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Cauchy problem for the complex Ginzburg-Landau equation with harmonic oscillator

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1. Introduction and results

Let $N \in \mathbb{N}$. This paper is concerned with the following Cauchy problem for the complex Ginzburg-Landau equation with Laplacian replaced with Hamiltonian for harmonic oscillator:

\[
(CGL)_{\mathbb{R}^N, \mu} \quad \begin{cases} 
\frac{\partial u}{\partial t} + (\lambda + i\alpha)(-\Delta + \mu^2 |x|^2)u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0 & \text{on } \mathbb{R}^N \times \mathbb{R}_+, \\
\quad u(x, 0) = u_0(x), & x \in \mathbb{R}^N,
\end{cases}
\]

where $\lambda, \kappa \in \mathbb{R}_+ := (0, \infty)$, $\alpha, \beta, \gamma \in \mathbb{R}$, $\mu > 0$ and $q \geq 2$ are constants, and $u = u(x, t)$ is a complex-valued unknown function. In particular, the case where $\mu = 0$, i.e., $(CGL)_{\mathbb{R}^N, 0}$ is a Cauchy problem for the usual complex Ginzburg-Landau equation which is also regarded as the special case of initial-boundary value problem of the form

\[
(CGL)_{\Omega, 0} \quad \begin{cases} 
\frac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0 & \text{on } \Omega \times \mathbb{R}_+, \\
\quad u = 0 & \text{on } \partial \Omega \times \mathbb{R}_+, \\
\quad u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
\]

where $\Omega \subset \mathbb{R}^N$ is a general domain with boundary $\partial \Omega$. For physical background of the complex Ginzburg-Landau equation see e.g., Aranson-Kramer [1].

The purpose of this paper is to discuss the following three problems.

1. **Problem 1** Existence of global strong solutions to $(CGL)_{\mathbb{R}^N, \mu}$.
2. **Problem 2** Uniqueness of global strong solutions to $(CGL)_{\mathbb{R}^N, \mu}$.
3. **Problem 3** Existence of global strong solutions to $(CGL)_{\mathbb{R}^N, 0}$ by letting $\mu \downarrow 0$ in $(CGL)_{\mathbb{R}^N, \mu}$.

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Figure 1: The boundary of $CGL(y_0)$ is given by a pair of hyperbolas.

To clarify the problem we review the known results. Ginibre-Velo [2] established the existence (except uniqueness) of global strong solutions to $(CGL)_{\mathbb{R}^N,0}$ with $u_0 \in H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ under the condition that

\[(\alpha/\lambda, \beta/\kappa) \in CGL(c_q^{-1}) := \{(x, y) \in \mathbb{R}^2; xy \geq 0 \text{ or } \frac{|xy| - 1}{|x| + |y|} < \frac{1}{c_q}\},\]

\[(1.2)\quad c_q := \frac{q-2}{2\sqrt{q-1}}\]

(see Figure 1). Condition (1.1) plays an essential role in deriving the estimates of

\[
\begin{align*}
\frac{\delta^2}{2} \|\nabla u(t)\|^2_{L^2} + (1/q) \|u(t)\|^q_{L^q}, \\
\int_0^t \{\delta^2 \|\Delta u(s)\|^2_{L^2} + \|u(s)\|^{2(q-1)}_{L^{2(q-1)}}\} \, ds
\end{align*}
\]

for some $\delta > 0$. In [2, Proof of Proposition 5.1] they used compactness methods; however, their proof is much complicated since both the nonlinear term and the initial data are regularized. The result is extended to problem $(CGL)_{\Omega,0}$ in a bounded domain $\Omega$ (see Okazawa-Yokota [5, Theorem 1.1 with $p = 2$]). However, when $\Omega$ is an unbounded general domain and $q \geq 2$ is not restricted by $\mathcal{N}$, there seems to be no work except the case where

\[
(\alpha/\lambda, \beta/\kappa) \in S(c_q^{-1}) := \{(x, y) \in \mathbb{R}^2; |y| \leq \frac{1}{c_q}\} \subset CGL(c_q^{-1}),
\]

\[
\left(\Leftrightarrow \frac{|eta|}{\kappa} \leq \frac{1}{c_q}\right).
\]

This implies that the mapping $u \mapsto -(\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u$ is accretive in $L^2(\Omega)$. In this case the existence and uniqueness of global strong solutions to $(CGL)_{\Omega,0}$ with
$u_0 \in L^2(\Omega)$ are obtained in [5, Theorem 1.3 with $p = 2$]. Therefore the problem lies in the case where $\Omega$ is unbounded and $(\alpha/\lambda, \beta/\kappa) \in CGL(c_q^{-1}) \setminus S(c_q^{-1})$. In this paper we give a partial answer to the case where $\Omega = \mathbb{R}^N$ via compactness methods by adding the harmonic oscillator $|x|^2$.

Before stating our results, we define a global strong solution to $(CGL)_{\mathbb{R}^N, \mu}$.

