数理解析研究所講究録
第 1746 巻 2011 年 100-111

Cauchy problem for the complex Ginzburg-Landau equation with harmonic oscillator

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1. Introduction and results

Let $N \in \mathbb{N}$. This paper is concerned with the following Cauchy problem for the complex Ginzburg-Landau equation with *Laplacian replaced with Hamiltonian for harmonic oscillator*:

$$(\mathrm{CGL})_{\mathbb{R}^N,\mu} \quad \begin{cases} \dfrac{\partial u}{\partial t} + (\lambda + i\alpha)(-\Delta + \mu^2|x|^2)u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0 & \text{on } \mathbb{R}^N \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}^N, \end{cases}$$

where $\lambda, \kappa \in \mathbb{R}_+ := (0,\infty)$, $\alpha, \beta, \gamma \in \mathbb{R}$, $\mu > 0$ and $q \geq 2$ are constants, and $u = u(x,t)$ is a complex-valued unknown function. In particular, the case where $\mu = 0$, i.e., $(\mathrm{CGL})_{\mathbb{R}^N,0}$ is a Cauchy problem for the *usual* complex Ginzburg-Landau equation which is also regarded as the special case of initial-boundary value problem of the form

$$(\mathrm{CGL})_{\Omega,0} \quad \begin{cases} \dfrac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0 & \text{on } \Omega \times \mathbb{R}_+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a general domain with boundary $\partial\Omega$. For physical background of the complex Ginzburg-Landau equation see e.g., Aranson-Kramer [**1**].

The purpose of this paper is to discuss the following three problems.

(**Problem 1**) Existence of global strong solutions to $(\mathrm{CGL})_{\mathbb{R}^N,\mu}$.
(**Problem 2**) Uniqueness of global strong solutions to $(\mathrm{CGL})_{\mathbb{R}^N,\mu}$.
(**Problem 3**) Existence of global strong solutions to $(\mathrm{CGL})_{\mathbb{R}^N,0}$ by letting $\mu \downarrow 0$ in $(\mathrm{CGL})_{\mathbb{R}^N,\mu}$.

*Partially supported by Grant-in-Aid for Scientific Research (C), No.20540190.
†Partially supported by Grant-in-Aid for Young Scientists Research (B), No.20740079.

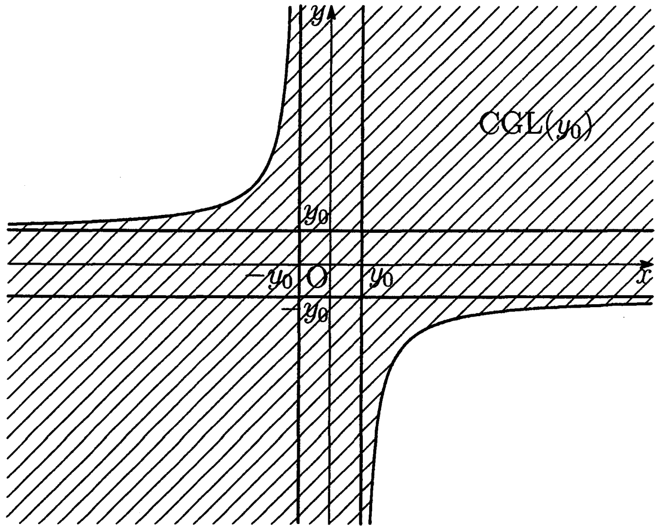

Figure 1: The boundary of $CGL(y_0)$ is given by a pair of hyperbolas.

To clarify the problem we review the known results. Ginibre-Velo [**2**] established the existence (except uniqueness) of global strong solutions to $(\mathrm{CGL})_{\mathbb{R}^N,0}$ with $u_0 \in H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ under the condition that

$$(1.1) \qquad \left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in CGL(c_q^{-1}) := \left\{(x,y) \in \mathbb{R}^2;\ xy \geq 0 \text{ or } \frac{|xy|-1}{|x|+|y|} < \frac{1}{c_q}\right\},$$

$$(1.2) \qquad c_q := \frac{q-2}{2\sqrt{q-1}}$$

(see Figure 1). Condition (1.1) plays an essential role in deriving the estimates of

$$(\delta^2/2)\|\nabla u(t)\|_{L^2}^2 + (1/q)\|u(t)\|_{L^q}^q,$$
$$\int_0^t \{\delta^2\|\Delta u(s)\|_{L^2}^2 + \|u(s)\|_{L^{2(q-1)}}^{2(q-1)}\}\, ds$$

for some $\delta > 0$. In [**2**, Proof of Proposition 5.1] they used compactness methods; however, their proof is much complicated since both the nonlinear term and the initial data are regularized. The result is extended to problem $(\mathrm{CGL})_{\Omega,0}$ in a *bounded* domain Ω (see Okazawa-Yokota [**5**, Theorem 1.1 with $p = 2$]). However, when Ω is an *unbounded* general domain and $q \geq 2$ is not restricted by N, there seems to be no work except the case where

$$\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in S(c_q^{-1}) := \left\{(x,y) \in \mathbb{R}^2;\ |y| \leq \frac{1}{c_q}\right\} \subset CGL(c_q^{-1}),$$
$$\left(\iff \frac{|\beta|}{\kappa} \leq \frac{1}{c_q}\right).$$

