1]). *The period map $\Psi'$ is injective.*

This theorem depends on the Torelli theorem for cubic 3-folds by Clemens-Griffiths [2] and Tjurin [6].

### 3 Relation between the period maps $\Psi$ and $\Psi'$

In this section, we will see that the polarized Hodge structure $(H^2_{\text{var}}(Y, \mathbb{Z}), \langle, \rangle_Y)$ is obtained from the Hodge structure of Allcock-Carlson-Toledo $(H^3(V, \mathbb{Z}), h_V)$. Let
\((H', h)\) be a Hermitian \(\mathbb{Z}[\omega]\)-lattice which is isomorphic to \((H^3(V, \mathbb{Z}), h_\nu)\). The cyclic group \(\mathbb{Z}/3\mathbb{Z}\) acts on \(H'\) by
\[
\mathbb{Z}/3\mathbb{Z} \times H' \longrightarrow H'; \quad ([m], u) \longmapsto \omega^m u,
\]
and we have an alternating form on \(H'\) by
\[
\bigwedge^2_{\mathbb{Z}} H' \longrightarrow \mathbb{Z}; \quad u \wedge v \longmapsto \frac{1}{\omega^2 - \omega} (h(u, v) - \overline{h(u, v)}).
\]
Let \(\alpha_0, \ldots, \alpha_4, \beta_0, \ldots, \beta_4\) be a symplectic basis of \(H'\). We set \(\theta = \sum_{i=0}^4 \alpha_i \wedge \beta_i \in \bigwedge^2_{\mathbb{Z}} H'\). Then we have a lattice \((\bigwedge^2_{\mathbb{Z}} H')_0^{\mathbb{Z}/3\mathbb{Z}}, \langle, \rangle_h\). We define a symmetric form on \(\bigwedge^2_{\mathbb{Z}} H'\) by
\[
\langle , \rangle_h : \bigwedge^2_{\mathbb{Z}} H' \times \bigwedge^2_{\mathbb{Z}} H' \longrightarrow \bigwedge^{10}_{\mathbb{Z}} H' \cong \mathbb{Z}; \quad (u_1 \wedge u_2, v_1 \wedge v_2) \longmapsto \frac{1}{6} \theta \wedge^3 u_1 \wedge u_2 \wedge v_1 \wedge v_2.
\]
We denote by \((\bigwedge^2_{\mathbb{Z}} H')_0\) the kernel of the alternating form \(\bigwedge^2_{\mathbb{Z}} H' \rightarrow \mathbb{Z}\). Then we have a lattice \((\bigwedge^2_{\mathbb{Z}} H')_0^{\mathbb{Z}/3\mathbb{Z}}, \langle, \rangle_h\). We set a \(\mathbb{C}\)-linear map \(j\) by
\[
j : \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H' \simeq H'_C, \omega \subset H'_C = \mathbb{C} \otimes_{\mathbb{Z}} H'; \quad 1 \otimes v \longmapsto \frac{1}{\omega^2 - \omega} (\omega^2 \otimes v - 1 \otimes \omega v).
\]
Then a Hodge structure of Alcock-Carlson-Toledo \(E \subset \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H'\) gives a polarized Hodge structure on \((\bigwedge^2_{\mathbb{Z}} H')_0^{\mathbb{Z}/3\mathbb{Z}}, \langle, \rangle_h\) by
\[
j(E) \wedge \overline{j(E^\perp)} \subset (\bigwedge^2 cH'_C)_0^{\mathbb{Z}/3\mathbb{Z}}.
\]
**Theorem 3.1 ([5]).** There is a natural isomorphism of polarized Hodge structures
\[
((\bigwedge^2 QH^3(V, Q)(1))_0^{\text{Gal}(\rho)}, \frac{1}{3} \langle, \rangle_{h^\nu}) \simeq (H^2_{\text{var}}(Y, Q), \langle, \rangle_Y).
\]
Let \((H, \langle, \rangle)\) be a lattice which is isomorphic to \((H^2_{\text{var}}(Y, \mathbb{Z}), \langle, \rangle_Y)\), and let
\[
\iota : ((\bigwedge^2 H'_Q)_0^{\mathbb{Z}/3\mathbb{Z}}, \frac{1}{3} \langle, \rangle_h) \simeq (H_Q, \langle, \rangle).
\]
be the isomorphism of lattices given by Theorem 3.1 for a base point \([F_0] \in C\). Then Theorem 3.1 means that the diagram
\[
\begin{array}{ccc}
& & \mathcal{M} \\
\Psi' \downarrow & \nearrow \psi & \\
\Gamma \backslash D' & \longrightarrow & \Gamma \backslash D
\end{array}
\]
is commutative, where the morphism \(\Phi\) is defined by
\[
\Phi : D' \longrightarrow D; \quad E \longmapsto \iota(j(E) \wedge \overline{j(E^\perp)}).
\]
And we can prove that \(\Phi : D' \rightarrow D\) is injective. These imply Theorem 1.1.
4 Geometry of lines

