数理解析研究所講究録
第 1746 巻 2011 年 69-87

On Steady Solutions for a Continuum Model with Density Gradient-Dependent Stress

北海道大学大学院理学研究科自然史科学部門
中野直人 (Naoto Nakano) *
Department of Natural History Sciences,
Hokkaido University

Abstract

Steady simple shear models of a continuum whose Cauchy stress depends also on the gradient of the density of the body are studied. The density gradient-dependent stress model arises from a study of granular flows. The problems under consideration are boundary value problems of the second order ordinary differential equations, and their existence results are proved in this paper.

1 Introduction

In this study we are concerned with steady simple shear flows of a continuum whose Cauchy stress depends also on the gradient of the density of the body. Such a model arises from a study of a flow of granular materials. Granular materials are some sorts of materials which consist of grains. In certain situations granular matter behaves in fluid-like manner, for example, quicksand, avalanches, and so on. Even it flows, however, the profile of the flow is completely different from that of usual liquids. Granular bodies are naturally inhomogeneous. Since effect of interstices of the particles on motion may not negligible, a term corresponding to inhomogeneity of the body should appear in the constitutive equation for a flow of such matter (see, for example, [7]).

A flow of granular materials as complex continua has been studied in fluid mechanics [1, 8, 5] and in mechanical engineering [4, 18, 20]. Rajagopal and Massoudi [20] proposed the constitutive equations of granular materials, and since then Boyle and Massoudi [4], Rajagopal, Troy and Massoudi [19], Yalamanchili, Gudhe and Rajagopal [22] *etc.* have studied in the framework of [20]. Mathematical

*e-mail: nakano.naoto@gmail.com

works (including mathematical modeling) have also been studied in the last decade [19, 13, 15, 16]. We call the body under consideration is of Korteweg type, since such a material was firstly considered by Korteweg [12].

In this paper we follow Rajagopal and Massoudi's model [20]. Especially, we focus further on simple shear models to investigate minute flow profiles derived by the density gradient-dependence of the stress.

2 Steady simple shear flow down on an inclined plane

2.1 Governing equations

Governing equations of a continuum model with the density gradient- dependent stress tensor are given as follows:

$$\begin{cases} \varrho_t + \mathrm{div}(\varrho \mathbf{v}) = 0 \quad \text{in } \Omega, \\ \varrho \dfrac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathrm{div}\mathbb{T} + \varrho \mathbf{b} \quad \text{in } \Omega, \\ \mathbb{T} = (-p + a_1 \mathrm{div}\mathbf{v} + a_2 \mathrm{tr}\mathbb{M})\mathbb{I} + a_3\mathbb{D} + a_4\mathbb{M} \quad \text{in } \Omega. \end{cases} \tag{2.1}$$

Here, $\Omega\,(\subset \mathbb{R}^3)$ is a domain where a material occupies; $\boldsymbol{x} = (x, y, z)^{\mathrm{T}} \in \Omega$; $\mathbf{v} = (v_1, v_2, v_3)^{\mathrm{T}}(\boldsymbol{x}, t)$ is the velocity vector field; $\varrho = \varrho(\boldsymbol{x}, t)$ is the density; $p(\varrho) = p(\varrho(\boldsymbol{x}, t))$ is the pressure of barotropic type; $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)$; $\mathbb{T}$ is the Cauchy stress tensor; $\mathbf{b} = (b_1, b_2, b_3)^{\mathrm{T}}(\boldsymbol{x}, t)$ is the external body forces; $(\mathrm{div}\mathbb{T})_i = \frac{\partial T_{i1}}{\partial x} + \frac{\partial T_{i2}}{\partial y} + \frac{\partial T_{i3}}{\partial z}$; $\mathbb{D} = \mathbb{D}(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + [\nabla \mathbf{v}]^{\mathrm{T}})$ is the symmetric part of the velocity gradient; $\mathbb{M} = \nabla\varrho \otimes \nabla\varrho$ the symmetric tensor corresponding to the density gradient; $a_j(\varrho) = a_j(\varrho(\boldsymbol{x}, t))$ is material moduli.

The constitutive equation $(2.1)_1$ depends on the density gradient through the symmetric tensor $\mathbb{M}$. This form of relation is a subclass of [20]. Furthermore, we assume that the stress under consideration is isotropic, therefore the stress $(2.1)_3$ satisfies $\mathbb{T} = \mathbb{T}^{\mathrm{T}}$. When $a_2 = a_4 \equiv 0$, (2.1) becomes the system of compressible Navier–Stokes equations. The coefficients a_2 and a_4 indicate the magnitude of the effect of material inhomogeneity on the motion.

2.2 Steady simple shear flow down on an inclined plane

We consider the steady planar flow model as follows. In this case we assume that $\Omega = \{(x, y, z) | 0 < y < h\}$, $\mathbf{v} = (u(y), 0, 0)^{\mathrm{T}}$, $\varrho = \varrho(y)$ and $\mathbf{b} = (g \sin\theta, -g\cos\theta, 0)^{\mathrm{T}}$ with the depth of the layer flow h, the acceleration gravity g and the angle of inclination θ. Then, the stress takes the form of

$$\mathbb{T} = \begin{pmatrix} -p + a_2(\varrho')^2 & a_3u'/2 & 0 \\ a_3u'/2 & -p + a(\varrho')^2 & 0 \\ 0 & 0 & -p + a_2(\varrho')^2 \end{pmatrix}, \tag{2.2}$$

where $a(\varrho) = a_2(\varrho) + a_4(\varrho)$. Thus, when a_4 does not vanish, this model can exhibit the normal-stress differences even in a simple shear flow.

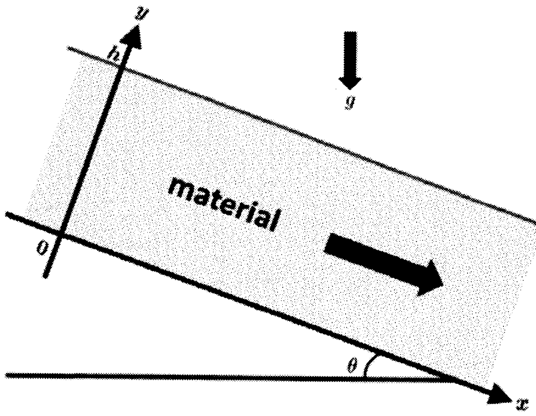

Figure 1: steady simple shear flow down on an inclined plane

For boundary conditions we assign the balance between the external pressure and the stress vector at the surface, and also assign so-calledNavier's slip on the bed, namely

$$\text{surface } (y = h) \quad \mathbb{T}\mathbf{n} = -p_e\mathbf{n}, \qquad \text{bed } (y = 0) \quad \mathbf{v} + k\Pi\mathbb{T}\mathbf{n} = \mathbf{0},$$

where p_e is the external pressure, k the slip rate, $\Pi\mathbf{f} = \mathbf{f} - (\mathbf{f}\cdot\mathbf{n})\mathbf{n}$ the tangential part, $\mathbf{n}$ the unit outward normal to the boundary.