**Definition 1.1.** A function $u(\cdot) \in C([0, \infty); L^2(\mathbb{R}^N))$ is said to be a global strong solution to $(CGL)_{\mathbb{R}^N, \mu}$ if $u(\cdot)$ has the following properties:

(a) $u(t) \in H^2(\mathbb{R}^N) \cap L^{2(q-1)}(\mathbb{R}^N)$, $|x|^2u(t) \in L^2(\mathbb{R}^N)$ a.a. $t > 0$;
(b) $(\partial u/\partial t)(\cdot), \Delta u(\cdot), |x|^2u(\cdot), |u|^{q-2}u(\cdot) \in L^2(0, T; L^2(\mathbb{R}^N))$ for every $T > 0$;
(c) $u(\cdot)$ satisfies the equation in $(CGL)_{\mathbb{R}^N, \mu}$ a.e. on $\mathbb{R}^N$ as well as the initial condition.

First we give an answer to Problem 1. Using the compactness of $(-\Delta + \mu^2|x|^2)^{-1}$ ($\mu > 0$) in $L^2(\mathbb{R}^N)$ (see Okazawa [4]), we can establish the existence of global strong solutions to $(CGL)_{\mathbb{R}^N, \mu}$ with $u_0 \in H^1(\mathbb{R}^N) \cap D(|x|) \cap L^q(\mathbb{R}^N)$ under condition (1.1). Here $D(|x|)$ is regarded as a Hilbert space given by

$$D(|x|) := \{u \in L^2(\mathbb{R}^N); |x|u \in L^2(\mathbb{R}^N)\},$$

$$(u, v)_{D(|x|)} := (u, v)_{L^2} + (|x|u, |x|v)_{L^2}, \quad u, v \in D(|x|).$$

**Theorem 1.1.** Let $N \in \mathbb{N}$, $\lambda > 0$, $\kappa > 0$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mu > 0$. Assume that condition (1.1) is satisfied. Then for any $u_0 \in H^1(\mathbb{R}^N) \cap D(|x|) \cap L^q(\mathbb{R}^N)$ there exists a global strong solution $u(\cdot) \in C([0, \infty); L^2(\mathbb{R}^N))$ to $(CGL)_{\mathbb{R}^N, \mu}$ such that

$$u(\cdot) \in C([0, \infty); H^1(\mathbb{R}^N) \cap D(|x|) \cap L^q(\mathbb{R}^N)),$$

with the estimates for every $t > 0$

$$\|u(t)\|_{L^2} \leq e^{\gamma t}\|u_0\|_{L^2},$$
$$E_\mu(u(t)) + \eta \int_0^t \{\delta^2 \||\Delta - \mu^2|x|^2|u(s)|\|_{L^2}^2 + \|u(s)\|_{L^2}^{2(q-1)}\} ds \leq e^{\gamma+qt}E_\mu(u_0),$$

where

$$E_\mu(u) := \frac{\delta^2}{2} \|\nabla u\|_{L^2}^2 + \mu^2 \||\Delta - \mu^2|x|^2|u\|_{L^2}^2 + \frac{1}{q}\|u\|_{L^q}^q,$$
$$\gamma_+ := \max\{\gamma, 0\} \text{ and } \delta > 0, \eta > 0 \text{ are constants depending only on } \lambda, \kappa, \alpha, \beta, q.$$

Secondly we give an answer to Problem 2 under the additional condition

$$2 \leq q < 2^* := \begin{cases} \frac{2 + \frac{4}{N - 2}}{N - 2} & (N \geq 3), \\ \infty & (N = 1, 2). \end{cases}$$

This condition appeared in proving the uniqueness of solutions to $(CGL)_{\mathbb{R}^N, 0}$ or $(CGL)_{\Omega, 0}$ (see Ginibre-Velo [3, Proposition 4.2] and Okazawa-Yokota [6, Theorem 1.2]).
Theorem 1.2. Let $N \in \mathbb{N}$, $\lambda > 0$, $\kappa > 0$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mu > 0$. Assume that (1.1) and (1.6) are satisfied. Then the solutions to $(\text{CGL})_{\mathbb{R}^N, \mu}$ in the sense of Definition 1.1 are unique. In fact, let $u(\cdot)$ and $v(\cdot)$ be global strong solutions to $(\text{CGL})_{\mathbb{R}^N, \mu}$ with initial data $u_0, v_0 \in H^1(\mathbb{R}^N) \cap D(|x|^2)$, respectively. Set $w(\cdot) := u(\cdot) - v(\cdot)$ and $w_0 := u_0 - v_0$. Then

\[ (1.7) \quad \|w(t)\|_{L^2}^2 + \lambda \int_0^t e^{\int_s^t K(r)dr} \left\{ \|\nabla w(s)\|_{L^2}^2 + \mu^2 \|x|w(s)\|_{L^2}^2 \right\} ds \leq e^{\int_0^t K(r)dr} \|w_0\|_{L^2}^2, \quad t > 0, \]

where $K(\cdot)$ is a continuous function depending only on $\lambda, \kappa, \beta, \gamma, q, E_{\mu}(u_0)$ and $E_{\mu}(v_0)$.