This implies that the mapping $u \mapsto -(\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u$ is accretive in $L^2(\Omega)$. In this case the existence and uniqueness of global strong solutions to $(\mathrm{CGL})_{\Omega,0}$ with

$u_0 \in L^2(\Omega)$ are obtained in [**5**, Theorem 1.3 with $p = 2$]. Therefore the problem lies in the case where Ω is unbounded and $(\alpha/\lambda, \beta/\kappa) \in CGL(c_q^{-1}) \setminus S(c_q^{-1})$. In this paper we give a partial answer to the case where $\Omega = \mathbb{R}^N$ via compactness methods by adding the harmonic oscillator $|x|^2$.

Before stating our results, we define a global strong solution to $(\mathrm{CGL})_{\mathbb{R}^N, \mu}$.

Definition 1.1. A function $u(\cdot) \in C([0,\infty); L^2(\mathbb{R}^N))$ is said to be a *global strong solution* to $(\mathrm{CGL})_{\mathbb{R}^N, \mu}$ if $u(\cdot)$ has the following properties:

(a) $u(t) \in H^2(\mathbb{R}^N) \cap L^{2(q-1)}(\mathbb{R}^N)$, $|x|^2 u(t) \in L^2(\mathbb{R}^N)$ a.a. $t > 0$;

(b) $(\partial u/\partial t)(\cdot)$, $\Delta u(\cdot)$, $|x|^2 u(\cdot)$, $|u|^{q-2}u(\cdot) \in L^2(0,T; L^2(\mathbb{R}^N))$ for every $T > 0$;

(c) $u(\cdot)$ satisfies the equation in $(\mathrm{CGL})_{\mathbb{R}^N, \mu}$ a.e. on $\mathbb{R}_+$ as well as the initial condition.

First we give an answer to **Problem 1**. Using the compactness of $(-\Delta + \mu^2|x|^2)^{-1}$ ($\mu > 0$) in $L^2(\mathbb{R}^N)$ (see Okazawa [**4**]), we can establish the existence of global strong solutions to $(\mathrm{CGL})_{\mathbb{R}^N, \mu}$ with $u_0 \in H^1(\mathbb{R}^N) \cap D(|x|) \cap L^q(\mathbb{R}^N)$ under condition (1.1). Here $D(|x|)$ is regarded as a Hilbert space given by

$$D(|x|) := \{u \in L^2(\mathbb{R}^N);\ |x|u \in L^2(\mathbb{R}^N)\},$$
$$(u,v)_{D(|x|)} := (u,v)_{L^2} + (|x|u, |x|v)_{L^2}, \quad u, v \in D(|x|).$$

Theorem 1.1. *Let $N \in \mathbb{N}$, $\lambda > 0$, $\kappa > 0$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mu > 0$. Assume that condition (1.1) is satisfied. Then for any $u_0 \in H^1(\mathbb{R}^N) \cap D(|x|) \cap L^q(\mathbb{R}^N)$ there exists a global strong solution $u(\cdot) \in C([0,\infty); L^2(\mathbb{R}^N))$ to $(\mathrm{CGL})_{\mathbb{R}^N, \mu}$ such that*

$$u(\cdot) \in C([0,\infty); H^1(\mathbb{R}^N) \cap D(|x|) \cap L^q(\mathbb{R}^N)), \tag{1.3}$$

with the estimates for every $t > 0$

$$\|u(t)\|_{L^2} \le e^{\gamma t}\|u_0\|_{L^2}, \tag{1.4}$$

$$E_\mu(u(t)) + \eta \int_0^t \{\delta^2 \|(\Delta - \mu^2|x|^2)u(s)\|_{L^2}^2 + \|u(s)\|_{L^{2(q-1)}}^{2(q-1)}\}\, ds \le e^{\gamma_+ qt} E_\mu(u_0), \tag{1.5}$$

where

$$E_\mu(u) := \frac{\delta^2}{2}\left[\|\nabla u\|_{L^2}^2 + \mu^2 \||x|u\|_{L^2}^2\right] + \frac{1}{q}\|u\|_{L^q}^q,$$

$\gamma_+ := \max\{\gamma, 0\}$ *and $\delta > 0$, $\eta > 0$ are constants depending only on $\lambda, \kappa, \alpha, \beta, q$.*

Secondly we give an answer to **Problem 2** under the additional condition

$$2 \le q < 2^* := \begin{cases} 2 + \dfrac{4}{N-2} & (N \ge 3), \\ \infty & (N = 1, 2). \end{cases} \tag{1.6}$$

This condition appeared in proving the uniqueness of solutions to $(\mathrm{CGL})_{\mathbb{R}^N, 0}$ or $(\mathrm{CGL})_{\Omega, 0}$ (see Ginibre-Velo [**3**, Proposition 4.2] and Okazawa-Yokota [**6**, Theorem 1.2]).