Consequently, we derive the boundary value problem of the second order ordinary differential equations for steady simple shear flows:

$$\begin{cases} (a_3(\varrho(y))u'(y))'/2 + \varrho(y)g\sin\theta = 0 & \text{for } 0 < y < h, \\ \{-p(\varrho(y)) + a(\varrho(y))(\varrho'(y))^2\}' - \varrho(y)g\cos\theta = 0 & \text{for } 0 < y < h, \\ a_3(\varrho(h))u'(h) = 0, \qquad -p(\varrho(h)) + a(\varrho(h))(\varrho'(h))^2 = -p_e, & \\ u(0) - ka_3(\varrho(0))u'(0) = 0. & \end{cases} \tag{2.3}$$

(2.3) is the coupled problem of u and ϱ, nevertheless we can decouple the problem by integrating equation $(2.3)_1$. Taking into account the boundary conditions $(2.3)_3$ and $(2.3)_5$, the velocity $u(y)$ can be uniquely determined by ϱ in the following formula:

$$u(y) = k\int_0^h 2\varrho(s)g\sin\theta \mathrm{d}s + \int_0^y \frac{\mathrm{d}\eta}{a_3(\varrho(\eta))}\int_\eta^h 2\varrho(s)g\sin\theta \mathrm{d}s. \tag{2.4}$$

Thus we only need to consider the problem for ϱ, *i.e.*,

$$\begin{cases} \{-p(\varrho(y)) + (a(\varrho(y)))(\varrho'(y))^2\}' - \varrho(y)g\cos\theta = 0 & \text{for } 0 < y < h, \\ -p(\varrho(h)) + a(\varrho(h))(\varrho'(h))^2 = -p_e. \end{cases} \tag{2.5}$$

Integrating $(2.5)_1$ and using the boundary condition $(2.5)_2$, we derive

$$-a(\varrho(y))(\varrho'(y))^2 = p_e - p(\varrho(y)) + g\cos\theta \int_y^h \varrho(s)\mathrm{d}s \quad \text{for } 0 < y < h.$$

Moreover, we assume the forms of the coefficient a and the pressure p as follows:

$$a(\varrho) = a < 0 : \text{constant}, \quad p(\varrho) = p_0\varrho, \quad p_0 > 0. \tag{2.6}$$

According to Boyle and Massoudi [4], the negative sign of coefficient a is adopted.

Taking into account (2.6), the problem becomes the following integro-differential equation.

$$-a(\varrho'(y))^2 = p_e - p_0\varrho(y) + g\cos\theta \int_y^h \varrho(s)\mathrm{d}s \quad \text{for } 0 < y < h. \tag{2.7}$$

Since the quadric dependence of the density gradient, (2.7) is a degenerate non-linear equation. In order to remove the difficulties caused by the degeneracy and non-linearity, we assume that the density function is monotonically decreasing. Of course, the density of a layer flow does not always behave monotonically. However, so-called stratified flow is often assumed to hold a monotonically decreasing density, we therefore seek a monotonically decreasing solution of (2.7) as the first step to examine this model.

Under the monotonic condition $\varrho'(y) \leq 0$ for $0 < y < h$, problem (2.7) is equivalent to

$$\varrho'(y) = -\sqrt{\frac{\mathrm{T}\varrho(y)}{-a}}, \tag{2.8}$$

where $\mathrm{T}\varrho$ denotes the right-hand side of (2.7), namely, we define the operator T as

$$\mathrm{T}f(y) = p_e - p_0 f(y) + g\cos\theta \int_y^h f(s)\mathrm{d}s \tag{2.9}$$

for $0 < y < h$.

When we know the boundary data $\varrho(h) = b$, then (2.8) is equivalent to the following integral equation.

$$\varrho(y) = b + \int_y^h \sqrt{\frac{\mathrm{T}\varrho(s)}{-a}}\mathrm{d}s.$$

We should remark that the boundary condition stated in $(2.5)_2$ cannot determine the Dirichlet deta $\varrho(h)$ uniquely by itself. Therefore we assign the Dirichlet condition for ϱ besides $(2.5)_2$, and in this case the problem becomes

$$\begin{cases} \varrho(y) = b + \displaystyle\int_y^h \sqrt{\frac{\mathrm{T}\varrho(s)}{-a}}\mathrm{d}s \quad \text{for } 0 < y < h, \\ \varrho(h) = b \geq 0, \quad \varrho'(h) = -d \leq 0, \end{cases} \tag{2.10}$$

under the compatibility condition

$$-ad^2 = p_{\mathrm{e}} - p_0 b. \tag{2.11}$$

Here, we note that the boundary condition $(2.10)_2 - (2.11)$ is equivalent to $(2.5)_2$. Holding the relation (2.11), the Dirichlet data and the Neumann data of ϱ can vary.

We proved the existence theorem for problem (2.10), thus we can ultimately obtain a solution for (2.3). The following theorem is proved in § 3.

Theorem 2.1 *Let $a < 0$, $h > 0$, $p_{\mathrm{e}} > 0$, $p_0 > 0$, $g > 0$, $0 \leq \theta < \pi/2$, $b \geq 0$, $d \geq 0$, and $-ad^2 = p_{\mathrm{e}} - p_0 b$. Problem* (2.10) *has a non-negative solution ϱ satisfying*

$$\varrho'(y) \leq 0, \quad \varrho(h) = b, \quad \varrho'(h) = -d, \quad \varrho \in C^1[0, h], \tag{2.12}$$

and a solution holding (2.12) *is unique.*

This existence theorem needs no additional conditions besides the compatibility condition, hence the monotonically decreasing density solution to the steady simple shear flow (2.3) always exists.

3 Proof of Theorem 2.1

3.1 Auxiliary estimates

First, we consider the problem (2.10) on some subinterval $[h_1, h]$ $(0 \leq h_1 < h)$, instead of $[0, h]$.

Let ϱ be a monotonically decreasing solution satisfying (2.7). Then $\mathrm{T}f(y) \geq 0$. It deduces that

$$\varrho(y) \leq \frac{p_{\mathrm{e}}}{p_0} + \frac{g\cos\theta}{p_0}\int_y^h \varrho(s)\mathrm{d}s.$$

Gronwall's inequality implies

$$\varrho(y) \leq \frac{p_{\mathrm{e}}}{p_0}\mathrm{e}^{\frac{g\cos\theta(h-y)}{p_0}}. \tag{3.1}$$

(3.1) therefore gives

$$
\begin{aligned}
\mathrm{T}\varrho(y) &\le p_{\mathrm{e}} - p_0\varrho(y) + g\cos\theta \int_y^h \frac{p_{\mathrm{e}}}{p_0} \mathrm{e}^{\frac{g\cos\theta(h-s)}{p_0}} \mathrm{d}s \\
&= p_{\mathrm{e}} \mathrm{e}^{\frac{g\cos\theta(h-y)}{p_0}} - p_0\varrho(y) \\
&\le p_{\mathrm{e}} \mathrm{e}^{\frac{g\cos\theta(h-y)}{p_0}} - p_0 b =: M(y),
\end{aligned} \tag{3.2}
$$

where $b = \varrho(h) > 0$.

