数理解析研究所講究録
第 1746 巻 2011 年 88-99

平面内の曲線の運動[1)]
On motion of curves in the plane

宮崎大学・工学部 矢崎 成俊 (Shigetoshi Yazaki)
Faculty of Engineering, University of Miyazaki[2)]

Contents.

1 Introduction

In this talk we study evolution of a family of closed smooth plane curves:

$$\boldsymbol{x}: \ [0,1]\times[0,T)\to\Gamma(t)=\{\boldsymbol{x}(u,t)\in\mathbb{R}^2;\ u\in[0,1]\subset\mathbb{R}/\mathbb{Z}\},$$

starting from a given initial curve $\Gamma(0)=\Gamma_0$, and driven by the evolution law:

$$\partial_t\boldsymbol{x}=\alpha\boldsymbol{t}+\beta\boldsymbol{n},$$

where $\boldsymbol{t}=\partial_u\boldsymbol{x}/|\partial_u\boldsymbol{x}|$ is the unit tangent vector, and $\boldsymbol{n}$ is the unit outward normal vector which satisfies $\det(\boldsymbol{n},\boldsymbol{t})=1$. Here and hereafter, we denote $\partial_\xi\mathsf{F}=\partial\mathsf{F}/\partial\xi$ and $|\boldsymbol{a}|=\sqrt{\boldsymbol{a}.\boldsymbol{a}}$ where $\boldsymbol{a}.\boldsymbol{b}$ is Euclidean inner product between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The solution curves are immersed or embedded such that $|\partial_u\boldsymbol{x}|>0$ holds.

We remark that the tangent velocity α has no effect of shape of solution curves and affect only parametrization. Therefore, the shape of solution curves are determined by the normal velocity β, and a nontrivial tangent velocity α will be chosen depending on the purpose.

The normal velocity β may depend on many factors, which are arising in various applied fields like e.g. the material sciences, dynamics of phase boundaries in thermomechanics, computational geometry, image processing and computer vision, fluid dynamics, the field of ice and snow crystal, etc. For the comprehensive overview of applications we refer the book by Sethian [14]. According to the book, β can be written as:

$$\beta=\beta(\mathcal{L};\mathcal{G};\mathcal{I}),$$

where

[1)]Manuscript for "現象の数理解析に向けた非線形発展方程式とその周辺", October 13 – 15, 2010 at RIMS, Kyoto University. The author is partially supported by Grant-in-Aid for Encouragement of Young Scientists (No.21740079).

[2)]1-1 Gakuen Kibanadai Nishi, Miyazaki 889-2192, JAPAN. *E-mail*: yazaki@cc.miyazaki-u.ac.jp

- $\mathcal{L}$ocal properties are those determined by local geometric information of Γ, such as the curvature k and the normal or the tangential direction.

- $\mathcal{G}$lobal properties of Γ are those that depend on the shape and the position of Γ, such as the position vector $\boldsymbol{x} \in \Gamma$, the length of Γ, and the integrals along Γ associated PDEs, etc.

- $\mathcal{I}$ndependent properties are those that are independent of the shape of Γ, such as an underlying fluid velocity, etc.

Here k is the curvature in the direction $-\boldsymbol{n}$, which is defined from $\partial_s \boldsymbol{t} = -k\boldsymbol{n}$, $\partial_s \boldsymbol{n} = k\boldsymbol{t}$, and described as $k = \det(\partial_s \boldsymbol{x}, \partial_{ss}\boldsymbol{x})$, and $\partial_s \boldsymbol{x} = \boldsymbol{t}$ is the unit tangent vector. Here and hereafter, we denote $\partial_{\xi\xi}\mathsf{F} = \partial(\partial_\xi \mathsf{F})/\partial\xi$. Note that ∂_s is not partial differentiation. It means the operator $\partial_s \mathsf{F}(u,t) = g(u,t)^{-1}\partial_u \mathsf{F}(u,t)$, where $g(u,t) = |\partial_u \boldsymbol{x}(u,t)| > 0$ is called the local length and s is the arc length parameter determined from $ds = g(u,t)\,du$.