Finally, combining Theorems 1.1 and 1.2, we can give an answer to Problem 3 under (1.6). The following theorem is the special case of [2, Proposition 5.1] concerning the existence; however, our approach here is much simpler than that in [2].

Theorem 1.3. Let $N \in \mathbb{N}$, $\lambda > 0$, $\kappa > 0$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mu > 0$. Assume that conditions (1.1) and (1.6) are satisfied. Let $\{u_{\mu}(\cdot)\}_{\mu > 0}$ be a family of unique global strong solutions to $(\text{CGL})_{\mathbb{R}^N, \mu}$ with initial data $u_0 \in H^1(\mathbb{R}^N) \cap D(|x|^2)$. Then

\[ u(\cdot) := \lim_{\mu \downarrow 0} u_{\mu}(\cdot) \]

gives a (unique) global strong solution to $(\text{CGL})_{\mathbb{R}^N, 0}$ with $u(0) = u_0$.

The proofs of Theorems 1.1, 1.2 and 1.3 are given in Sections 2, 3 and 4, respectively.

2. Answer to Problem 1

First we review an abstract theorem in [5] toward Theorem 1.1. Let $X$ be a complex Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Let $\varphi, \psi : X \rightarrow [0, \infty]$ be proper lower semicontinuous convex functions on $X$. We assume for simplicity that the subdifferentials $\partial \varphi, \partial \psi$ are single-valued. Then we consider the abstract Cauchy problem in $X$:

\[ (\text{ACP}) \quad \left\{ \begin{array}{l} \frac{\partial u}{\partial t} + (\lambda + i\alpha)\partial \varphi(u) + (\kappa + i\beta)\partial \psi(u) - \gamma u = 0, \\ u(0) = u_0, \end{array} \right. \]

where $\lambda, \kappa \in \mathbb{R}_+$, $\alpha, \beta, \gamma \in \mathbb{R}$ are constants. We need the following conditions on $\varphi, \psi$:

(A1) The sublevel set $\{u \in D(\varphi); \varphi(u) \leq c\}$ is compact in $X$ for each $c > 0$.

(A2) $\exists p \in [2, \infty)$ such that $\varphi(\zeta u) = |\zeta|^p \varphi(u)$, $u \in D(\varphi)$, $\zeta \in \mathbb{C}$, $\text{Re} \zeta > 0$.

(A3) $\exists q \in [2, \infty)$ such that $\psi(\zeta u) = |\zeta|^q \psi(u)$, $u \in D(\psi)$, $\zeta \in \mathbb{C}$, $\text{Re} \zeta > 0$.

(A4) $\exists c_p \geq 0$ such that for $u, v \in D(\partial \varphi)$ and $\varepsilon > 0$,

\[ |\text{Im}(\partial \varphi(u) - \partial \varphi(v), u - v)| \leq c_p \text{Re}(\partial \varphi(u) - \partial \varphi(v), u - v). \]

(A5) $\exists c_q \geq 0$ such that for $u \in D(\partial \varphi)$ and $\varepsilon > 0$,

\[ |\text{Im}(\partial \varphi(u), \partial \psi(u))| \leq c_q \text{Re}(\partial \varphi(u), \partial \psi(u)), \]

where $\partial \psi_\varepsilon$ is the Yosida approximation of $\partial \psi$: $\partial \psi_\varepsilon := \varepsilon^{-1}(1 - (1 + \varepsilon \partial \psi)^{-1})$. 

The following theorem is established in [5].

**Theorem 2.1 ([5, Theorem 4.1]).** Assume that (A1)–(A5) are satisfied. Assume that $\alpha/\lambda$ and $\beta/\kappa$ satisfy
\[
\frac{|\alpha|}{\lambda} \leq c_p^{-1}, \quad \left(\frac{\alpha}{\lambda \cdot \kappa}\right) \in CGL(c_q^{-1}).
\]
Then for any $u_0 \in D(\varphi) \cap D(\psi)$ there exists a global strong solution $u(\cdot) \in C([0, \infty); X)$ to (ACP) such that

(a) $u(\cdot) \in C^{0,1/2}([0, T]; X), \quad T > 0$,  
(b) $(du/dt)(\cdot), \partial\varphi(u(\cdot)), \partial\psi(u(\cdot)) \in L^2(0, T; X), \quad T > 0$,  
(c) $\varphi(u(\cdot))$ and $\psi(u(\cdot))$ are absolutely continuous on $[0, T]$ for every $T > 0$, with the estimates

\[
\|u(t)\| \leq e^{\gamma t}\|u_0\|, \quad t > 0, \tag{2.1}
\]
\[
E(u(t)) + \eta \int_0^t (\delta^2 \|\varphi(u(s))\|^2 + \|\psi(u(s))\|^2) \, ds \leq e^{\gamma r t} E(u_0), \quad t > 0, \tag{2.2}
\]
where
\[
E(u) := \delta^2 \varphi(u) + \psi(u),
\]
\[
\gamma := \max\{\gamma, 0\}, \quad r := \max\{p, q\} \quad \text{and} \quad \delta, \eta > 0 \quad \text{are constants.}
\]

