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A period map for cubic surfaces

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1 Construction of the period map

In this report, we construct a period map for cubic surfaces, and we prove the injectivity of the period map. Let $X = X_F \subset \mathbb{P}^3$ be a nonsingular cubic surface defined by $F \in \mathbb{C}[x_0, \ldots, x_3]$. We remark that the cubic surface $X$ has no holomorphic 2-form, therefore we cannot have a nontrivial Hodge structure by the period integral on $X$ itself. We will consider another variety associated with $X$. Let $B = B_F$ be the zeros of the Hessian of the cubic polynomial $F$;

$$B_F = \{ p \in X \mid \det \left( \frac{\partial^2 F}{\partial x_i \partial x_j}(p) \right)_{0 \leq i,j \leq 3} = 0 \},$$

and let $\phi : Y = Y_F \to X$ be the double cover branched along $B$. We remark that $B$ has at most node as its singularity, and $Y$ is the canonical resolution of the finite double cover. Here we want to classify the isomorphism class of $X$ by using period integral on $Y$.

The double cover $Y$ is a minimal surface of general type with the geometric genus $p_g(Y) = 4$, the irregularity $q(Y) = 0$ and the square of the canonical divisor $K_Y^2 = 6$. Then the second Betti number is $h^2(Y, \mathbb{Z}) = 52$. The Néron-Severi group $\text{NS}(Y)$ is contained in $H^2(Y, \mathbb{Z})$, and the Picard number depends on the equation $F$. We can prove that the Picard number of $Y$ for the generic equation is 28. We denote by $H^2_{\text{inv}}(Y, \mathbb{Z})$ the subgroup of rank 28 in $H^2(Y, \mathbb{Z})$ which corresponds to the Néron-Severi group of the generic equation, and we denote by $H^2_{\text{var}}(Y, \mathbb{Z})$ the subgroup of rank 24 in $H^2(Y, \mathbb{Z})$ orthogonal to $H^2_{\text{inv}}(Y, \mathbb{Z})$ by the symmetric form

$$\langle \ , \ \rangle_Y : H^2(Y, \mathbb{Z}) \times H^2(Y, \mathbb{Z}) \to H^4(Y, \mathbb{Z}) \simeq \mathbb{Z}.$$

We study the Hodge structure defined on $H^2_{\text{var}}(Y, \mathbb{Z})$.

Let $(H, \langle \ , \ \rangle)$ be a lattice which is isomorphic to $(H^2_{\text{var}}(Y, \mathbb{Z}), \langle \ , \ \rangle_Y)$. We have an Hermitian form on $H_C = \mathbb{C} \otimes_{\mathbb{Z}} H$ by

$$\langle \ , \ \rangle : H_C \times H_C \to \mathbb{C}; (\alpha, \beta) \mapsto \langle \alpha, \overline{\beta} \rangle,$$
where $\overline{\beta}$ denotes the complex conjugate of $\beta \in H_C$. We define the classifying space of the polarized Hodge structure by

$$D = \{ W \in \text{Grass}(4, H_C) \mid W \subset W^\perp, \langle \cdot, \cdot \rangle|_W > 0 \},$$

and we call elements of $D$ polarized Hodge structure on $H$. We define the polarized Hodge structure on $H_{\text{var}}^2(Y, Z)$ by the image of the injective homomorphism

$$H^0(Y, \Omega^2_Y) \longrightarrow \text{Hom}\left(\frac{H^2(Y, Z)}{H^2_{\text{inv}}(Y, Z)}, C\right) \simeq H_{\text{var}}^2(Y, C); \eta \mapsto \left[ \alpha \mapsto \int_{\alpha} \eta \right].$$