In this note, we will focus on the utilization of a nontrivial tangential velocity α, and will mention on two applications in the case $\beta = \beta(\mathcal{L};\mathcal{G};\mathcal{I})$: one is image segmentation with $\mathcal{L}(k,\boldsymbol{n})$; $\mathcal{G}(\boldsymbol{x})$ or $\mathcal{L}(k)$; $\mathcal{G}(\boldsymbol{x})$; $\mathcal{I}(const.)$, and the other one is numerical computation of Hele-Shaw flow in a time-dependent gap with $\mathcal{G}$(integral of a PDE).

2 Curvature adjusted tangential velocity

As mentioned in the previous section, the tangential velocity functional α has no effect of the shape of evolving curves [5, Proposition 2.4], and the shape is determined by the value of the normal velocity β only. Hence the simplest setting $\alpha \equiv 0$ can be chosen. Dziuk [4] studied a numerical scheme for $\beta = -k$ in this case. In the case general β, however, such a choice of α may lead to various numerical instabilities caused by either undesirable concentration and/or extreme dispersion of numerical grid points. Therefore, to obtain stable numerical computation, several choices of a nontrivial tangential velocity have been emphasized and developed by many authors. We will present a brief review of development of nontrivial tangential velocities.

Kimura [7, 8] proposed a uniform redistribution scheme in the case $\beta = -k$ by using a special choice of α which satisfies discretization of an average condition and the uniform distribution condition:

$$\text{(U)}\quad r(u,t) = \frac{g(u,t)}{L(\Gamma(t))} \equiv 1 \quad (\forall u).$$

Hou, Lowengrub and Shelley [6] utilized condition (U) directly (especially for $\beta = -k$) starting from $r(u,0) \equiv 1$, and derived

$$\partial_s \alpha = \langle k\beta \rangle - k\beta, \tag{2.1}$$

which comes from

$$\partial_t r = \frac{g}{L}\left(\partial_s\alpha + k\beta - \langle k\beta\rangle\right) \equiv 0 \quad (\forall u).$$

It was proposed independently by Mikula and Ševčovič [10]. In [6, Appendix 2], Hou et al. also pointed out generalization of (2.1) as follows:

$$\frac{\partial_s(\varphi(k)\alpha)}{\varphi(k)} = \frac{\langle f\rangle}{\langle\varphi(k)\rangle} - \frac{f}{\varphi(k)}, \quad f = \varphi(k)k\beta - \varphi'(k)\left(\partial_{ss}\beta + k^2\beta\right) \tag{2.2}$$

for a given function φ. If $\varphi \equiv 1$, then this is nothing but (2.1). (2.2) is derived from the following calculation. Let a generalized relative local length be

$$r_\varphi(u,t) = r(u,t)\frac{\varphi(k(u,t))}{\langle\varphi(k(\cdot,t))\rangle}.$$

Then preserving condition $\partial_t r_\varphi(u,t) \equiv 0$ leads (2.2).

As mentioned above, in the paper [10] the authors arrived (2.1) in general frame work of the so-called intrinsic heat equation for $\beta = \beta(\theta, k)$, where θ is the angle of $\boldsymbol{n}$, i.e., $\boldsymbol{n} = (\cos\theta, \sin\theta)$ and $\boldsymbol{t} = (-\sin\theta, \cos\theta)$. This result was improvement of [9] in which satisfactory results were obtained only in the case $\beta = \beta(k)$ being linear and sublinear with respect to k. After these results, in the series of the paper [11, 12, 13], they proposed method of asymptotically uniform redistribution, i.e., derived

$$\partial_s\alpha = \langle k\beta\rangle - k\beta + \left(r^{-1} - 1\right)\omega(t) \tag{2.3}$$

for quite general normal velocity $\beta = \beta(\boldsymbol{x}, \theta, k)$, where $\omega \in L^1_{loc}[0,T)$ is a relaxation function satisfying $\int_0^T \omega(t)\,dt = +\infty$. Their method succeeded and was applied to geodesic curvature flows and image segmentation, etc.