Next we apply Theorem 2.1 to (CGL)$_{\mathbb{R}^N, \mu}$. In the complex Hilbert space $X := L^2(\mathbb{R}^N)$ we introduce two convex functions on $X$

\[
\varphi(u) := \begin{cases}
\frac{1}{2} (\|\nabla u\|_{L^2}^2 + \mu^2 \|x\|_{L^2}^2) & \text{if } u \in D(\varphi) := H^1(\mathbb{R}^N) \cap D(|x|), \\
\infty & \text{otherwise},
\end{cases} \tag{2.3}
\]
\[
\psi(u) := \begin{cases}
\frac{1}{q} \|u\|_{L^q}^q & \text{if } u \in D(\psi) := X \cap L^q(\mathbb{R}^N), \\
\infty & \text{otherwise}.
\end{cases} \tag{2.4}
\]

Then their subdifferentials are given by
\[
\partial\varphi(u) = -\Delta u + \mu^2 |x|^2 u, \quad u \in D(\partial\varphi) = H^2(\mathbb{R}^N) \cap D(|x|^2),
\]
\[
\partial\psi(u) = |u|^{q-2} u, \quad u \in D(\partial\psi) = X \cap L^{2(q-1)}(\mathbb{R}^N).
\]

To apply Theorem 2.1 with those $X, \varphi$ and $\psi$, we prepare some lemmas.

**Lemma 2.2.** Let $N \in \mathbb{N}$ and $\mu > 0$. Then for every $u \in H^1(\mathbb{R}^N) \cap D(|x|)$,

\[
\|u\|_{L^2}^2 \leq \frac{2}{N} \|\nabla u\|_{L^2}^2 \|x\|_{L^2}^2; \tag{2.5}
\]
in particular,

\[
N \mu \|u\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 + \mu^2 \|x\|_{L^2}^2. \tag{2.6}
\]
\textbf{Proof.} Let $u \in C_0^\infty(\mathbb{R}^N)$ and $\epsilon > 0$. Let $|x|_\epsilon := |x|(1 + \epsilon|x|)^{-1}$ be the Yosida approximation of $|x|$ and $x_\epsilon := x(1 + \epsilon|x|)^{-1}$. Then we can obtain

\begin{equation}
N \int_{\mathbb{R}^N} \frac{|u(x)|^2}{1 + \epsilon|x|} \, dx \leq 2\|\nabla u\|_{L^2} \|x|_\epsilon u\|_{L^2} + \epsilon \|u\|_{L^2} \|x|_\epsilon u\|_{L^2}.
\end{equation}

In fact, observing

\begin{equation}
N(1 + \epsilon|x|)^{-1} = \text{div} x_\epsilon + \epsilon |x|_\epsilon (1 + \epsilon|x|)^{-1}
\end{equation}

\begin{align*}
\leq \text{div} x_\epsilon + \epsilon |x|_\epsilon,
\end{align*}

we see from integration by parts that

\begin{align*}
N \int_{\mathbb{R}^N} \frac{|u(x)|^2}{1 + \epsilon|x|} \, dx &\leq \int_{\mathbb{R}^N} \text{div} x_\epsilon |u(x)|^2 \, dx + \epsilon \int_{\mathbb{R}^N} |x|_\epsilon |u(x)|^2 \, dx \\
&= -2 \int_{\mathbb{R}^N} x_\epsilon \cdot \text{Re} (u(x) \nabla \overline{u(x)}) \, dx + \epsilon \|u\|_{L^2} \|x|_\epsilon u\|_{L^2} \\
&\leq 2 \|\nabla u\|_{L^2} \|x|_\epsilon u\|_{L^2} + \epsilon \|u\|_{L^2} \|x|_\epsilon u\|_{L^2}.
\end{align*}

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, (2.7) is true also for $u \in H^1(\mathbb{R}^N) \cap D(|x|)$. Letting $\epsilon \downarrow 0$ in (2.7) for $u \in H^1(\mathbb{R}^N) \cap D(|x|)$, we obtain (2.5).

\begin{lemma}[5, Lemma 6.2] Let $q \geq 2$. Then for $u \in H^2(\mathbb{R}^N)$ and $\epsilon > 0$,

\begin{equation}
|\text{Im} \langle -\Delta u, \partial\psi_\epsilon(u) \rangle_{L^2}| \leq \frac{q - 2}{2\sqrt{q - 1}} \text{Re} \langle -\Delta u, \partial\psi_\epsilon(u) \rangle_{L^2}.
\end{equation}

\end{lemma}

\begin{lemma} Let $V : \mathbb{R}^N \rightarrow \mathbb{R}$ be a nonnegative function. Then for $\epsilon > 0$ and $u \in L^2(\mathbb{R}^N)$ with $Vu \in L^2(\mathbb{R}^N)$,

\begin{equation}
(Vu, \partial\psi_\epsilon(u))_{L^2} = \int_{\mathbb{R}^N} V|u_\epsilon|^q \, dx + \epsilon \int_{\mathbb{R}^N} V|u_\epsilon|^{2(q-1)} \, dx
\end{equation}

where $u_\epsilon := (1 + \epsilon \partial\psi)^{-1}u$. Consequently, $(Vu, \partial\psi_\epsilon(u))_{L^2}$ is real and nonnegative.