Theorem 1.2. *Let $N \in \mathbb{N}$, $\lambda > 0$, $\kappa > 0$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mu > 0$. Assume that* (1.1) *and* (1.6) *are satisfied. Then the solutions to* $(\mathrm{CGL})_{\mathbb{R}^N,\mu}$ *in the sense of* Definition 1.1 *are unique. In fact, let $u(\cdot)$ and $v(\cdot)$ be global strong solutions to $(\mathrm{CGL})_{\mathbb{R}^N,\mu}$ with initial data $u_0, v_0 \in H^1(\mathbb{R}^N) \cap D(|x|)$, respectively. Set $w(\cdot) := u(\cdot) - v(\cdot)$ and $w_0 := u_0 - v_0$. Then*

$$(1.7)\quad \|w(t)\|_{L^2}^2 + \lambda \int_0^t e^{\int_s^t K(r)dr}\{\|\nabla w(s)\|_{L^2}^2 + \mu^2 \||x|w(s)\|_{L^2}^2\}ds \le e^{\int_0^t K(r)dr}\|w_0\|_{L^2}^2,\ t > 0,$$

where $K(\cdot)$ is a continuous function depending only on $\lambda, \kappa, \beta, \gamma, q, E_\mu(u_0)$ and $E_\mu(v_0)$.

Finally, combining Theorems 1.1 and 1.2, we can give an answer to **Problem 3** under (1.6). The following theorem is the special case of [**2**, Proposition 5.1] concerning the existence; however, our approach here is *much simpler* than that in [**2**].

Theorem 1.3. *Let $N \in \mathbb{N}$, $\lambda > 0$, $\kappa > 0$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mu > 0$. Assume that conditions* (1.1) *and* (1.6) *are satisfied. Let $\{u_\mu(\cdot)\}_{\mu > 0}$ be a family of unique global strong solutions to $(\mathrm{CGL})_{\mathbb{R}^N,\mu}$ with initial data $u_0 \in H^1(\mathbb{R}^N) \cap D(|x|^2)$. Then*

$$u(\cdot) := \lim_{\mu \downarrow 0} u_\mu(\cdot)$$

gives a (unique) global strong solution to $(\mathrm{CGL})_{\mathbb{R}^N,0}$ with $u(0) = u_0$.

The proofs of Theorems 1.1, 1.2 and 1.3 are given in Sections 2, 3 and 4, respectively.

2. Answer to Problem 1

First we review an abstract theorem in [**5**] toward Theorem 1.1. Let X be a complex Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $\varphi, \psi : X \to [0, \infty]$ be proper lower semicontinuous convex functions on X. We assume for simplicity that the subdifferentials $\partial\varphi$, $\partial\psi$ are single-valued. Then we consider the abstract Cauchy problem in X:

$$(\mathrm{ACP})\qquad \begin{cases} \dfrac{\partial u}{\partial t} + (\lambda + i\alpha)\partial\varphi(u) + (\kappa + i\beta)\partial\psi(u) - \gamma u = 0, \\ \hfill u(0) = u_0, \end{cases}$$

where $\lambda, \kappa \in \mathbb{R}_+$, $\alpha, \beta, \gamma \in \mathbb{R}$ are constants. We need the following conditions on φ, ψ:

(A1) The sublevel set $\{u \in D(\varphi);\ \varphi(u) \le c\}$ is compact in X for each $c > 0$.

(A2) $\exists p \in [2, \infty)$ such that $\varphi(\zeta u) = |\zeta|^p \varphi(u)$, $u \in D(\varphi)$, $\zeta \in \mathbb{C}$, $\mathrm{Re}\,\zeta > 0$.

(A3) $\exists q \in [2, \infty)$ such that $\psi(\zeta u) = |\zeta|^q \psi(u)$, $u \in D(\psi)$, $\zeta \in \mathbb{C}$, $\mathrm{Re}\,\zeta > 0$.

(A4) $\exists c_p \ge 0$ such that for $u, v \in D(\partial\varphi)$ and $\varepsilon > 0$,

$$|\mathrm{Im}\,(\partial\varphi(u) - \partial\varphi(v), u - v)| \le c_p \mathrm{Re}\,(\partial\varphi(u) - \partial\varphi(v), u - v).$$

(A5) $\exists c_q \ge 0$ such that for $u \in D(\partial\varphi)$ and $\varepsilon > 0$,

$$|\mathrm{Im}\,(\partial\varphi(u), \partial\psi_\varepsilon(u))| \le c_q \mathrm{Re}\,(\partial\varphi(u), \partial\psi_\varepsilon(u)),$$

where $\partial\psi_\varepsilon$ is the Yosida approximation of $\partial\psi$: $\partial\psi_\varepsilon := \varepsilon^{-1}(1 - (1 + \varepsilon\partial\psi)^{-1})$.

The following theorem is established in [**5**].