Besides these uniform distribution method, under the so-called crystalline curvature flow, grid points are distributed dense (resp. sparse) on the subarc where the absolute value of curvature is large (resp. small). Although this redistribution is far from uniform, numerical computation is quite stable. One of the reason is that polygonal curves are restricted in an admissible class. To apply the essence of crystalline curvature flow to a general discretization model of motion of smooth curves, the tangential velocity $\alpha = \partial_s\beta/k$ was extracted, which is utilized in crystalline curvature flow equation implicitly [18].

The asymptotically uniform redistribution is quite effective and valid for wide range of application. However, from approximation point of view, unless solution curve is a circle, there is no reason to take uniform redistribution. Hence the redistribution will be desired in a way of taking into account the shape of evolution curves, i.e., depending on size of curvatures. In the paper [15, 16], it is proposed that a method of redistribution which takes into account the shape of limiting curve such as

$$\frac{\partial_s(\varphi(k)\alpha)}{\varphi(k)} = \frac{\langle f\rangle}{\langle\varphi(k)\rangle} - \frac{f}{\varphi(k)} + \left(r_\varphi^{-1} - 1\right)\omega(t).$$

If $\varphi \equiv 1$, then this is nothing but (2.3), and if $\varphi = k$ and Γ is convex, we have $\alpha = \partial_s \beta / k$ in the case $\omega = 0$. Therefore, this is a combination of method of asymptotic uniform redistribution and the crystalline tangential velocity as mentioned above. Notice that this method was applied to an image segmentation and nice results were confirmed [2].

To complete the overview of various tangential redistribution method we also mention a locally dependent tangential velocity. For the case $\beta = -k$ it was proposed by Deckelnick [3] who used $\alpha = -\partial_u(g^{-1})$. Then the evolution equation becomes a simple parabolic PDE $\partial_t \boldsymbol{x} = g^{-2}\partial_u^2 \boldsymbol{x}$.

As far as 3D implementation of tangential redistribution is concerned, in a recent paper by Barrett, Garcke and Nürnberg [1] the authors proposed and studied a new efficient numerical scheme for evolution of surfaces driven by the Laplacian of the mean curvature. It turns out, that their numerical scheme has implicitly built in a uniform redistribution tangential velocity vector.

3 Image segmentation

The gradient flow $\beta = -\gamma(\boldsymbol{x})k - \nabla\gamma(\boldsymbol{x}).\boldsymbol{n}$ is utilized for image segmentation as follows. Let an image intensity function be $I : \mathbb{R}^2 \supset \Omega \to [0,1]$. Here $I = 0$ (resp. $I = 1$) corresponds to black (resp. white) color and $I \in (0,1)$ corresponds to gray colors. For simplicity, we assume that our target figures are given in white color with black background. Then the image outline or edge correspond to the region where $|\nabla I(\boldsymbol{x})|$ is quite large. Let us introduce an auxiliary function $\gamma(\boldsymbol{x}) = f(|\nabla I(\boldsymbol{x})|)$ where f is a smooth edge detector function like e.g. $f(s) = 1/(1+s^2)$ or $f(s) = e^{-s}$. Hence the solution curve $\Gamma(t)$ of $\beta = -\gamma(\boldsymbol{x})k - \nabla\gamma(\boldsymbol{x}).\boldsymbol{n}$ makes the energy $E_F(\Gamma(t))$ smaller and smaller, in other words, its moves toward the edge where $|\nabla I(\boldsymbol{x})|$ is large. This is a fundamental idea of image segmentation, and it has developed to a sophisticated scheme [12, 13].

The following scheme is more simple [2]. In the following computations, the target figure is given by a digital gray scale bitmap image represented by integer values between 0 and 255 on some prescribed pixels. The values 0 and 255 correspond to black and white colors, respectively, whereas the values between 0 and 255 correspond to gray colors.

Given a figure, we can construct its image intensity function $I : \mathbb{R}^2 \supset \Omega \to \{0, \ldots, 255\} \subset \mathbb{Z}$. Note that $I(\boldsymbol{x})$ is piecewise constant in each pixel.