Consequently,

$$
0 \le -\varrho'(y) = \sqrt{\frac{\mathrm{T}\varrho(y)}{-a}} \le \sqrt{\frac{M(y)}{-a}}. \tag{3.3}
$$

Since (3.3), we have

$$
\begin{aligned}
\mathrm{T}\varrho(y) &= p_{\mathrm{e}} - p_0\varrho(h) + p_0(\varrho(h) - \varrho(y)) + g\cos\theta \int_y^h \varrho(s)\mathrm{d}s \\
&= -ad^2 + p_0 \int_y^h \varrho'(s)\mathrm{d}s + g\cos\theta \int_y^h \varrho(s)\mathrm{d}s \\
&\ge -ad^2 - p_0 \int_y^h \sqrt{\frac{M(s)}{-a}}\mathrm{d}s + g\cos\theta \int_y^h \varrho(s)\mathrm{d}s \\
&\ge -ad^2 - p_0\sqrt{\frac{M(y)}{-a}}(h-y) + g\cos\theta b(h-y),
\end{aligned} \tag{3.4}
$$

where $d = -\varrho'(h) \ge 0$. Moreover,

$$
M(y) = p_{\mathrm{e}}\left(\mathrm{e}^{\frac{g\cos\theta(h-y)}{p_0}} - 1\right) + p_{\mathrm{e}} - p_0 b \le p_{\mathrm{e}} \cdot \frac{g\cos\theta(h-y)}{p_0} \cdot \mathrm{e}^{\frac{g\cos\theta(h-y)}{p_0}} - ad^2. \tag{3.5}
$$

Thus,

$$
\begin{aligned}
\mathrm{T}\varrho(y) &\ge -ad^2 - p_0\sqrt{\frac{p_{\mathrm{e}} \cdot \frac{g\cos\theta(h-y)}{p_0} \cdot \mathrm{e}^{\frac{g\cos\theta(h-y)}{p_0}} - ad^2}{-a}}(h-y) + g\cos\theta b(h-y) \\
&\ge -ad^2 - p_0\left\{\sqrt{\frac{p_{\mathrm{e}}}{-a} \cdot \frac{g\cos\theta(h-y)}{p_0} \cdot \mathrm{e}^{\frac{g\cos\theta(h-y)}{p_0}}} + d\right\}(h-y) \\
&\quad + g\cos\theta b(h-y) \\
&\ge -ad^2 - p_0 d(h-y) \\
&\quad + \left\{g\cos\theta b - p_0\sqrt{\frac{p_{\mathrm{e}}}{-a} \cdot \frac{g\cos\theta(h-y)}{p_0} \cdot \mathrm{e}^{\frac{g\cos\theta(h-y)}{p_0}}}\right\}(h-y).
\end{aligned} \tag{3.6}
$$

Here,

$$
\begin{aligned}
-ad^2 - p_0 d(h-y) &= -a\left(d^2 - \frac{p_0 d(h-y)}{-a} + \frac{p_0^2(h-y)^2}{4(-a)^2} - \frac{p_0^2(h-y)^2}{4(-a)^2}\right) \\
&= -a\left(d - \frac{p_0(h-y)}{-2a}\right)^2 - \frac{p_0^2(h-y)^2}{-4a} \\
&\geq -\frac{p_0^2(h-y)^2}{-4a},
\end{aligned}
$$

therefore (3.6) leads

$$
\begin{aligned}
\mathrm{T}\varrho(y) &\geq p_0\left\{\frac{g\cos\theta b}{p_0} - \sqrt{\frac{p_e}{-a}\cdot\frac{g\cos\theta(h-y)}{p_0}\cdot \mathrm{e}^{\frac{g\cos\theta(h-y)}{p_0}}} - \frac{p_0(h-y)}{-4a}\right\}(h-y) \\
&=: m(y)(h-y).
\end{aligned}
\tag{3.7}
$$

Here,

$$
\begin{aligned}
m(y) &= p_0\left\{\frac{g\cos\theta b}{p_0} - \sqrt{\frac{p_e}{-a}\cdot\frac{g\cos\theta(h-y)}{p_0}\cdot \mathrm{e}^{\frac{g\cos\theta(h-y)}{p_0}}} - \frac{p_0(h-y)}{-4a}\right\} \\
&= \frac{g\cos\theta b}{2} \\
&\quad + p_0\left\{\frac{g\cos\theta b}{2p_0} - \sqrt{\frac{p_e}{-a}\cdot\frac{g\cos\theta(h-y)}{p_0}\cdot \mathrm{e}^{\frac{g\cos\theta(h-y)}{p_0}}} - \frac{p_0(h-y)}{-4a}\right\} \\
&=: \frac{g\cos\theta b}{2} + m_1(y).
\end{aligned}
$$

It is easy to see that $m(y) \geq m(h_1)$ and $m_1(y) \geq m_1(h_1)$.

Let $\alpha = \frac{g\cos\theta}{2p_0}$, $A = p_0/(-4a)$, $B = \sqrt{p_e\alpha e^{\alpha}/(-a)}$, $C = \alpha b$ and $X = \sqrt{h-h_1}$. If $h - h_1 \leq 1$, then $m_1(h_1) \geq 0$ is equivalent to

$$AX^2 + BX - C \leq 0.$$

Let

$$X_{\pm} = \frac{-B \pm \sqrt{B^2 + 4AC}}{2A}.$$

Hence, if X satisfies $X < X_+$, then we have $AX^2 + BX - C \leq 0$.

Consequently, we obtain the following lemma.

Lemma 3.1 Let $h > 0$, $0 \leq h_1 < h$, $p_e > 0$, $p_0 > 0$, $-a > 0$, $g > 0$, $0 \leq \theta < \pi/2$, $b > 0$, $d \geq 0$, $-ad^2 = p_e - p_0 b$ and $\alpha = g\cos\theta/p_0$. If $h - h_1$ satisfies

$$
h - h_1 \leq \delta_1 := \min\left\{\frac{4(-a)^2}{p_0^2}\left(-\sqrt{\frac{p_e\alpha e^{\alpha}}{-a}} + \sqrt{\frac{p_e\alpha e^{\alpha}}{-a} + \frac{g\cos\theta b}{-a}}\right)^2, 1\right\},
\tag{3.8}
$$

then $m_1(h_1) > 0$ and $m(h_1) \geq g\cos\theta b/2$.