Theorem 2.1 ([**5**, Theorem 4.1]). *Assume that* (**A1**)-(**A5**) *are satisfied. Assume that* α/λ *and* β/κ *satisfy*

$$\frac{|\alpha|}{\lambda} \le c_p^{-1}, \quad \left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in CGL(c_q^{-1}).$$

Then for any $u_0 \in D(\varphi) \cap D(\psi)$ *there exists a global strong solution* $u(\cdot) \in C([0,\infty); X)$ *to* (ACP) *such that*

(a) $u(\cdot) \in C^{0,1/2}([0,T]; X), \quad T > 0,$

(b) $(du/dt)(\cdot), \partial\varphi(u(\cdot)), \partial\psi(u(\cdot)) \in L^2(0,T; X), \quad T > 0,$

(c) $\varphi(u(\cdot))$ *and* $\psi(u(\cdot))$ *are absolutely continuous on* $[0,T]$ *for every* $T > 0$,

with the estimates

$$\|u(t)\| \le e^{\gamma t}\|u_0\|, \quad t > 0, \tag{2.1}$$

$$E(u(t)) + \eta \int_0^t (\delta^2\|\partial\varphi(u(s))\|^2 + \|\partial\psi(u(s))\|^2)\, ds \le e^{\gamma + rt}E(u_0), \quad t > 0, \tag{2.2}$$

where

$$E(u) := \delta^2\varphi(u) + \psi(u),$$

$\gamma := \max\{\gamma, 0\}$, $r := \max\{p, q\}$ *and* $\delta, \eta > 0$ *are constants.*

Next we apply Theorem 2.1 to $(\mathrm{CGL})_{\mathbb{R}^N,\mu}$. In the complex Hilbert space $X := L^2(\mathbb{R}^N)$ we introduce two convex functions on X:

$$\varphi(u) := \begin{cases} \frac{1}{2}(\|\nabla u\|_{L^2}^2 + \mu^2\||x|u\|_{L^2}^2) & \text{if } u \in D(\varphi) := H^1(\mathbb{R}^N) \cap D(|x|), \\ \infty & \text{otherwise,} \end{cases} \tag{2.3}$$

$$\psi(u) := \begin{cases} \frac{1}{q}\|u\|_{L^q}^q & \text{if } u \in D(\psi) := X \cap L^q(\mathbb{R}^N), \\ \infty & \text{otherwise.} \end{cases} \tag{2.4}$$

Then their subdifferentials are given by

$$\partial\varphi(u) = -\Delta u + \mu^2|x|^2 u, \quad u \in D(\partial\varphi) = H^2(\mathbb{R}^N) \cap D(|x|^2),$$
$$\partial\psi(u) = |u|^{q-2}u, \quad u \in D(\partial\psi) = X \cap L^{2(q-1)}(\mathbb{R}^N).$$

To apply Theorem 2.1 with those X, φ and ψ, we prepare some lemmas.

Lemma 2.2. *Let* $N \in \mathbb{N}$ *and* $\mu > 0$. *Then for every* $u \in H^1(\mathbb{R}^N) \cap D(|x|)$,

$$\|u\|_{L^2}^2 \le \frac{2}{N}\|\nabla u\|_{L^2}\||x|u\|_{L^2}; \tag{2.5}$$

in particular,

$$N\mu\|u\|_{L^2}^2 \le \|\nabla u\|_{L^2}^2 + \mu^2\||x|u\|_{L^2}^2. \tag{2.6}$$

Proof. Let $u \in C_0^\infty(\mathbb{R}^N)$ and $\varepsilon > 0$. Let $|x|_\varepsilon := |x|(1+\varepsilon|x|)^{-1}$ be the Yosida approximation of $|x|$ and $x_\varepsilon := x(1+\varepsilon|x|)^{-1}$. Then we can obtain

$$N\int_{\mathbb{R}^N} \frac{|u(x)|^2}{1+\varepsilon|x|}\,dx \le 2\|\nabla u\|_{L^2}\big\||x|_\varepsilon u\big\|_{L^2} + \varepsilon\|u\|_{L^2}\big\||x|_\varepsilon u\big\|_{L^2}. \tag{2.7}$$

In fact, observing

$$\begin{aligned} N(1+\varepsilon|x|)^{-1} &= \operatorname{div} x_\varepsilon + \varepsilon|x|_\varepsilon(1+\varepsilon|x|)^{-1} \\ &\le \operatorname{div} x_\varepsilon + \varepsilon|x|_\varepsilon, \end{aligned}$$

we see from integration by parts that

$$\begin{aligned} N\int_{\mathbb{R}^N} \frac{|u(x)|^2}{1+\varepsilon|x|}\,dx &\le \int_{\mathbb{R}^N} (\operatorname{div} x_\varepsilon)|u(x)|^2\,dx + \varepsilon\int_{\mathbb{R}^N} |x|_\varepsilon|u(x)|^2\,dx \\ &= -2\int_{\mathbb{R}^N} x_\varepsilon\cdot \operatorname{Re}\,(u(x)\nabla\overline{u(x)})\,dx + \varepsilon\|u\|_{L^2}\big\||x|_\varepsilon u\big\|_{L^2} \\ &\le 2\|\nabla u\|_{L^2}\big\||x|_\varepsilon u\big\|_{L^2} + \varepsilon\|u\|_{L^2}\big\||x|_\varepsilon u\big\|_{L^2}. \end{aligned}$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, (2.7) is true also for $u \in H^1(\mathbb{R}^N)$. Letting $\varepsilon \downarrow 0$ in (2.7) for $u \in H^1(\mathbb{R}^N)\cap D(|x|)$, we obtain (2.5). (2.6) is a consequence of (2.5). □