\end{lemma}

\begin{proof} Let $\epsilon > 0$ and $u \in L^2(\mathbb{R}^N)$ with $Vu \in L^2(\mathbb{R}^N)$. Setting $u_\epsilon := (1 + \epsilon \partial\psi)^{-1}u$, we see that

\begin{align*}
u = u_\epsilon + \epsilon |u_\epsilon|^{q-2}u_\epsilon, \quad \partial\psi_\epsilon(u) = |u_\epsilon|^{q-2}u_\epsilon.
\end{align*}

Substituting these identities into $(Vu, \partial\psi_\epsilon(u))_{L^2}$, we can obtain (2.9).

\end{proof}

\begin{lemma} Let $q \geq 2$. Then for $u \in D(\partial\varphi)$ and $\epsilon > 0$,

\begin{equation}
|\text{Im} \langle \partial\varphi(u), \partial\psi_\epsilon(u) \rangle_{L^2}| \leq \frac{q - 2}{2\sqrt{q - 1}} \text{Re} \langle \partial\varphi(u), \partial\psi_\epsilon(u) \rangle_{L^2}.
\end{equation}

\end{lemma}

Lemma 2.5 is a consequence of Lemmas 2.3 and 2.4 with $V(x) := \mu^2|x|^2$; note that $\partial\varphi = -\Delta + V(x)$.
**Proof of Theorem 1.1.** Let \( X := L^2(\mathbb{R}^N) \). Let \( \varphi \) and \( \psi \) be defined as (2.3) and (2.4). We see from (2.6) that \((-\Delta + \mu^2|x|^2)^{-1}\) is bounded. In fact, (2.6) implies that for every \( u \in H^2(\mathbb{R}^N) \cap D(|x|^2) \),

\[
N\mu \|u\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 + \mu^2 \|x|u\|_{L^2}^2
= ((-\Delta + \mu^2|x|^2)u, u)_{L^2}
\leq \|(-\Delta + \mu^2|x|^2)u\|_{L^2} \|u\|_{L^2}.
\]

Since the potential \(|x|^2\) blows up as \(|x| \to \infty\), it follows from [4, Theorem 4.1] that \((-\Delta + \mu^2|x|^2)^{-1}\) is compact in \( X \) and hence (A1) is satisfied. (A2) (with \( p = 2 \)) and (A3) are trivial by definition. Since \( \partial \varphi \) is nonnegative selfadjoint in \( X \), (A4) is satisfied with \( c_p = 0 \). Lemma 2.4 implies that (A5) is satisfied with

\[
c_q := \frac{q - 2}{2\sqrt{q - 1}}.
\]

Therefore we can apply Theorem 2.1 with those \( X, \varphi \). Consequently, we obtain the existence part of Theorem 1.1. As in the proof of [5, Theorem 1.1], we can prove (1.3) by virtue of Theorem 2.1 (c). Moreover, (1.4) and (1.5) follow from (2.1) and (2.2), respectively (see Remark 2.1 below). This completes the proof of Theorem 1.1.

**Remark 2.1.** By the definition of \( \varphi \) in (2.3), Theorem 2.1 (b) asserts that

\[
u(\cdot), (\Delta - \mu^2|x|^2)u(\cdot) \in L^2(0, T; L^2(\mathbb{R}^N)), \ T > 0.
\]

This fact implies that

\[
\Delta u(\cdot), |x|^2u(\cdot) \in L^2(0, T; L^2(\mathbb{R}^N)), \ T > 0.
\]

This is a direct consequence of the following inequality (see Okazawa [4]):

\[
\|\Delta u\|_{L^2}^2 + \mu^4 \|x|^2u\|_{L^2}^2 \leq \|(\Delta - \mu^2|x|^2)u\|_{L^2}^2 + 2N\mu^2 \|u\|_{L^2}^2, \ u \in H^2(\mathbb{R}^N) \cap D(|x|^2).
\]