3.2 Proof of local existence theorem

Let h, h_1, p_e, p_0, $-a$, g, θ, b, d and α be the same as in Lemma 3.1. Moreover, taking (3.3) into account, we define the function space as follows.

$$X = \left\{ f(y) \geq 0,\ 0 \leq -f'(y) \leq \sqrt{\frac{M(y)}{-a}} \text{ for } h_1 \leq y \leq h,\ f(h) = b \right\}. \tag{3.9}$$

Applying the same way as that used in (3.4), (3.6) and (3.7), for any $f \in X$ it holds that

$$\mathrm{T}f(y) \geq m(y)(h-y) \quad \text{for } h_1 \leq y \leq h, \tag{3.10}$$

and $m(y) \geq m(h_1) > 0$. Thus the operator J can be defined over X. Let $F = \mathrm{J}f$, namely,

$$F(y) \equiv \mathrm{J}f(y) = b + \int_y^h \sqrt{\frac{\mathrm{T}f(s)}{-a}} \mathrm{d}s \quad \text{for } h_1 \leq y \leq h.$$

Here,

$$0 \leq f(y) \leq \frac{p_e}{p_0} \mathrm{e}^{\frac{g\cos\theta(h-y)}{p_0}} \quad \text{for } h_1 \leq y \leq h \tag{3.11}$$

follows from (3.10). Repeating the same calculation carried out in (3.2) and (3.3), we obtain

$$0 \leq -F'(y) = \sqrt{\frac{\mathrm{T}f(y)}{-a}} \leq \sqrt{\frac{M(y)}{-a}} \quad \text{for } h_1 \leq y \leq h.$$

Besides, F clearly satisfies $F \geq 0$ and $F(h) = b$. After all, F also belongs to X. The operator J is well-defined over X.

Moreover, we obtain the following estimates concerning J.

Lemma 3.2 Let h, h_1, p_e, p_0, $-a$, g, θ, b, d and α be the same as in Lemma 3.1. For any f, $g \in X$ it holds that

$$\|\mathrm{J}f - \mathrm{J}g\| \leq \frac{\{p_0 + g\cos\theta(h-h_1)\}\sqrt{h-h_1}}{\sqrt{-am(h_1)}} \|f-g\|, \tag{3.12}$$

$$\|(\mathrm{J}f)' - (\mathrm{J}g)'\| \leq \frac{\{p_0 + g\cos\theta(h-h_1)\}}{\sqrt{-am(h_1)}} \|f-g\|, \tag{3.13}$$

where $\|f\| = \sup_{h_1 \leq y \leq h} |f(y)|$.

Proof.
From Lemma 3.1 and (3.10) we have $\mathrm{T}f(y) \geq m(h_1)(h-y)$ and $\mathrm{T}g(y) \geq m(h_1)(h-y)$ for f, $g \in X$. These estimates lead

$$|\mathrm{J}f(y) - \mathrm{J}g(y)| \leq \int_y^h \left| \sqrt{\frac{\mathrm{T}f(s)}{-a}} - \sqrt{\frac{\mathrm{T}g(s)}{-a}} \right| \mathrm{d}s = \frac{1}{\sqrt{-a}} \int_y^h \frac{|\mathrm{T}f(s) - \mathrm{T}g(s)|}{\sqrt{\mathrm{T}f(s)} + \sqrt{\mathrm{T}g(s)}} \mathrm{d}s$$

$$\leq \frac{\|\mathrm{T}f - \mathrm{T}g\|}{\sqrt{-a}} \int_y^h \frac{\mathrm{d}s}{2\sqrt{m(h_1)(h-s)}} = \frac{\|\mathrm{T}f - \mathrm{T}g\|}{\sqrt{-am(h_1)}}\sqrt{h-y}, \quad (3.14)$$

$$|\mathrm{T}f(y) - \mathrm{T}g(y)| \leq \left| -p_0(f(y) - g(y)) + g\cos\theta \int_y^h (f(s) - g(s))\mathrm{d}s \right|$$
$$\leq \{p_0 + g\cos\theta(h-y)\} \|f - g\|. \quad (3.15)$$

(3.14) and (3.15) therefore conclude (3.12).

Since

$$|(\mathrm{J}f)'(y) - (\mathrm{J}g)'(y)| = \frac{1}{\sqrt{-a}} \cdot \frac{|\mathrm{T}f(y) - \mathrm{T}g(y)|}{\sqrt{\mathrm{T}f(y)} + \sqrt{\mathrm{T}g(y)}} \leq \frac{\|\mathrm{T}f - \mathrm{T}g\|}{\sqrt{-am(h_1)}},$$

(3.13) also follows from (3.15). ∎

Furthermore, Lemma 3.1 leads the following lemma.

Lemma 3.3 Let h, h_1, p_e, p_0, $-a$, g, θ, b, d and α be the same as in Lemma 3.1, and let $0 < C < 1$. If $h - h_1$ satisfies

$$h - h_1 \leq \delta_2 := \frac{-ag\cos\theta bC}{2(p_0 + g\cos\theta)^2} \quad (3.16)$$

besides (3.8), then the operator J is contraction, *i.e.*,

$$\|\mathrm{J}f - \mathrm{J}g\| \leq C\|f - g\|.$$

Lemmata 3.2 and 3.3 derive the following local existence theorem.

Theorem 3.1 *Let h, h_1, p_e, p_0, $-a$, g, θ, b, d, α and C be the same as in Lemma 3.3. The integral equation* (2.10) *has a unique solution on $[h_1, h]$, which belongs to X.*

Proof.

Let f_0 be an arbitrary function in X, and we define $f_{j+1} = \mathrm{J}f_j$ ($j = 0, 1, 2, \ldots$) successively. Lemma 3.1 deduces $\{f_j\}_{j=0}^{\infty} \subset X$.