Lemma 2.3 ([**5**, Lemma 6.2]). *Let $q \ge 2$. Then for $u \in H^2(\mathbb{R}^N)$ and $\varepsilon > 0$,*

$$|\operatorname{Im}\,(-\Delta u, \partial\psi_\varepsilon(u))_{L^2}| \le \frac{q-2}{2\sqrt{q-1}}\operatorname{Re}\,(-\Delta u, \partial\psi_\varepsilon(u)). \tag{2.8}$$

Lemma 2.4. *Let $V : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative function. Then for $\varepsilon > 0$ and $u \in L^2(\mathbb{R}^N)$ with $Vu \in L^2(\mathbb{R}^N)$,*

$$(Vu, \partial\psi_\varepsilon(u))_{L^2} = \int_{\mathbb{R}^N} V|u_\varepsilon|^q\,dx + \varepsilon\int_{\mathbb{R}^N} V|u_\varepsilon|^{2(q-1)}\,dx \tag{2.9}$$

where $u_\varepsilon := (1+\varepsilon\partial\psi)^{-1}u$. Consequently, $(Vu, \partial\psi_\varepsilon(u))_{L^2}$ is real and nonnegative.

Proof. Let $\varepsilon > 0$ and $u \in L^2(\mathbb{R}^N)$ with $Vu \in L^2(\mathbb{R}^N)$. Setting $u_\varepsilon := (1+\varepsilon\partial\psi)^{-1}u$, we see that

$$u = u_\varepsilon + \varepsilon|u_\varepsilon|^{q-2}u_\varepsilon, \quad \partial\psi_\varepsilon(u) = |u_\varepsilon|^{q-2}u_\varepsilon.$$

Substituting these identities into $(Vu, \partial\psi_\varepsilon(u))_{L^2}$, we can obtain (2.9). □

Lemma 2.5. *Let $q \ge 2$. Then for $u \in D(\partial\varphi)$ and $\varepsilon > 0$,*

$$|\operatorname{Im}\,(\partial\varphi(u), \partial\psi_\varepsilon(u))_{L^2}| \le \frac{q-2}{2\sqrt{q-1}}\operatorname{Re}\,(\partial\varphi(u), \partial\psi_\varepsilon(u))_{L^2}. \tag{2.10}$$

Lemma 2.5 is a consequence of Lemmas 2.3 and 2.4 with $V(x) := \mu^2|x|^2$; note that $\partial\varphi = -\Delta + V(x)$.

Proof of Theorem 1.1. Let $X := L^2(\mathbb{R}^N)$. Let φ and ψ be defined as (2.3) and (2.4). We see from (2.6) that $(-\Delta + \mu^2|x|^2)^{-1}$ is bounded. In fact, (2.6) implies that for every $u \in H^2(\mathbb{R}^N) \cap D(|x|^2)$,

$$\begin{aligned} N\mu\|u\|_{L^2}^2 &\le \|\nabla u\|_{L^2}^2 + \mu^2\big\||x|u\big\|_{L^2}^2 \\ &= ((-\Delta + \mu^2|x|^2)u, u)_{L^2} \\ &\le \|(-\Delta + \mu^2|x|^2)u\|_{L^2}\|u\|_{L^2}. \end{aligned}$$

Since the potential $|x|^2$ blows up as $|x| \to \infty$, it follows from [**4**, Theorem 4.1] that $(-\Delta + \mu^2|x|^2)^{-1}$ is compact in X and hence (**A1**) is satisfied. (**A2**) (with $p = 2$) and (**A3**) are trivial by definition. Since $\partial\varphi$ is nonnegative selfadjoint in X, (**A4**) is satisfied with $c_p = 0$. Lemma 2.4 implies that (**A5**) is satisfied with

$$c_q := \frac{q-2}{2\sqrt{q-1}}.$$

Therefore we can apply Theorem 2.1 with those X, φ. Consequently, we obtain the existence part of Theorem 1.1. As in the proof of [**5**, Theorem 1.1], we can prove (1.3) by virtue of Theorem 2.1 (c). Moreover, (1.4) and (1.5) follow from (2.1) and (2.2), respectively (see Remark 2.1 below). This completes the proof of Theorem 1.1. □

Remark 2.1. By the definition of φ in (2.3), Theorem 2.1 (b) asserts that

$$u(\cdot),\ (\Delta - \mu^2|x|^2)u(\cdot) \in L^2(0,T;\ L^2(\mathbb{R}^N)), \quad T > 0.$$

This fact implies that

$$\Delta u(\cdot), |x|^2u(\cdot) \in L^2(0,T;\ L^2(\mathbb{R}^N)), \quad T > 0.$$

This is a direct consequence of the following inequality (see Okazawa [**4**]):

$$\text{(2.11)} \quad \|\Delta u\|_{L^2}^2 + \mu^4\big\||x|^2u\big\|_{L^2}^2 \le \|(\Delta - \mu^2|x|^2)u\|_{L^2}^2 + 2N\mu^2\|u\|_{L^2}^2, \quad u \in H^2(\mathbb{R}^N)\cap D(|x|^2).$$

3. Answer to Problem 2

In this section we give the proof of Theorem 1.2.