Let $C \subset \text{Grass}(1, H^0(P^3, \mathcal{O}_{P^3}(3)))$ be the space of smooth cubic surfaces. We fix a base point $[F_0] \in C$ and an isomorphism $(H_{\text{var}}^2(Y_{F_0}, Z), \langle \cdot, \cdot \rangle_{Y_{F_0}}) \simeq (H, \langle \cdot, \cdot \rangle)$. Then the monodromy group $\Gamma$ is defined as the image of the monodromy representation

$$\pi_1(C, [F_0]) \longrightarrow \text{Aut}(H, \langle \cdot, \cdot \rangle),$$

and we have a period map

$$C \longrightarrow \Gamma \backslash D; \ [F] \mapsto [H^0(Y_F, \Omega^2_{Y_F}) \subset H_{\text{var}}^2(Y_F, C) \simeq H_{\text{var}}^2(Y_{F_0}, C) \simeq H_C],$$

where the isomorphism $H_{\text{var}}^2(Y_F, C) \simeq H_{\text{var}}^2(Y_{F_0}, C)$ is defined by a path from $[F_0]$ to $[F]$ in $C$. This map gives the period map $\Psi: \mathcal{M} \to \Gamma \backslash D$, where $\mathcal{M} = C/\text{PGL}(4)$ is the moduli space of nonsingular cubic surfaces.

**Theorem 1.1.** The period map $\Psi$ is injective.

Indeed, this theorem depends on the injectivity of another period map constructed by Allcock-Carlson-Toledo [1]. In the next section, we review the work of Allcock-Carlson-Toledo.

## 2 The period map by Allcock-Carlson-Toledo

Let $X = X_F \subset P^3$ be a nonsingular cubic surface defined by $F(x_0, \ldots, x_3)$, and let $V = V_F \subset P^4$ be the cubic 3-fold defined by the equation $F(x_0, \ldots, x_3) = x_4^3$. Then the natural projection

$$\rho: V \longrightarrow P^3; \ [x_0 : \cdots : x_3 : x_4] \mapsto [x_0 : \cdots : x_3]$$

is the triple Galois cover branched along $X$. Let $\sigma$ be a generator of the Galois group. Since $H^3(V, Z)$ has no invariant vector by the Galois action, we consider $H^3(V, Z)$ as a $Z[\omega]$-module of rank 5 by $\omega \alpha = \sigma^*(\alpha)$ for $\alpha \in H^3(V, Z)$, where $\omega \in C$ denotes a primitive 3-rd root of unity. By using the alternating form

$$\langle \cdot, \cdot \rangle_V : H^3(V, Z) \times H^3(V, Z) \longrightarrow H^6(V, Z) \simeq Z,$$
we define a Hermitian form on $H^3(V, \mathbb{Z})$ by

$$h_V : H^3(V, \mathbb{Z}) \times H^3(V, \mathbb{Z}) \rightarrow \mathbb{Z}[\omega]; \ (\alpha, \beta) \mapsto \langle \alpha, \omega \beta \rangle_V - \omega \langle \alpha, \beta \rangle_V.$$ 

Then we have a natural isomorphism of Hermitian space $H^3(V, \mathbb{C})_\omega \simeq \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H^3(V, \mathbb{Z})$, where

$$H^3(V, \mathbb{C})_\omega = \{ \alpha \in H^3(V, \mathbb{C}) \mid \sigma^*(\alpha) = \omega \alpha \}$$

is the eigenspace of $\omega$ in $H^3(V, \mathbb{C})$ by the action $\sigma^*$, and the Hermitian form on $H^3(V, \mathbb{C})_\omega$ is defined by

$$H^3(V, \mathbb{C})_\omega \times H^3(V, \mathbb{C})_\omega \rightarrow \mathbb{C}; \ (\alpha, \beta) \mapsto (\omega^2 - \omega) \langle \alpha, \bar{\beta} \rangle_V.$$

Let $(H', h)$ be a Hermitian $\mathbb{Z}[\omega]$-lattice which is isomorphic to $(H^3(V, \mathbb{Z}), h_V)$. The period domain of Allcock-Carlson-Toledo is defined by

$$D' = \{ E \in \text{Grass}(4, \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H') \mid h|_E > 0 \},$$

which is isomorphic to the 4-dimensional complex ball

$$\Delta = \{ (a_1, \ldots, a_4) \in \mathbb{C}^4 \mid |a_1| + \cdots + |a_4| < 1 \}.$$ 