From Lemma 3.3 and (3.11) it holds that for any $j = 0, 1, 2, \ldots$

$$\|f_{j+1} - f_j\| \leq C\|f_j - f_{j-1}\| \leq \cdots \leq C^j\|f_1 - f_0\| \leq C^j \cdot \frac{p_e}{p_0} \mathrm{e}^{\frac{g\cos\theta(h-h_1)}{p_0}}.$$

Consequently, we obtain the convergence result, namely, for m, $n \in \mathbb{N}$ ($m > n$)

$$\|f_m - f_n\| \leq \sum_{j=n}^{m-1} \|f_{j+1} - f_j\| \leq \sum_{j=n}^{m-1} C^j \cdot \frac{p_e}{p_0} \mathrm{e}^{\frac{g\cos\theta(h-h_1)}{p_0}} \longrightarrow 0 \quad (m, n \to 0). \quad (3.17)$$

Moreover, from (3.12) and (3.13) we have

$$\|f'_{j+1} - f'_j\| \leq \frac{C\|f_j - f_{j-1}\|}{\sqrt{h-h_1}} \leq \cdots \leq \frac{C^j\|f_1 - f_0\|}{\sqrt{h-h_1}} \leq C^j \cdot \frac{p_e \mathrm{e}^{\frac{g\cos\theta(h-h_1)}{p_0}}}{p_0\sqrt{h-h_1}}.$$

Similarly, it holds that for $m,\, n \in \mathbb{N}$ $(m > n)$

$$\|f'_m - f'_n\| \le \sum_{j=n}^{m-1} \|f'_{j+1} - f'_j\| \le \sum_{j=n}^{m-1} C^j \cdot \frac{p_e e^{\frac{g\cos\theta(h-h_1)}{p_0}}}{p_0\sqrt{h-h_1}} \longrightarrow 0 \quad (m,n \to 0). \quad (3.18)$$

(3.17) and (3.18) indicate that $\{f_j\}$ converges to a function in $C^1[h_1, h]$, and the limit solves the integral equation (2.10) on $[h_1, h]$. The uniqueness of the solution is derived by the contraction property of J. The necessary conditions obtained in § 3.1 imply the solution belongs to X. ∎

Remark 3.1 Theorem 3.1 implies that the integro-differential equation (2.7) has a unique monotonically decreasing solution on $[h_1, h]$.

3.3 Proof of Global existence therorem

In this section we shall extend the interval the solution exsists. From the result obtained in § 3.2 the integro-differential equation

$$\begin{cases} -a(\varrho'(y))^2 = p_e - p_0\varrho(y) + g\cos\theta \displaystyle\int_y^h \varrho(s)\mathrm{d}s \quad \text{for } h_1 \le y \le h, \\ -a(\varrho'(h))^2 = p_e - p_0\varrho(h), \\ \varrho(h) = b \end{cases} \quad (3.19)$$

has a unique monotonically decreasing solution on $[h_1, h]$, where $h - h_1$ satisfies (3.8) and (3.16), *i.e.*, $h - h_1 \le \min\{\delta_1, \delta_2\}$.

In order to obtain the uniform estimate used later, we take smaller $h - h_1$ for problem (3.19) so that $h - h_1 \le \min\{\delta'_1, \delta_2\}$, where

$$\delta'_1 = \min\left\{\frac{4(-a)^2}{p_0^2}\left(-\sqrt{\frac{p_e\alpha e^{\alpha(1+h)}}{-a}} + \sqrt{\frac{p_e\alpha e^{\alpha(1+h)}}{-a} + \frac{g\cos\theta b}{-a}}\right)^2,\ 1\right\}. \quad (3.20)$$

Obviously, $\delta'_1 < \delta_1$, therefore problem (3.19) has a unique monotonically decreasing solution on $[h_1, h]$.

In the case $h_1 > 0$, we next consider the following problem

$$\begin{cases} -a(\eta'(y))^2 = \mathrm{T}^*\eta(y) \equiv p_e^* - p_0\eta(y) + g\cos\theta \displaystyle\int_y^{h^*} \eta(s)\mathrm{d}s \quad \text{for } h_1^* \le y \le h^*, \\ \eta'(y) \le 0 \quad \text{for } h_1^* \le y \le h^*, \\ \eta'(h^*) = \varrho'(h^*), \quad \eta(h^*) = \varrho(h^*), \end{cases} \quad (3.21)$$

where $h^* = (h + h_1)/2$, $h_1^* = \max\{(3h_1 - h)/2,\ 0\}$ and

$$p_e^* = p_e + g\cos\theta \int_{h^*}^h \varrho(s)\mathrm{d}s \quad (3.22)$$

with ϱ being a solution of (3.19). Obviously, h^* is the middle point of h and h_1, and $h^* - h_1^* \le h - h_1$.

We should remark that the boundary condition $(3.21)_3$-$(3.21)_4$ is equivalent to

$$\begin{cases} -a(\eta'(h^*))^2 = p_e^* - p_0\eta(h^*), \\ \eta(h^*) = b^* := \varrho(h^*), \end{cases}$$

since (3.19). Hence, problem (3.21) takes the same form as problem (3.19). According to Theorem 3.1, the integral equation corresponding to problem (3.21)

$$\mathrm{J}^* f(y) \equiv f(h^*) + \int_y^{h^*} \sqrt{\frac{\mathrm{T}^* f(s)}{-a}} \mathrm{d}s \tag{3.23}$$

is uniquely solvable on $[h_2, h^*]$ $(h_1^* \le h_2)$, if $h^* - h_2 \le \min\{\delta_1^*, \delta_2^*\}$ holds, where δ_j^* $(j = 1, 2)$ are δ_j in (3.8) and (3.16) for parameters appeared in problem (3.21), respectively.

Here,

$$\begin{aligned} p_e^* &= p_e + g\cos\theta \int_{h^*}^{h} \varrho(s)\mathrm{d}s \le p_e + g\cos\theta \int_{h^*}^{h} \frac{p_e}{p_0} \mathrm{e}^{\frac{g\cos\theta(h-s)}{p_0}} \mathrm{d}s \\ &\le p_e + g\cos\theta \int_0^h \frac{p_e}{p_0} \mathrm{e}^{\frac{g\cos\theta(h-s)}{p_0}} \mathrm{d}s = p_e \mathrm{e}^{\frac{g\cos\theta h}{p_0}} = p_e \mathrm{e}^{\alpha h}. \end{aligned} \tag{3.24}$$

Consequently, from (3.24) and $b^* \ge b$ we have

$$\begin{aligned} h^* - h_1^* \le h - h_1 \le \delta_1' &\le \frac{4(-a)^2}{p_0^2} \left(-\sqrt{\frac{p_e\alpha \mathrm{e}^{\alpha(1+h)}}{-a}} + \sqrt{\frac{p_e\alpha \mathrm{e}^{\alpha(1+h)}}{-a} + \frac{g\cos\theta b}{-a}} \right)^2 \\ &\le \frac{4(-a)^2}{p_0^2} \left(-\sqrt{\frac{p_e^*\alpha \mathrm{e}^{\alpha}}{-a}} + \sqrt{\frac{p_e^*\alpha \mathrm{e}^{\alpha}}{-a} + \frac{g\cos\theta b}{-a}} \right)^2 \\ &\le \frac{4(-a)^2}{p_0^2} \left(-\sqrt{\frac{p_e^*\alpha \mathrm{e}^{\alpha}}{-a}} + \sqrt{\frac{p_e^*\alpha \mathrm{e}^{\alpha}}{-a} + \frac{g\cos\theta b^*}{-a}} \right)^2. \end{aligned} \tag{3.25}$$

This implies that

$$h^* - h_1^* \le \min\{\delta_1^*, \delta_2^*\}. \tag{3.26}$$

Due to (3.26), repeaing the same argument carried out in § 3.2, we can obtain a unique solution η for problem (3.21) on $[h_1^*, h^*]$.