Proof of Theorem 1.2. It suffices to prove (1.7). Let $q < 2^*$. Then $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$. Let $u(\cdot)$ and $v(\cdot)$ be the global strong solutions to $(\mathrm{CGL})_{\mathbb{R}^N,\mu}$ with initial data $u_0, v_0 \in H^1(\mathbb{R}^N) \cap D(|x|)$, respectively. Then $w(\cdot) := u(\cdot) - v(\cdot)$ satisfies

$$\text{(3.1)} \qquad \frac{\partial w}{\partial t} + (\lambda + i\alpha)(-\Delta + \mu^2|x|^2)w + (\kappa + i\beta)(|u|^{q-2}u - |v|^{q-2}v) = \gamma w.$$

Making the L^2-inner product of (3.1) with w, we have

$$\text{(3.2)} \qquad \frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \lambda(\|\nabla w\|_{L^2}^2 + \mu^2\big\||x|w\big\|_{L^2}^2) + I = \gamma\|w\|_{L^2}^2,$$

where

$$I := \mathrm{Re}\,[(\kappa + i\beta)(|u|^{q-2}u - |v|^{q-2}v, w)_{L^2}].$$

Since $||u|^{q-2}u - |v|^{q-2}v| \le (q-1)(|u|^{q-2} + |v|^{q-2})|w|$, we have

$$\begin{aligned} |I| &\le (q-1)\sqrt{\kappa^2+\beta^2}\int_{\mathbb{R}^N}(|u|^{q-2}+|v|^{q-2})|w|^2\,dx \\ &\le (q-1)\sqrt{\kappa^2+\beta^2}(\|u\|_{L^q}^{q-2}+\|v\|_{L^q}^{q-2})\|w\|_{L^q}^2, \end{aligned} \tag{3.3}$$

where we used the Hölder inequality in the second inequality. We see from (1.5) that

$$\|u(t)\|_{L^q}^q \le qe^{\gamma_+ qt}E_\mu(u_0), \quad \|v(t)\|_{L^q}^q \le qe^{\gamma_+ qt}E_\mu(v_0).$$

Hence we have

$$\|u(t)\|_{L^q}^{q-2} + \|v(t)\|_{L^q}^{q-2} \le K_1 e^{\gamma_+(q-2)t}, \tag{3.4}$$

where

$$K_1 := q^{1-2/q}\Big[E_\mu(u_0)^{1-2/q} + E_\mu(v_0)^{1-2/q}\Big].$$

On the other hand, we use the Gagliardo-Nirenberg inequality

$$\|w\|_{L^q} \le C\|w\|_{L^2}^{1-a}\|\nabla w\|_{L^2}^a, \tag{3.5}$$

where $a := N(1/2 - 1/q) \in [0,1)$ and $C = C(q,N)$ is a positive constant. Applying (3.4) and (3.5) to (3.3), we see by the Young inequality that

$$\begin{aligned} |I| &\le (q-1)\sqrt{\kappa^2+\beta^2}CK_1e^{\gamma_+(q-2)t}\|w\|_{L^2}^{2(1-a)}\|\nabla w\|_{L^2}^{2a} \\ &\le K_2 e^{\frac{\gamma_+(q-2)}{1-a}t}\|w\|_{L^2}^2 + \frac{\lambda}{2}\|\nabla w\|_{L^2}^2, \end{aligned}$$

where

$$K_2 := \Big(\frac{2}{\lambda}\Big)^{a/(1-a)}\Big[(q-1)\sqrt{\kappa^2+\beta^2}CK_1\Big]^{1/(1-a)}.$$

Plugging this inequality with (3.2), we obtain

$$\frac{d}{dt}\|w\|_{L^2}^2 + \lambda(\|\nabla w\|_{L^2}^2 + \mu^2\||x|w\|_{L^2}^2) \le 2\Big(\gamma + K_2e^{\frac{\gamma_+(q-2)}{1-a}t}\Big)\|w\|_{L^2}^2. \tag{3.6}$$

Setting

$$K(t) := 2\Big(\gamma + K_2 e^{\frac{\gamma_+(q-2)}{1-a}t}\Big),$$

we have

$$\frac{d}{ds}\Big[e^{-\int_0^s K(r)\,dr}\|w(s)\|_{L^2}^2\Big] + \lambda e^{-\int_0^s K(r)\,dr}(\|\nabla w(s)\|_{L^2}^2 + \mu^2\||x|w(s)\|_{L^2}^2) \le 0.$$

Integrating this inequality on $[0,t]$ for $t > 0$, we obtain (1.7). $\square$

4. Answer to Problem 3

Let $u_\mu(\cdot)$ be the unique global strong solution to $(\mathrm{CGL})_{\mathbb{R}^N,\mu}$ ($\mu > 0$) constructed in Theorems 1.1 and 1.2. To prove Theorem 1.3 we need a priori estimate of $\big\||x|u_\mu(\cdot)\big\|_{L^2}$ independent of μ.