We fix an isomorphism $(H^3(V_{F_0}, \mathbb{Z}), h_{V_{F_0}}) \simeq (H', h)$. Then an element of $D'$ is defined by

$$H^{2,1}(V_F)_\omega \subset H^3(V_F, \mathbb{C})_\omega \simeq \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H^3(V_F, \mathbb{Z}) \simeq \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H^3(V_{F_0}, \mathbb{Z}) \simeq \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H',$$

where the isomorphism $H^3(V_F, \mathbb{Z}) \simeq H^3(V_{F_0}, \mathbb{Z})$ is defined by a path from $[F_0]$ to $[F]$ in $\mathbb{C}$. This gives a period map

$$\Psi' : \mathcal{M} \rightarrow \Gamma' \backslash D'; \ [F] \mapsto [H^{2,1}(V)_\omega \subset \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H'],$$

where $\Gamma'$ is the monodromy group.

**Theorem 2.1** (Hartling [4], Allcock-Carlson-Toledo [1]). *The period map $\Psi'$ is injective.*

This theorem depends on the Torelli theorem for cubic 3-folds by Clemens-Griffiths [2] and Tjurin [6].

### 3 Relation between the period maps $\Psi$ and $\Psi'$

In this section, we will see that the polarized Hodge structure $(H^3_{\text{var}}(Y, \mathbb{Z}), \langle , \rangle_Y)$ is obtained from the Hodge structure of Allcock-Carlson-Toledo $(H^3(V, \mathbb{Z}), h_V)$. Let
$(H', h)$ be a Hermitian $\mathbb{Z}[\omega]$-lattice which is isomorphic to $(H^3(V, \mathbb{Z}), h_V)$. The cyclic group $\mathbb{Z}/3\mathbb{Z}$ acts on $H'$ by

$$\mathbb{Z}/3\mathbb{Z} \times H' \to H'; ([m], u) \mapsto \omega^m u,$$

and we have a alternating form on $H'$ by

$$\bigwedge_2^2 H' \to \mathbb{Z}; u \wedge v \mapsto \frac{1}{\omega^2 - \omega}(h(u, v) - \overline{h(u, v)}).$$

Let $\alpha_0, \ldots, \alpha_4, \beta_0, \ldots, \beta_4$ be a symplectic basis of $H'$. We set $\theta = \sum_{i=0}^4 \alpha_i \wedge \beta_i \in \bigwedge_2^2 H'$, which does not depend on the choice of the symplectic basis. We define a symmetric form on $\bigwedge_2^2 H'$ by

$$j : \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H^f \simeq H_{C, \omega}^f \subset H' = \mathbb{C} \otimes_{\mathbb{Z}} H'; 1 \otimes v \mapsto \frac{1}{\omega^2 - \omega}(\omega^2 \otimes v - 1 \otimes \omega v).$$

Then a Hodge structure of Allcock-Carlson-Toledo $E \subset \mathbb{C} \otimes_{\mathbb{Z}[\omega]} H'$ gives a polarized Hodge structure on $((\bigwedge_2^2 H')_{0}^{Z/3Z}, \langle, \rangle_h)$ by

$$j(E) \wedge \overline{j(E^\perp)} \subset \bigwedge_2^2 H_{C, \omega}^f \subset \bigwedge_2^2 H'_{C, \omega} \subset \bigwedge_2^2 H' \simeq \mathbb{Z}. $$

Theorem 3.1 ([5]). There is a natural isomorphism of polarized Hodge structures

$$\left((\bigwedge_2^2 \mathbb{Q}H^3(V, \mathbb{Q})(1))_{0}^{\mathbb{Q}^{\text{Gal}(\rho)}} \frac{1}{3} \langle, \rangle_{h_V} \right) \simeq (H^2_{\text{var}}(Y, \mathbb{Q}), \langle, \rangle_Y).$$

Let $(H, \langle, \rangle)$ be a lattice which is isomorphic to $(H^2_{\text{var}}(Y, \mathbb{Z}), \langle, \rangle_Y)$, and let

$$\iota : \left((\bigwedge_2^2 H'_{0}^{Z/3Z}) \frac{1}{3} \langle, \rangle_h \right) \simeq (H_{Q}, \langle, \rangle_{Y}).$$

be the isomorphism of lattices given by Theorem 3.1 for a base point $[F_0] \in \mathbb{C}$. Then Theorem 3.1 means that the diagram