Since ϱ also solves problem (3.21) on $[h_1, h^*]$, namely,

$$-a(\varrho'(y))^2 = p_e - p_0\varrho(y) + g\cos\theta \int_y^h \varrho(s)\mathrm{d}s = p_e^* - p_0\varrho(y) + g\cos\theta \int_y^{h^*} \varrho(s)\mathrm{d}s$$

for $h_1 \le y \le h^*$, the uniqueness property derives $\varrho \equiv \eta$ on $[h_1, h^*]$.

Accordingly, let

$$\varrho_1(y) = \begin{cases} \varrho(y) & (y \in [h^*, h]), \\ \eta(y) & (y \in [h_1^*, h^*]), \end{cases} \tag{3.27}$$

then ϱ_1 satisfies

$$\begin{aligned}
&-a(\varrho_1(y))^2 \\
&= \begin{cases} -a(\varrho'(y))^2 & (y \in [h^*, h]) \\ -a(\eta'(y))^2 & (y \in [h_1^*, h^*]) \end{cases} \\
&= \begin{cases} p_e - p_0\varrho(y) + g\cos\theta \displaystyle\int_y^h \varrho(s)\mathrm{d}s & (y \in [h^*, h]) \\ p_e^* - p_0\eta(y) + g\cos\theta \displaystyle\int_y^{h^*} \eta(s)\mathrm{d}s & (y \in [h_1^*, h^*]) \end{cases} \\
&= \begin{cases} p_e - p_0\varrho(y) + g\cos\theta \displaystyle\int_y^h \varrho(s)\mathrm{d}s & (y \in [h^*, h]) \\ p_e - p_0\eta(y) + g\cos\theta \left(\displaystyle\int_y^{h^*} \eta(s)\mathrm{d}s + \int_{h^*}^h \varrho(s)\mathrm{d}s\right) & (y \in [h_1^*, h^*]) \end{cases} \\
&= \begin{cases} p_e - p_0\varrho_1(y) + g\cos\theta \displaystyle\int_y^h \varrho_1(s)\mathrm{d}s & (y \in [h^*, h]) \\ p_e - p_0\varrho_1(y) + g\cos\theta \left(\displaystyle\int_y^{h^*} \varrho_1(s)\mathrm{d}s + \int_{h^*}^h \varrho_1(s)\mathrm{d}s\right) & (y \in [h_1^*, h^*]) \end{cases} \\
&= p_e - p_0\varrho_1(y) + g\cos\theta \int_y^h \varrho_1(s)\mathrm{d}s.
\end{aligned} \tag{3.28}$$

When $h_1^* = 0$, (3.28) gives the conclusion. If not, (3.28) means the interval, which the solution exsits, is extended by $(h - h_1)/2$. By iterating the same argument carried out above, we ultimately obtain the desired unique monotonically decreasing solution to problem (2.7) on $[0, h]$. This completes the proof of Thorem 2.1.

4 Steady simple shear flow between vertical planes

4.1 Set up and mathematical issues

We next consider the steady simple shear flow between vertical planes as follows. In this case, $\Omega = \{(x, y, z) | -h < y < h\}$, $\mathbf{v} = (u(y), 0, 0)^{\mathrm{T}}$, $\varrho = \varrho(y)$ and $\mathbf{b} = (g, 0, 0)^{\mathrm{T}}$. Here, the vertical planes are set at intervals of $2h$, g denotes the acceleration gravity. The stress takes the same form as (2.2).

Hence, the governing equations (2.1) are reduced to

$$\begin{cases} (a_3(\varrho(y))u'(y))'/2 + \varrho(y)g = 0 & \text{for } -h < y < h, \\ \{-p(\varrho(y)) + a(\varrho(y))(\varrho'(y))^2\}' = 0 & \text{for } -h < y < h. \end{cases} \tag{4.1}$$

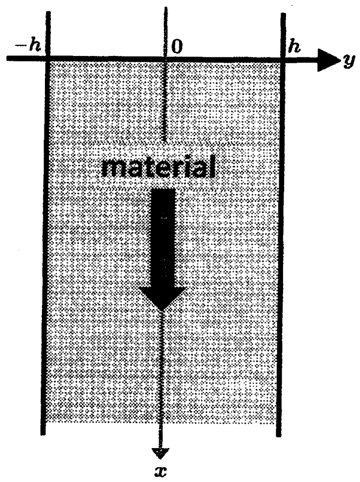

Figure 2: steady simple shear flow between vertical planes

In this study we seek a symmetric solution, namely,

$$\varrho(y) = \varrho(-y), \quad u(y) = u(-y) \quad \text{for } -h \le y \le h. \tag{4.2}$$

Then the coupled problem (4.1) can be decoupled, by repeating the similar argument carried for (2.3). We obtain

$$u(y) = u(-h) + \int_{-h}^{y} \frac{r(h) - 2r(s)}{a_3(\varrho(s))} ds, \tag{4.3}$$

where

$$r(y) = \int_{-h}^{y} \varrho(s) g ds.$$

Hence, what we consider is the ordinary differential equation $(4.1)_2$. We also assume that the pressure and the coefficient a take the form of

$$a(\varrho) = a_0 \varrho, \quad a_0 < 0, \quad p(\varrho) = p_0 \varrho, \quad p_0 > 0. \tag{4.4}$$

The reason for the sign of a_0 is the same as in (2.6). Consequently, the equation is reduced to

$$\{-p_0 \varrho(y) + a_0 \varrho(y)(\varrho'(y))^2\}' = 0 \quad \text{for } -h < y < h.$$

Integrating the equation above from $-h$ to y, we derive

$$-p_0 \varrho(y) + a_0 \varrho(y)(\varrho'(y))^2 = -p_0 \varrho(-h) + a_0 \varrho(-h)(\varrho'(-h))^2 =: C \quad \text{for } -h < y < h. \tag{4.5}$$

Here, C is a non-positive constant by virtue of the signs of p_0, ϱ and a_0. This means $(2,2)$-component of the stress $\mathbb{T}$ does not vary in y direction, and the pressure holds $p_0 \varrho(y) = -C + a_0 \varrho(y)(\varrho'(y))^2$. In this model the pressure is not constant even in a simple shear flow, and this feature is a typical property of the density-gradient dependent stress model.