Lemma 4.1. *Let $N, \lambda + i\alpha, \kappa + i\beta, \gamma, \mu$ be the same as in* Theorem 1.2. *Let $u_\mu(\cdot)$ be the solution to $(\mathrm{CGL})_{\mathbb{R}^N,\mu}$ with $u_\mu(0) = u_0 \in H^1(\mathbb{R}^N) \cap D(|x|^2)$. Then for every $t > 0$,*

$$\big\||x|^2 u_\mu(t)\big\|_{L^2} \le e^{\gamma t}\Big(ct\|u_0\|_{L^2} + \big\||x|^2 u_0\big\|_{L^2}\Big), \tag{4.1}$$

where $c > 0$ is a constant depending only on $\lambda + i\alpha$.

Proof. We give a formal proof. The proof can be justified by using the Yosida approximation of $|x|^2$. Making the inner product of the equation in $(\mathrm{CGL})_{\mathbb{R}^N,\mu}$ with $|x|^4 u_\mu(\cdot)$, we have

$$\frac{1}{2}\frac{d}{dt}\big\||x|^2 u_\mu\big\|_{L^2}^2 + J - \gamma\big\||x|^2 u_\mu\big\|_{L^2}^2 \le 0, \tag{4.2}$$

where

$$J := \mathrm{Re}\,[(\lambda + i\alpha)(-\Delta u_\mu + \mu^2|x|^2 u_\mu, |x|^4 u_\mu)_{L^2}].$$

Applying integration by parts and the Schwarz inequality, we obtain

$$\begin{aligned} J &\ge \lambda\big\||x|^2\nabla u_\mu\big\|_{L^2}^2 - 4\sqrt{\lambda^2 + \alpha^2}\big\||x|^2\nabla u_\mu\big\|_{L^2}\big\||x|u_\mu\big\|_{L^2} \\ &\ge -c\big\||x|u_\mu\big\|_{L^2}^2, \end{aligned} \tag{4.3}$$

where $c := (4/\lambda)(\lambda^2 + \alpha^2)$. On the other hand, it follows from the Schwarz inequality and (1.4) that

$$\big\||x|u_\mu(t)\big\|_{L^2}^2 \le e^{\gamma t}\|u_0\|_{L^2}\big\||x|^2 u_\mu(t)\big\|_{L^2}.$$

Applying this inequality to (4.3), we see from (4.2) that

$$\frac{1}{2}\frac{d}{dt}\big\||x|^2 u_\mu(t)\big\|_{L^2}^2 - ce^{\gamma t}\|u_0\|_{L^2}\big\||x|^2 u_\mu(t)\big\|_{L^2} - \gamma\big\||x|^2 u_\mu(t)\big\|_{L^2}^2 \le 0,$$

which implies that

$$\frac{d}{dt}\Big(e^{-\gamma t}\big\||x|^2 u_\mu(t)\big\|_{L^2}\Big) \le c\|u_0\|_{L^2}.$$

Integrating this inequality on $[0, t]$ yields (4.1). □

Now we are in position to complete the proof of Theorem 1.3 which answers to **Problem 3**.

Proof of Theorem 1.3. Let $u_\mu(\cdot)$ be the unique global strong solution to $(\mathrm{CGL})_{\mathbb{R}^N,\mu}$ with $u_\mu(0) = u_0 \in H^1(\mathbb{R}^N) \cap D(|x|^2)$. Set $w_{\mu,\nu}(\cdot) := u_\mu(\cdot) - u_\nu(\cdot)$ for $\mu, \nu \in (0, 1]$. Similarly in deriving (3.6), we have

$$\frac{1}{2}\frac{d}{dt}\|w_{\mu,\nu}\|_{L^2}^2 + \frac{\lambda}{2}\|\nabla w_{\mu,\nu}\|_{L^2}^2 + I_{\mu,\nu} \le \frac{K(t)}{2}\|w_{\mu,\nu}\|_{L^2}^2,$$

where

$$I_{\mu,\nu} := \mathrm{Re}\,[(\lambda + i\alpha)(\mu^2|x|^2u_\mu - \nu^2|x|^2u_\nu, w_{\mu,\nu})_{L^2}]$$
$$= \lambda\mu^2\||x|w_{\mu,\nu}\|_{L^2}^2 + (\mu^2 - \nu^2)\mathrm{Re}\,[(\lambda + i\alpha)(|x|^2u_\nu, w_{\mu,\nu})_{L^2}],$$

and $K(\cdot)$ is the same function as in Theorem 1.2. From (4.1) we have

$$I_{\mu,\nu} \geq -\sqrt{\lambda^2 + \alpha^2}|\mu^2 - \nu^2|\left\||x|u_\nu\right\|_{L^2}\|w_{\mu,\nu}\|_{L^2}$$
$$\geq -M(t)|\mu^2 - \nu^2|\|w_{\mu,\nu}\|_{L^2},$$

where

$$M(t) := \sqrt{\lambda^2 + \alpha^2}e^{\gamma t}\Big(ct\|u_0\|_{L^2} + \left\||x|^2u_0\right\|_{L^2}\Big).$$