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\psi'} & \mathcal{M} \\
\Gamma \backslash D' & \xrightarrow{\Phi} & \Gamma \backslash D
\end{array}$$

is commutative, where the morphism $\Phi$ is defined by

$$\Phi : D' \to D; E \mapsto \iota(j(E) \wedge \overline{j(E^\perp)}).$$

And we can prove that $\Phi : D' \to D$ is injective. These imply Theorem 1.1.
4 Geometry of lines

In this section, we explain the isomorphism in Theorem 3.1. Let \( \Lambda(P^n) \) be the Grassmannian variety of lines in \( P^n \). Let \( X \) be a nonsingular cubic surface. We define a subvariety of \( P^3 \times \Lambda(P^3) \) by

\[
Y = \{(p, L) \in P^3 \times \Lambda(P^3) \mid \text{mult}_p(L.X) \geq 3\},
\]

which is the double cover branched along \( B \) by the first projection

\[
\phi: Y \to X; \ (p, L) \mapsto p.
\]

We define a divisor on \( Y \) by

\[
Y_\infty = \{(p, L) \in P^3 \times \Lambda(P^3) \mid p \in L \subset X\} = L_1^+ \sqcup \cdots \sqcup L_{27}^+.
\]

We remark that \( X \) contains 27 lines \( L_1, \ldots, L_{27} \) in \( P^3 \), and \( L_i^+ \) denotes the component of \( Y_\infty \) which corresponds to \( L_i \). Then we can prove that

\[
H^{2}_{\text{inv}}(Y, Z) = \phi^*H^{2}(X, Z) + \sum_{i=1}^{27}Z[L_i^+].
\]

Let

\[
\pi: Y \to Z \subset \Lambda(P^3); \ (p, L) \mapsto L
\]

be the second projection, where \( Z \) denotes its image. Then \( \pi \) is the birational morphism which contracts curves \( L_i^+ \), and we have an isomorphism of Hodge structures

\[
H^{2}_{\text{prim}}(Z, \mathbb{Q}) \simeq H^{2}_{\text{var}}(Y, \mathbb{Q}),
\]

where \( H^{2}_{\text{prim}}(Z, \mathbb{Q}) \) is the subspace in \( H^2(Z, \mathbb{Q}) \) orthogonal to the class of the hyperplane section by the Plücker embedding of \( \Lambda(P^3) \).

Next we review some results on Fano surface of lines on cubic 3-folds by Clemens-Griffiths [2] and Tjurin [6]. Let \( V \subset P^4 \) be a nonsingular cubic 3-fold, and let

\[
S = \{L \in \Lambda(P^4) \mid L \subset V\}
\]

be the Fano surface of lines on \( V \). Then there are isomorphisms of Hodge structures

\[
\wedge^2_{\mathbb{Q}}H^{1}(S, \mathbb{Q}) \simeq H^{2}(S, \mathbb{Q}),
\]

\[
H^{3}(V, \mathbb{Q})(1) \simeq H^{1}(S, \mathbb{Q})
\]

by [2], [3] or [6].
Let $X \subset \mathbb{P}^3$ be a nonsingular cubic surface, and let $V \subset \mathbb{P}^4$ be the cubic 3-fold which is the triple Galois cover $\rho : V \rightarrow \mathbb{P}^3$ branched along $X$. If $L$ is a line on $V$, then the image of the projection $\rho(V)$ is a line of $\mathbb{P}^3$ which is contained in $X$ or intersects $X$ with the multiplicity 3. Therefore we have the triple Galois cover

$$S \rightarrow Z; \ L \mapsto \rho(L),$$

and we have an isomorphism of Hodge structures

$$H^2(S, \mathbb{Q})^{\text{Gal}(\rho)} \simeq H^2(Z, \mathbb{Q}). \quad (4)$$

By these isomorphisms (1) – (4), we have the isomorphism in Theorem 3.1.

References