**3. Answer to Problem 2**

In this section we give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** It suffices to prove (1.7). Let \( q < 2^* \). Then \( H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \). Let \( u(\cdot) \) and \( v(\cdot) \) be the global strong solutions to \((\text{CGL})_{\mathbb{R}^N, \mu}\) with initial data \( u_0, v_0 \in H^1(\mathbb{R}^N) \cap D(|x|) \), respectively. Then \( w(\cdot) := u(\cdot) - v(\cdot) \) satisfies

\[
\frac{\partial w}{\partial t} + (\lambda + i\alpha)(-\Delta + \mu^2|x|^2)w + (\kappa + i\beta)(|u|^{q-2}u - |v|^{q-2}v) = \gamma w.
\]

Making the \( L^2 \)-inner product of (3.1) with \( w \), we have

\[
\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 + \lambda \|\nabla w\|_{L^2}^2 + \mu^2 \|x|w\|_{L^2}^2 + I = \gamma \|w\|_{L^2}^2,
\]
where \( I := \text{Re} [(\kappa + i\beta)(|u|^{q-2}u - |v|^{q-2}v, w)_{L^2}] \).

Since \( ||u|^{q-2}u - |v|^{q-2}v|| \leq (q-1)(||u||^{q-2} + ||v||^{q-2})||w|| \), we have

\[
|I| \leq (q-1)\sqrt{\kappa^2 + \beta^2} \int_{\mathbb{R}^N} (||u||^{q-2} + ||v||^{q-2}) ||w||^2 \, dx
\]

\[
\leq (q-1)\sqrt{\kappa^2 + \beta^2} ||u||_s^{q-2} + ||v||_s^{q-2} ||w||_s^2,
\]

where we used the Hölder inequality in the second inequality. We see from (1.5) that

\[
||u||_s^q \leq q e^{\gamma+qt} E_{\mu}(u_0), \quad ||v||_s^q \leq q e^{\gamma+qt} E_{\mu}(v_0).
\]

Hence we have

\[
(u(t)||^{q-2}_s + ||v(t)||^{q-2}_s \leq K_1 e^{\gamma+(q-2)t},
\]

where

\[
K_1 := q^{1-2/q}[E_{\mu}(u_0)^{1-2/q} + E_{\mu}(v_0)^{1-2/q}].
\]

On the other hand, we use the Gagliardo-Nirenberg inequality

\[
\|w\|_{L^s} \leq C\|w\|_2^{1-a}\|\nabla w\|_2^a,
\]

where \( a := N(1/2 - 1/q) \in [0, 1) \) and \( C = C(q, N) \) is a positive constant. Applying (3.4) and (3.5) to (3.3), we see by the Young inequality that

\[
|I| \leq (q-1)\sqrt{\kappa^2 + \beta^2} C K_1 e^{\gamma+(q-2)t} ||w||_s^{2(1-a)}\|\nabla w\|_2^{2a}
\]

\[
\leq K_2 e^{\gamma+(q-2)t} ||w||_2^2 + \frac{\lambda}{2} ||\nabla w||_2^2,
\]

where

\[
K_2 := \left( \frac{2}{\lambda} \right)^{a/(1-a)} [(q-1)\sqrt{\kappa^2 + \beta^2} C K_1]^{1/(1-a)}.
\]

Plugging this inequality with (3.2), we obtain

\[
\frac{d}{dt} ||w||_2^2 + \lambda (\|\nabla w\|^2_2 + \mu^2 \|x|w||^2_2) \leq 2 \left( \gamma + K_2 e^{\gamma+(q-2)t} \right) ||w||_2^2.
\]

Setting

\[
K(t) := 2 \left( \gamma + K_2 e^{\gamma+(q-2)t} \right),
\]

we have

\[
\frac{d}{ds} \left[ e^{-\int_{s_0}^{s} K(r) \, dr} ||w(s)||_2^2 \right] + \lambda e^{-\int_{s_0}^{s} K(r) \, dr} (\|\nabla w(s)\|^2_2 + \mu^2 \|x|w(s)||^2_2) \leq 0.
\]

Integrating this inequality on \([0, t]\) for \( t > 0 \), we obtain (1.7). \( \square \)
4. Answer to Problem 3

Let $u_\mu(\cdot)$ be the unique global strong solution to $(\text{CGL})_{\mathbb{R}^N, \mu}$ ($\mu > 0$) constructed in Theorems 1.1 and 1.2. To prove Theorem 1.3 we need a priori estimate of $\||x|u_\mu(\cdot)\|_{L^2}$ independent of $\mu$.

**Lemma 4.1.** Let $N, \lambda + i\alpha, \kappa + i\beta, \gamma, \mu$ be the same as in Theorem 1.2. Let $u_\mu(\cdot)$ be the solution to $(\text{CGL})_{\mathbb{R}^N, \mu}$ with $u_\mu(0) = u_0 \in H^1(\mathbb{R}^N) \cap D(|x|^2)$. Then for every $t > 0$,

$$\||x|^2u_\mu(t)\|_{L^2} \leq e^{\gamma t} \left( ct\|u_0\|_{L^2} + \||x|^2u_0\|_{L^2} \right),$$

where $c > 0$ is a constant depending only on $\lambda + i\alpha$.