Hence we obtain

$$\frac{d}{dt}\|w_{\mu,\nu}\|_{L^2} \leq \frac{K(t)}{2}\|w_{\mu,\nu}\|_{L^2} + M(t)|\mu^2 - \nu^2|. \tag{4.4}$$

Applying the Gronwall lemma to (4.4) yields

$$\|w_{\mu,\nu}(t)\|_{L^2} \leq |\mu^2 - \nu^2| \int_0^t e^{\int_s^t \frac{K(r)}{2}\,dr} M(s)\,ds.$$

This inequality implies that for every $T > 0$,

$$\sup_{0<t<T} \|w_{\mu,\nu}(t)\|_{L^2} \leq |\mu^2 - \nu^2| \int_0^T e^{\int_s^T \frac{K(r)}{2}\,dr} M(s)\,ds.$$

This implies that $\{u_\mu(\cdot)\}$ satisfies the Cauchy condition in $C([0,T];\, L^2(\mathbb{R}^N))$ and hence there exists $u \in C([0,\infty);\, L^2(\mathbb{R}^N))$ such that

$$u_\mu(\cdot) \to u(\cdot) \quad (\mu \downarrow 0) \quad \text{strongly in } C([0,T];\, L^2(\mathbb{R}^N)).$$

We see from (1.4), (1.5) and (2.11) that

$$\{\Delta u_\mu(\cdot)\} \text{ and } \{|u_\mu|^{q-2}u_\mu(\cdot)\} \text{ are bounded in } L^2(0,T;\, L^2(\mathbb{R}^N)).$$

Moreover, (4.1) implies that

$$\{|x|^2u_\mu(\cdot)\} \text{ is also bounded in } L^2(0,T;\, L^2(\mathbb{R}^N)).$$

Since Δ, $|x|^2$ and $\partial/\partial t$ are weakly closed as operators in $L^2(0,T;\, L^2(\mathbb{R}^N))$, it follows that $\Delta u(\cdot)$, $|x|^2u(\cdot)$, $(\partial u/\partial t)(\cdot) \in L^2(0,T;\, L^2(\mathbb{R}^N))$ and

$$\Delta u_\mu(\cdot) \to \Delta u(\cdot) \quad \text{weakly in } L^2(0,T;\, L^2(\mathbb{R}^N)),$$
$$\mu^2|x|^2u_\mu(\cdot) \to 0 \quad \text{weakly in } L^2(0,T;\, L^2(\mathbb{R}^N)),$$
$$(\partial u_\mu/\partial t)(\cdot) \to (\partial u/\partial t)(\cdot) \quad \text{weakly in } L^2(0,T;\, L^2(\mathbb{R}^N)).$$

We can also see from the demiclosedness of $\partial\psi$ as operators in $L^2(0,T;\, L^2(\mathbb{R}^N))$ that $|u|^{q-2}u(\cdot) \in L^2(0,T;\, L^2(\mathbb{R}^N))$ and

$$|u_\mu|^{q-2}u_\mu(\cdot) \to |u|^{q-2}u(\cdot) \quad \text{weakly in } L^2(0,T;\, L^2(\mathbb{R}^N)).$$

Therefore $u(\cdot)$ is a global strong solution to $(\mathrm{CGL})_{\mathbb{R}^N,0}$. □

5. Concluding remarks

We have proved the existence of global strong solutions to $(\mathrm{CGL})_{\mathbb{R}^N,0}$ under the conditions that

$$\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in CGL(c_q^{-1}),$$
$$2 \leq q < 2^*,$$
$$u_0 \in H^1(\mathbb{R}^N) \cap D(|x|^2).$$

There are two comments; one is about the initial data u_0 and the other is about the exponent q.

(I) If $u_0 \in H^1(\mathbb{R}^N)$, then we can approximate u_0 by

$$u_{0,n} := (1 + n^{-1}|x|^2)^{-1} u_0.$$

As in the proof of Theorem 1.3 we can see that the corresponding solution $u_n(\cdot)$ with $u_n(0) = u_{0,n}$ converges to the desired solution.

(II) For the uniqueness we assumed that $2 \leq q < 2^*$; and hence we obtain the solution to $(\mathrm{CGL})_{\mathbb{R}^N,0}$ for such exponent q. On the other hand, Ginibre-Velo [**2**] have already proved the existence of solutions to $(\mathrm{CGL})_{\mathbb{R}^N,0}$ under the mild condition that "$2 \leq q < \infty$". The key of their proof lies in the compactness of $H^1(\Omega) \hookrightarrow L^2(\Omega)$ for a bounded domain $\Omega \subset \mathbb{R}^N$. Our method lies in another compactness $H^1(\mathbb{R}^N) \cap D(|x|) \hookrightarrow L^2(\mathbb{R}^N)$. In the future we shall improve our method by using the compactness $H^1(\mathbb{R}^N) \cap D(V) \hookrightarrow L^2(\mathbb{R}^N)$, where $V : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative function satisfying

$$\lim_{|x| \to \infty} V(x) = \infty.$$

Choosing V properly, we would show the existence under the condition that $2 \leq q < \infty$.

Acknowledgments

The authors would like to express their sincere gratitude to Professor Toyohiko Aiki for giving them an opportunity to talk about this work.

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