We consider the flow $\beta = -k + F$ and define the forcing term $F(\boldsymbol{x})$ as follows:

$$F(\boldsymbol{x}) = (F_{max} - F_{min})\frac{I(\boldsymbol{x})}{255} - F_{max} \quad (\boldsymbol{x} \in \Omega),$$

where $F_{max} > 0$ corresponds to purely black color (background) and $F_{min} < 0$ corresponds to purely white color (the object to be segmented). Maximal and minimal values determine the final shape because in general $1/F$ is equivalent to the minimal radius the curve can attain. The choice of small values of F causes the final shape to be rounded or the curve can not pass through narrow gaps.

4 Hele-Shaw flow in a time-dependent gap

The so-called Hele-Shaw flow is flow of viscous liquid which is contained in the narrow gap between two parallel plates, that is, in the Hele-Shaw cell. Figure 4.1 indicates the Hele-Shaw cell settled in the xyz-coordinate.

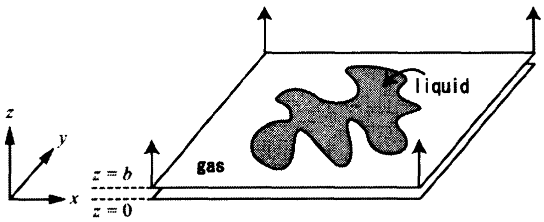

Figure 4.1: Hele-Shaw cell

Let b be the gap between two parallel plates in the z-direction. In classical Hele-Shaw experiments, b is fixed and taken around 1mm. Shelley, Tian and Wlodarski [17] proposed a problem in the case where b depends on the time t, i.e., the upper plate is being lifted uniformly at a specific rate. They established the existence, uniqueness and regularity of solutions in the case where the surface tension is zero. They also studied numerical computation by means of ODE for the time discretization.

In this section, we will propose the boundary element method with a curvature adjusted tangential velocity.

We assume that there are no effect of external force like gravity in the Hele-Shaw cell and the liquid is governed by the Navier-Stokes equation:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \nu \Delta \boldsymbol{v}, \tag{4.1}$$

where ρ is density, ν is kinetic viscosity. Unknown functions are the pressure p and the velocity $\boldsymbol{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$. In what follows, we will simplify this equations.

Firstly, we require the following.

(A1) the velocity of water is very slow, and the flow is stationary.

By this assumption, we operate so-called Stokes approximation for stationary flow, and the LHS of (4.1) is neglected:

$$\mathbf{0} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\Delta \boldsymbol{v}, \tag{4.2}$$

where $\mu = \rho\nu$ is viscosity.

Secondly, we assume that

(A2) the fluid does not move in the vertical direction, i.e., $w = 0$.

Then we have

$$\nabla p = \begin{pmatrix} \partial_x p \\ \partial_y p \\ \partial_z p \end{pmatrix} = \mu \Delta \boldsymbol{v}, \quad \boldsymbol{v} = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}.$$

From this, $p = p(x, y, t)$ holds.

Thirdly, we assume the following profile of u and v (Figure 4.2):

(A3) graphs of u and v with respect to a variable z draw a parabola satisfying "$u = v = 0$ at $z = 0$ and $z = b$."

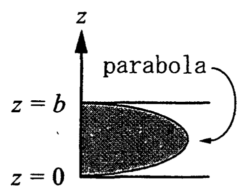

Figure 4.2: Velocity profile of u and v

Note that this assumption for u (same as for v) is equivalent to that the terms $\partial_{xx}u$ and $\partial_{yy}u$ are negligible and the term $\partial_{zz}u$ is dominant in Δu. Then u and v can be expressed as

$$u(x, y, z, t) = \varphi(x, y, t)z(z - b), \quad v(x, y, z, t) = \psi(x, y, t)z(z - b)$$

with functions φ and ψ. Hence we have

$$\partial_x p = \mu((\partial_{xx}\varphi + \partial_{yy}\varphi)z(z - b) + 2\varphi), \quad \partial_y p = \mu((\partial_{xx}\psi + \partial_{yy}\psi)z(z - b) + 2\psi).$$