In fact, (4.5) is related to the brachistochrone curve in the following manner. Deforming (4.5), we have

$$\frac{-a_0}{p_0}(\varrho')^2 = \frac{-C}{p_0\varrho} - 1. \tag{4.6}$$

Let $y = \sqrt{\frac{-a_0}{p_0}}\,Y$, $\varrho(y) = \varrho(\sqrt{\frac{-a_0}{p_0}}\,Y) = \eta(Y)$ and $R = \frac{-C}{2p_0}$, then η satisfies the following ordinary differential equaiton.

$$\left(\frac{\mathrm{d}\eta(Y)}{\mathrm{d}Y}\right)^2 = \frac{2R}{\eta(Y)} - 1. \tag{4.7}$$

It is well-known that (4.7) is integrable and its solution describes a cycloid, therefore η can be written by parametric representation

$$\begin{cases} \eta = R(1-\cos\theta), \\ Y = R(\theta - \sin\theta). \end{cases} \tag{4.8}$$

Hence the solution of (4.6) can be represented by the following formula

$$\begin{cases} \varrho = R(1-\cos\theta), \\ y = \kappa R(\theta - \sin\theta), \end{cases} \tag{4.9}$$

where $\kappa = \sqrt{\frac{-a_0}{p_0}}$. The domain of θ can be determined by the boundary conditions for ϱ, and more importantly, it is not necessary that the domain of θ is connected.

Moreover, due to the quadric dependence of the density gradient, the equation admits a sectionally C^1 solution if it holds

$$|\varrho'(y^* - 0)| = |\varrho'(y^* + 0)|$$

at the point of discontinuity of ϱ'. If ϱ' satisfies the above condition, $(\varrho')^2$ is differentiable almost everywhere on $[-h, h]$. Namely, even though ϱ is not C^1 and satisfies the equation on $(-h, h)$ only sectionally, ϱ can solve the problem if $|\varrho'|$ is absolutely continuous. This implies that a sectionally C^1 solution spontaneously holds no uniqueness (examples are shown in figures 3 and 4). We need more physically meaningful conditions to determine the one sectionally C^1 solution, it is however still open.

If we assume the regularity of the solution, then we obtain the uniqueness result.

Theorem 4.1 *If a solution of problem* (4.1) *is* C^2 *under the symmetric condition* (4.2) *and the boundary condition*

$$u(h) = u_0, \quad \varrho(h) = \varrho_0, \quad \varrho'(h) = 0,$$

then the solution is unique.

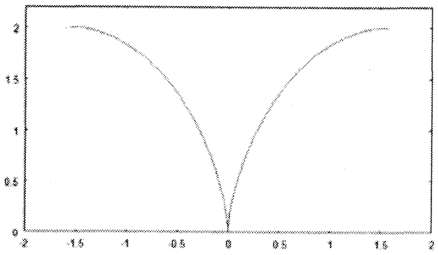

Figure 3: $h = 0.5\pi$, $\kappa = 0.5$, $R = 1$, $\varrho'(h) = 0$

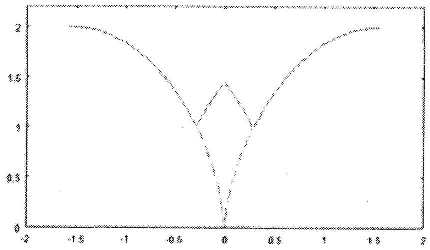

Figure 4: $h = 0.5\pi$, $\kappa = 0.5$, $R = 1$, $\varrho'(h) = 0$ with three folding points

Remark 4.1 The symmetric condition is not necessary to obtain the uniqueness. When we obtain the C^2 solution under the appropriate boundary conditions, it would be unique.

4.2 Numerics

When the domain of θ is connected, then we can obtain the exact solution formula for velocity u in some cases.

In order to calcurate $r(y)$, first we consider an indefinite integral of ϱ. Taking into account the parametric representation of ϱ (4.9), we have

$$\frac{\mathrm{d}\varrho}{\mathrm{d}y} = \frac{\frac{\mathrm{d}\varrho}{\mathrm{d}\theta}}{\frac{\mathrm{d}y}{\mathrm{d}\theta}} = \frac{R\sin\theta}{\kappa R(1-\cos\theta)} = \frac{R\sin\theta}{\kappa\varrho}, \tag{4.10}$$

and then

$$\begin{aligned}\int \varrho g \mathrm{d}y &= \int R(1-\cos\theta)g\cdot\kappa R(1-\cos\theta)\mathrm{d}\theta = \kappa R^2 g\int(1-\cos\theta)^2\mathrm{d}\theta \\ &= \kappa R^2 g\left(\frac{3\theta}{2} - 2\sin\theta + \frac{\sin\theta\cos\theta}{2}\right) \\ &= \kappa R^2 g\left\{\frac{3}{2}(\theta-\sin\theta) - \frac{\sin\theta}{2}(1-\cos\theta)\right\} = \frac{Rg}{2}(3y - \kappa\varrho\sin\theta) \\ &= \frac{Rg}{2}\left(3y - \frac{\kappa^2\varrho^2\varrho'}{R}\right). \end{aligned} \tag{4.11}$$

Here, we omit the integral constant.

Consequently, the symmetric conditions

$$\varrho(-y) = \varrho(y), \quad \varrho'(-y) = -\varrho'(y)$$

lead the following formula.

$$r^*(y) := r(h) - 2r(y) = -Rg\left(3y - \frac{\kappa^2\varrho(y)^2\varrho'(y)}{R}\right).$$

Here, we assume the form of a_3 as

$$a_3(\varrho) = a_{30} + a_{31}\varrho$$

with $a_{3j} \geq 0$ $(j = 0, 1)$.

Especially, when $a_{30} = 0$ we have

$$\begin{aligned}
\int \frac{r^*(y)}{a_3(\varrho(y))}dy &= -\frac{Rg}{a_{31}}\int\left(\frac{3y}{\varrho} - \frac{\kappa^2\varrho\varrho'}{R}\right)dy \\
&= -\frac{Rg}{a_{31}}\left\{3\int\frac{\kappa R(\theta - \sin\theta)}{R(1-\cos\theta)}\cdot\kappa R(1-\cos\theta)d\theta - \frac{\kappa^2\varrho^2}{2R}\right\} \\
&= -\frac{Rg}{a_{31}}\left\{3\kappa^2 R\left(\frac{\theta^2}{2} + \cos\theta\right) - \frac{\kappa^2\varrho^2}{2R}\right\} \\
&= -\frac{Rg}{a_{31}}\left[3\kappa^2 R\left\{\frac{(\theta - \sin\theta)^2}{2} + \sin\theta\,(\theta - \sin\theta) - \frac{(1-\cos\theta)^2}{2}\right\}\right. \\
&\qquad \left. - \frac{\kappa^2\varrho^2}{2R}\right] \\
&= -\frac{g}{a_{31}}\left\{\frac{3y^2}{2} + 3\kappa^2\varrho\varrho' y - 2\kappa^2\varrho^2\right\}.
\end{aligned}$$

Thus, in this case u can be represented by

$$\begin{aligned}
u(y) = u(-h) &- \frac{g}{a_{31}}\left(\frac{3y^2}{2} + 3\kappa^2\varrho(y)\varrho'(y)y - 2\kappa^2\varrho(y)^2\right) \\
&+ \frac{g}{a_{31}}\left(\frac{3h^2}{2} - 3\kappa^2\varrho(-h)\varrho'(-h)h - 2\kappa^2\varrho(-h)^2\right). \quad (4.12)
\end{aligned}$$

Interestingly, the flow profiles are obtained by the use of numerical calculation of the formula (4.12). In the case $g = 1$, $R = 1$, $h = \pi$, $a_{30} = 0$, $a_{31} = 1$, $\theta \in [-\pi/\kappa, \pi/\kappa]$, we obtain the density and the velocity profiles as follows. The figures show the comparison of cases $a_0 = 0$ and $a_0 \neq 0$, and consider the limit $a_0 \to 0$. We remark that if $a_0 = 0$ then $\kappa = 0$, and when $a_0 = 0$ the solution must be constant in this problem.