**Proof.** We give a formal proof. The proof can be justified by using the Yosida approximation of $|x|^2$. Making the inner product of the equation in $(\text{CGL})_{\mathbb{R}^N}$ with $|x|^4u_\mu(\cdot)$, we have

$$\frac{1}{2}\frac{d}{dt}\||x|^2u_\mu\|_{L^2}^2 + J - \gamma \||x|^2u_\mu\|_{L^2}^2 \leq 0,$$

where

$$J := \text{Re}[(\lambda + i\alpha)(-\Delta u_\mu + \mu^2|x|^2u_\mu, |x|^4u_\mu)_{L^2}].$$

Applying integration by parts and the Schwarz inequality, we obtain

$$J \geq \gamma \||x|^2\nabla u_\mu\|_{L^2}^2 - 4\sqrt{\lambda^2 + \alpha^2}\||x|^2\nabla u_\mu\|_{L^2}\|u_\mu\|_{L^2} \geq -c\|u_\mu\|_{L^2}^2,$$

where $c := (4/\lambda)(\lambda^2 + \alpha^2)$. On the other hand, it follows from the Schwarz inequality and (1.4) that

$$\||x|u_\mu(t)\|_{L^2} \leq e^{\gamma t}\|u_0\|_{L^2}\||x|^2u_\mu(t)\|_{L^2}.$$

Applying this inequality to (4.3), we see from (4.2) that

$$\frac{1}{2}\frac{d}{dt}\||x|^2u_\mu(t)\|_{L^2}^2 - ce^{\gamma t}\|u_0\|_{L^2}\||x|^2u_\mu(t)\|_{L^2}^2 - \gamma \||x|^2u_\mu(t)\|_{L^2}^2 \leq 0,$$

which implies that

$$\frac{d}{dt} \left( e^{-\gamma t}\||x|^2u_\mu(t)\|_{L^2} \right) \leq c\|u_0\|_{L^2}.$$

Integrating this inequality on $[0, t]$ yields (4.1). \hfill \Box

Now we are in position to complete the proof of Theorem 1.3 which answers to Problem 3.

**Proof of Theorem 1.3.** Let $u_\mu(\cdot)$ be the unique global strong solution to $(\text{CGL})_{\mathbb{R}^N, \mu}$ with $u_\mu(0) = u_0 \in H^1(\mathbb{R}^N) \cap D(|x|^2)$. Set $w_{\mu, \nu}(\cdot) := u_\mu(\cdot) - u_\nu(\cdot)$ for $\mu, \nu \in (0, 1]$. Similarly in deriving (3.6), we have