Since the pressure is $p = p(x, y, t)$, we obtain

$$\partial_{xx}\varphi + \partial_{yy}\varphi = 0, \quad \partial_{xx}\psi + \partial_{yy}\psi = 0.$$

Therefore from

$$\partial_{xx}u + \partial_{yy}u = 0, \quad \partial_{xx}v + \partial_{yy}v = 0,$$

we have

$$\begin{cases} \partial_x p = \mu \partial_{zz} u = 2\mu\varphi = \dfrac{2\mu}{z(z - b)}u, \\ \partial_y p = \mu \partial_{zz} v = 2\mu\psi = \dfrac{2\mu}{z(z - b)}v. \end{cases} \Leftrightarrow \begin{cases} u = \dfrac{p_x}{2\mu}z(z - b), \\ v = \dfrac{p_y}{2\mu}z(z - b). \end{cases}$$

Fourthly, we require the following.
(A4) to take average of u and v in z-direction.

Then, the two dimensional average velocity vector is expressed by a gradient of pressure:

$$\overline{u} = \frac{1}{b}\int_0^b u\,dz = \frac{\partial_x p}{2\mu b}\left[\frac{z^3}{3} - \frac{bz^2}{2}\right]_0^b = -\frac{b^2}{12\mu}\partial_x p, \quad \overline{v} = -\frac{b^2}{12\mu}\partial_y p.$$

Hence the average velocity $\overline{u}$, $\overline{v}$ and the pressure p are functions of three variables (x, y, t), respectively. We define the two dimensional velocity such as

$$\boldsymbol{u} = \begin{pmatrix}\overline{u}\\ \overline{v}\end{pmatrix}.$$

Then we have

$$\boldsymbol{u} = -\frac{b^2}{12\mu}\nabla p, \quad \nabla p = \begin{pmatrix}\partial_x p\\ \partial_y p\end{pmatrix},$$

and from the incompressibility

$$0 = \frac{1}{b}\int_0^b \operatorname{div}\boldsymbol{v}\,dz = \frac{1}{b}\int_0^b (\partial_x u + \partial_y v + 0)\,dz = \partial_x\overline{u} + \partial_y\overline{v} = -\frac{b^2}{12\mu}(\partial_{xx}p + \partial_{yy}p).$$

Taking average of fluid region in z-direction, we deduce the problem to two dimensional problem like Figure 4.3. Then the pressure $p = p(x, y, t)$ satisfies

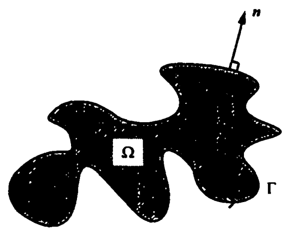

Figure 4.3: Liquid in the Hele-Shaw cell

$$\Delta p = 0, \quad (x, y) \in \Omega, \quad t > 0$$

in the interior of two dimensional fluid region. Here we have defied two dimensional Laplacian such as $\Delta p = \partial_{xx}p + \partial_{yy}p$. Since the boundary Γ moves with the fluid, deformation velocity β in the normal direction $\boldsymbol{n}$ of Γ is given as

$$\beta = \boldsymbol{u}\cdot\boldsymbol{n} = -\frac{b^2}{12\mu}\frac{\partial p}{\partial \boldsymbol{n}}, \quad \frac{\partial p}{\partial \boldsymbol{n}} = \nabla p\cdot\boldsymbol{n}.$$

Here the normal velocity of $\Gamma = \Gamma(t)$ is defined as

$$\beta = \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}\cdot\boldsymbol{n}, \quad (x, y) \in \Gamma.$$

Here and hereafter, we denote $\dot{\mathsf{F}} = \partial_t \mathsf{F}$.

On the boundary Γ, we use the Laplace's relation

$$p - p_* = \tau k, \quad \tau > 0.$$

Here p_* is the atmospheric pressure and τ is a surface tension coefficient. Since p_* is a constant, we have $\nabla(p - p_*) = \nabla p$ and $\Delta(p - p_*) = \Delta p$. Then we can assume $p := p - p_*$.