In this section, we explain the isomorphism in Theorem 3.1. Let $\Lambda(P^n)$ be the Grassmannian variety of lines in $P^n$. Let $X$ be a nonsingular cubic surface. We define a subvariety of $P^3 \times \Lambda(P^3)$ by

$$ Y = \{(p, L) \in P^3 \times \Lambda(P^3) \mid \text{mult}_p(L.X) \geq 3\}, $$

which is the double cover branched along $B$ by the first projection

$$ \phi : Y \rightarrow X; \ (p, L) \mapsto p. $$

We define a divisor on $Y$ by

$$ Y_{\infty} = \{(p, L) \in P^3 \times \Lambda(P^3) \mid p \in L \subset X\} = L_1^+ \cup \cdots \cup L_{27}^+. $$

We remark that $X$ contains 27 lines $L_1, \ldots, L_{27}$ in $P^3$, and $L_i^+$ denotes the component of $Y_{\infty}$ which corresponds to $L_i$. Then we can prove that

$$ H_{\text{inv}}^2(Y, Z) = \phi^*H^2(X, Z) + \sum_{i=1}^{27} Z[L_i^+]. $$

Let

$$ \pi : Y \rightarrow Z \subset \Lambda(P^3); \ (p, L) \mapsto L $$

be the second projection, where $Z$ denotes its image. Then $\pi$ is the birational morphism which contracts curves $L_i^+$, and we have an isomorphism of Hodge structures

$$ H_{\text{prim}}^2(Z, Q) \simeq H_{\text{var}}^2(Y, Q), \quad (1) $$

where $H_{\text{prim}}^2(Z, Q)$ is the subspace in $H^2(Z, Q)$ orthogonal to the class of the hyperplane section by the Plücker embedding of $\Lambda(P^3)$.

Next we review some results on Fano surface of lines on cubic 3-folds by Clemens-Griffiths [2] and Tjurin [6]. Let $V \subset P^4$ be a nonsingular cubic 3-fold, and let

$$ S = \{L \in \Lambda(P^4) \mid L \subset V\} $$

be the Fano surface of lines on $V$. Then there are isomorphisms of Hodge structures

$$ \bigwedge^2 QH^1(S, Q) \simeq H^2(S, Q), \quad (2) $$

$$ H^3(V, Q)(1) \simeq H^1(S, Q) \ (3) $$

by [2], [3] or [6].
Let $X \subset \mathbb{P}^{3}$ be a nonsingular cubic surface, and let $V \subset \mathbb{P}^{4}$ be the cubic 3-fold which is the triple Galois cover $\rho : V \rightarrow \mathbb{P}^{3}$ branched along $X$. If $L$ is a line on $V$, then the image of the projection $\rho(V)$ is a line of $\mathbb{P}^{3}$ which is contained in $X$ or intersects $X$ with the multiplicity 3. Therefore we have the triple Galois cover

$$S \rightarrow Z; \; L \mapsto \rho(L),$$

and we have an isomorphism of Hodge structures

$$H^{2}(S, \mathbb{Q})^{\text{Gal}(\rho)} \simeq H^{2}(Z, \mathbb{Q}). \quad (4)$$

By these isomorphisms (1) – (4), we have the isomorphism in Theorem 3.1.

References