$$\frac{1}{2}\frac{d}{dt}\|w_{\mu, \nu}\|_{L^2}^2 + \frac{\lambda}{2}\|\nabla w_{\mu, \nu}\|_{L^2}^2 + I_{\mu, \nu} \leq \frac{K(t)}{2}\|w_{\mu, \nu}\|_{L^2}^2,$$
where
\[
I_{\mu, \nu} := \text{Re} \left[ (\lambda + i\alpha)(\mu^{2}|x|^{2}u_{\mu} - \nu^{2}|x|^{2}u_{\nu}, w_{\mu, \nu})_{L^{2}} \right]
\]
\[
= \lambda \mu^{2}\|x|w_{\mu, \nu}\|_{L^{2}}^{2} + (\mu^{2} - \nu^{2})\text{Re} \left[ (\lambda + i\alpha)(|x|^{2}u_{\nu}, w_{\mu, \nu})_{L^{2}} \right],
\]
and \(K(\cdot)\) is the same function as in Theorem 1.2. From (4.1) we have
\[
I_{\mu, \nu} \geq -\sqrt{\lambda^{2} + \alpha^{2}}|\mu^{2} - \nu^{2}|\|x|u_{\mu}\|_{L^{2}}\|w_{\mu, \nu}\|_{L^{2}}
\]
\[
\geq -M(t)|\mu^{2} - \nu^{2}|\|w_{\mu, \nu}\|_{L^{2}},
\]
where
\[
M(t) := \sqrt{\lambda^{2} + \alpha^{2}}e^{ct\|u_{0}\|_{L^{2}} + \||x|^{2}u_{0}\|_{L^{2}}},
\]
Hence we obtain
\[
\frac{d}{dt}\|w_{\mu, \nu}\|_{L^{2}} \leq \frac{K(t)}{2}\|w_{\mu, \nu}\|_{L^{2}} + M(t)|\mu^{2} - \nu^{2}|.
\]
Applying the Gronwall lemma to (4.4) yields
\[
\|w_{\mu, \nu}(t)\|_{L^{2}} \leq |\mu^{2} - \nu^{2}|\int_{0}^{t}e^{\int_{s}^{t}K(r)dr}M(s)ds.
\]
This inequality implies that for every \(T > 0\),
\[
\sup_{0 < t < T}\|w_{\mu, \nu}(t)\|_{L^{2}} \leq |\mu^{2} - \nu^{2}|\int_{0}^{T}e^{\int_{s}^{T}K(r)dr}M(s)ds.
\]
This implies that \(\{u_{\mu}(\cdot)\}\) satisfies the Cauchy condition in \(C([0, T]; L^{2}(\mathbb{R}^{N}))\) and hence there exists \(u \in C([0, \infty); L^{2}(\mathbb{R}^{N}))\) such that
\[
u_{\mu}(\cdot) \rightharpoonup u(\cdot) \quad (\mu \downarrow 0) \quad \text{strongly in } C([0, T]; L^{2}(\mathbb{R}^{N})).
\]
We see from (1.4), (1.5) and (2.11) that
\(
\{\Delta u_{\mu}(\cdot)\} \text{ and } \{|u_{\mu}|^{q-2}u_{\mu}(\cdot)\} \text{ are bounded in } L^{2}(0, T; L^{2}(\mathbb{R}^{N})).
\)
Moreover, (4.1) implies that
\[
\{|x|^{2}u_{\mu}(\cdot)\} \text{ is also bounded in } L^{2}(0, T; L^{2}(\mathbb{R}^{N})).
\]
Since \(\Delta, |x|^{2}\) and \(\partial/\partial t\) are weakly closed as operators in \(L^{2}(0, T; L^{2}(\mathbb{R}^{N}))\), it follows that
\[
\Delta u(\cdot), |x|^{2}u(\cdot), (\partial u/\partial t)(\cdot) \in L^{2}(0, T; L^{2}(\mathbb{R}^{N}))\text{ and}
\]
\[
\Delta u_{\mu}(\cdot) \rightharpoonup \Delta u(\cdot) \text{ weakly in } L^{2}(0, T; L^{2}(\mathbb{R}^{N})),
\]
\[
|x|^{2}u_{\mu}(\cdot) \rightharpoonup 0 \text{ weakly in } L^{2}(0, T; L^{2}(\mathbb{R}^{N})),
\]
\[
(\partial u_{\mu}/\partial t)(\cdot) \rightharpoonup (\partial u/\partial t)(\cdot) \text{ weakly in } L^{2}(0, T; L^{2}(\mathbb{R}^{N})).
\]
We can also see from the demiclosedness of \(\partial\psi\) as operators in \(L^{2}(0, T; L^{2}(\mathbb{R}^{N}))\) that
\[
|u|^{q-2}u(\cdot) \in L^{2}(0, T; L^{2}(\mathbb{R}^{N})),
\]
and
\[
|u_{\mu}|^{q-2}u_{\mu}(\cdot) \rightharpoonup |u|^{q-2}u(\cdot) \text{ weakly in } L^{2}(0, T; L^{2}(\mathbb{R}^{N})).
\]
Therefore \(u(\cdot)\) is a global strong solution to \((CGL)_{\mathbb{R}^{N}, 0}\). \(\Box\)
5. Concluding remarks

We have proved the existence of global strong solutions to \((CGL)_{\mathbb{R}^{N},0}\) under the conditions that
\[
\begin{align*}
\left( \frac{\alpha}{\lambda}, \frac{\beta}{\kappa} \right) & \in CGL(c_{q}^{-1}), \\
2 & \leq q < 2^*, \\
u_0 & \in H^1(\mathbb{R}^{N}) \cap D(|x|^2).
\end{align*}
\]
There are two comments; one is about the initial data \(u_0\) and the other is about the exponent \(q\).

(I) If \(u_0 \in H^1(\mathbb{R}^{N})\), then we can approximate \(u_0\) by
\[
u_{0,n} := (1 + n^{-1}|x|^2)^{-1}u_0.
\]
As in the proof of Theorem 1.3 we can see that the corresponding solution \(u_n(\cdot)\) with \(u_n(0) = u_{0,n}\) converges to the desired solution.

(II) For the uniqueness we assumed that \(2 \leq q < 2^*\); and hence we obtain the solution to \((CGL)_{\mathbb{R}^{N},0}\) for such exponent \(q\). On the other hand, Ginibre-Velo [2] have already proved the existence of solutions to \((CGL)_{\mathbb{R}^{N},0}\) under the mild condition that \(2 \leq q < \infty\). The key of their proof lies in the compactness of \(H^1(\Omega) \hookrightarrow L^2(\Omega)\) for a bounded domain \(\Omega \subset \mathbb{R}^{N}\). Our method lies in another compactness \(H^1(\mathbb{R}^{N}) \cap D(|x|) \hookrightarrow L^2(\mathbb{R}^{N})\). In the future we shall improve our method by using the compactness \(H^1(\mathbb{R}^{N}) \cap D(V) \hookrightarrow L^2(\mathbb{R}^{N})\), where \(V : \mathbb{R}^{N} \rightarrow \mathbb{R}\) is a nonnegative function satisfying
\[
\lim_{|x| \rightarrow \infty} V(x) = \infty.
\]
Choosing \(V\) properly, we would show the existence under the condition that \(2 \leq q < \infty\).

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References