Consequently, we have the following classical Hele-Shaw problem:

$$\text{(CHS)} \qquad \begin{cases} \Delta p = 0, & (x,y) \in \Omega(t), \quad t > 0, \\ p = \tau k, & (x,y) \in \Gamma(t), \quad t > 0, \\ \beta = -\dfrac{b^2}{12\mu}\dfrac{\partial p}{\partial \boldsymbol{n}}, & (x,y) \in \Gamma(t), \quad t > 0. \end{cases}$$

In our case, the gap b is lifted uniformly at a specific rate. Then instead of (A2), we assume that

(A2)' w is a linear function with respect to z, i.e., $w = \eta(t)z + \xi(t)$.

At the bottom $z = 0$, the fluid does not move. Then $\xi(t) = 0$. At the top $z = b(t)$, the fluid moves with the plate. Then $\eta(t) = \dot{b}(t)/b(t)$. Hence we have

$$w = \frac{\dot{b}(t)}{b(t)} z.$$

In this case, the pressure p does not depend on z, since

$$p_z = \mu \Delta w = 0.$$

We can discuss the same argument of u and v as above, except the contribution from incompressibility:

$$\begin{aligned} 0 = \frac{1}{b(t)} \int_0^{b(t)} \operatorname{div} \boldsymbol{v}\, dz &= \frac{1}{b(t)} \int_0^{b(t)} (u_x + v_y + w_z)\, dz \\ &= \overline{u}_x + \overline{v}_y + \frac{\dot{b}(t)}{b(t)} = -\frac{b(t)^2}{12\mu}(p_{xx} + p_{yy}) + \frac{\dot{b}(t)}{b(t)}. \end{aligned}$$

Hence we have

$$\Delta p = 12\mu \frac{\dot{b}(t)}{b(t)^3},$$

and the following Hele-Shaw problem in a time-dependent gap $b(t)$:

$$\text{(TDHS)} \qquad \begin{cases} \Delta p = 12\mu \dfrac{\dot{b}(t)}{b(t)^3}, & (x,y) \in \Omega(t), \quad t > 0, \\ p = \tau k, & (x,y) \in \Gamma(t), \quad t > 0, \\ \beta = -\dfrac{b(t)^2}{12\mu}\dfrac{\partial p}{\partial \boldsymbol{n}}, & (x,y) \in \Gamma(t), \quad t > 0. \end{cases}$$

In the case the plates are fixed, then $\dot{b}(t) = 0$ and the above problem is nothing but the classical Hele-Shaw problem.

Now we will arrange the above problem by dimensionalization as follows. The variables x, y and t are scaled by a characteristic rate $l_0 > 0$ and $t_0 > 0$, respectively:

$$\tilde{x} = \frac{x}{l_0}, \quad \tilde{y} = \frac{y}{l_0}, \quad \tilde{t} = \frac{t}{t_0}.$$

Then the curvature k and the normal velocity β are scaled by $\tilde{k} = k/k_0$, $k_0 = l_0^{-1}$ and $\tilde{\beta} = \beta/\beta_0$, $\beta_0 = l_0/t_0$, respectively. The pressure p, the gap b and surface tension coefficient τ are scaled by a characteristic rate $p_0 > 0$, $b_0 > 0$ and $\tau_0 > 0$, respectively:

$$\tilde{p}(\tilde{x}, \tilde{y}, \tilde{t}) = \frac{p(x, y, t)}{p_0}, \quad \tilde{b}(\tilde{t}) = \frac{b(t)}{b_0}, \quad \tilde{\tau} = \frac{\tau}{\tau_0}.$$

If we take

$$p_0 = \frac{12\mu l_0^2}{b_0^2 t_0}, \quad \tau_0 = p_0 l_0,$$

then retaining the same variable names, the nondimensional (TDHS) becomes

$$\text{(NDHS)} \qquad \begin{cases} \Delta p = \dfrac{\dot{b}(t)}{b(t)^3}, & (x, y) \in \Omega(t), \quad t > 0, \\ p = \tau k, & (x, y) \in \Gamma(t), \quad t > 0, \\ \beta = -b(t)^2 \dfrac{\partial p}{\partial \boldsymbol{n}}, & (x, y) \in \Gamma(t), \quad t > 0. \end{cases}$$

Note that RHS of the Poisson equation depends only on time. Then RHS can be erased by means of a special solution $p_\star$ satisfying $\Delta p_\star = \dot{b}(t)/b(t)^3$. If we put $\hat{p} = p - p_\star$, then (NDHS) becomes

$$\begin{cases} \Delta \hat{p} = 0, & (x, y) \in \Omega(t), \quad t > 0, \\ \hat{p} = \tau k - p_\star, & (x, y) \in \Gamma(t), \quad t > 0, \\ \beta = -b(t)^2 \dfrac{\partial}{\partial \boldsymbol{n}}(\hat{p} + p_\star), & (x, y) \in \Gamma(t), \quad t > 0. \end{cases}$$

For instance, in the case

$$p_\star = \frac{\dot{b}(t)}{4b(t)^3} |\boldsymbol{x}|^2,$$

we have

$$\text{(HS)} \qquad \begin{cases} \Delta p = 0, & (x, y) \in \Omega(t), \quad t > 0, \\ p = \tau k - \dfrac{\dot{b}(t)}{4b(t)^3} |\boldsymbol{x}|^2, & (x, y) \in \Gamma(t), \quad t > 0, \\ \beta = -b(t)^2 \dfrac{\partial p}{\partial \boldsymbol{n}} - \dfrac{\dot{b}(t)}{2b(t)} \boldsymbol{x} \cdot \boldsymbol{n}, & (x, y) \in \Gamma(t), \quad t > 0. \end{cases}$$

Here we denoted $p = \hat{p}$.

Properties. It is easy to check that the time transition of enclosed area $|\Omega(t)|$ is

$$\partial_t |\Omega(t)| = \int_{\Gamma(t)} \beta\, ds = -\frac{\dot{b}(t)}{b(t)} |\Omega(t)|.$$

Hence the volume is preserved in the following sense:

$$b(t)|\Omega(t)| \equiv b(0)|\Omega(0)|.$$

One more important property is preserving the center of mass:

$$\boldsymbol{c} = \frac{1}{|\Omega|} \iint_\Omega \boldsymbol{x}\, d\Omega.$$

The time derivative of $\boldsymbol{c}$ is

$$\dot{\boldsymbol{c}} = \frac{\dot{b}}{b}\boldsymbol{c} - \frac{b^2}{|\Omega|} \int_\Gamma \boldsymbol{x} \frac{\partial p}{\partial \boldsymbol{n}}\, ds.$$

Here we have used the solution p of (NDHS) and

$$\partial_t \iint_\Omega \boldsymbol{x}\, d\Omega = \int_\Gamma \boldsymbol{x}\beta\, ds.$$

Therefore the following equations imply $\dot{\boldsymbol{c}} = \boldsymbol{0}$.

$$\begin{aligned}
\int_\Gamma \boldsymbol{x} \frac{\partial p}{\partial \boldsymbol{n}}\, ds &= \int_\Gamma p \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{n}}\, ds + \iint_\Omega (\boldsymbol{x} \triangle p - p \triangle \boldsymbol{x})\, d\Omega \\
&= \int_\Gamma p\boldsymbol{n}\, ds + \frac{\dot{b}}{b^3} \iint_\Omega \boldsymbol{x}\, d\Omega \\
&= \tau \int_\Gamma k\boldsymbol{n}\, ds + \frac{\dot{b}}{b^3} \iint_\Omega \boldsymbol{x}\, d\Omega \\
&= -\tau \int_\Gamma \partial_s \boldsymbol{t}\, ds + \frac{\dot{b}}{b^3} |\Omega| \boldsymbol{c} \\
&= \frac{\dot{b}}{b^3} |\Omega| \boldsymbol{c}.
\end{aligned}$$

Remark. In the presentation talk, we showed a numerical simulation of (HS) by means of boundary element method (BEM) and technique of curvature adjusted tangential velocity. It is to be desired that numerical scheme should satisfy the above two properties in some sense, e.g. in discrete sense. However, it is still open problem.

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