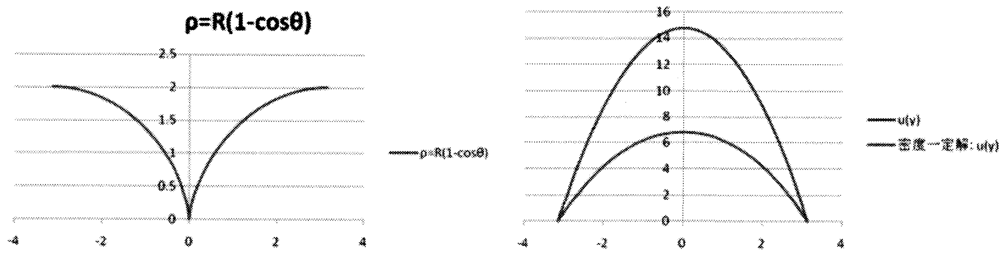

Figure 5: $\kappa = 1$

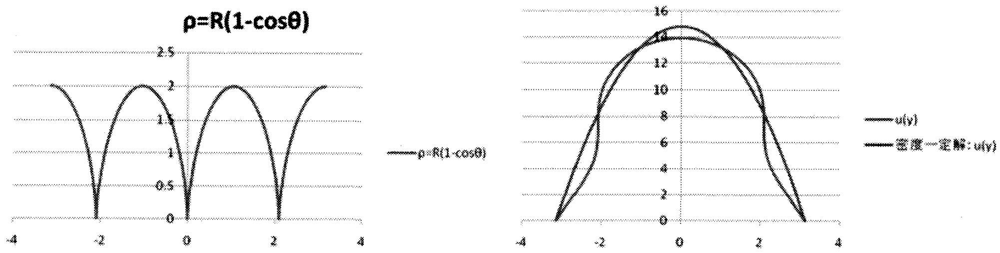

Figure 6: $\kappa = 1/3$

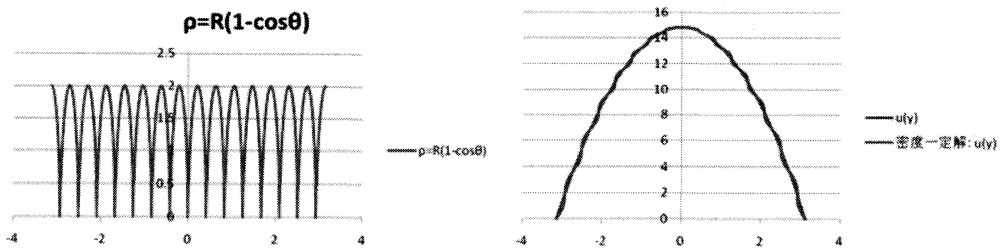

Figure 7: $\kappa = 1/15$

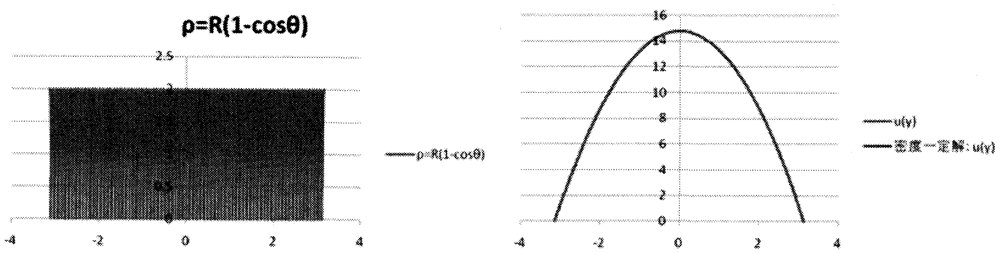

Figure 8: $\kappa = 1/77$

The figures imply that the velocity of the problem with $a_0 < 0$ converges to that of case $a_0 = 0$, even though the density function does not show the pointwise convergence to the constant density case. Hence, the limit for the density of this problem is likely the singular limit. We need more investigation into the density-gradient dependent stress model.

References

[1] G. AHMADI: *A generalized continuum theory for granular materials.* Int. J. Non-Linear Mech. **17**, 21-33 (1982).

[2] R. A. BAGNOLD: The Physics of Blown Sand and Desert Dunes, *London, Methuen*, 1941.

[3] O. V. BESOV, V. P. IL'IN AND S. M. NIKOL'SKIĬ: Integral representations of functions and imbedding theorems. *"Nauka", Moscow*, 1975; English transl., Vols. 1, 2, Wiley, 1979.

[4] E. J. BOYLE AND M. MASSOUDI: *A theory for granular materials exhibiting normal stress effects based on Enskog's dense gas theory.* Int. J. Eng. Sci. **28**, 1261-1275 (1990).

[5] J. E. DUNN AND J. SERRIN: *On the thermomechanics of interstitial working.* Arch. Rational Mech. Anal. **88**, 95-133 (1985).

[6] J. DURAN: Sands, Powders, and Grains: An Introduction to the Physics of Granular Materials. *Springer*, 1991.

[7] M. A. GOODMAN AND S. C. COWIN: *Two problems in the gravity flow of granular materials.* J. Fluid. Mech. **45**, 321-339 (1971).

[8] M. A. GOODMAN AND S. C. COWIN: *A continuum theory for granular materials.* Arch. Rational Mech. Anal. **44**, 249-266 (1972).

[9] H. HAYAKAWA, S. NASUNO: Funtaino Butsuri, *Kyōritsu Shuppan, Gendaibutsuri Saizensen* **1** (2000), 49-115.

[10] H. HAYAKAWA: *Sanitsu ryushikeino rikigaku. Iwanamishoten*, 2003.

[11] K. HUTTER AND K. R. RAJAGOPAL: *On flows of granular materials.* Continuum. Mech. Thermodyn. **6**, 81-139 (1994).

[12] D. J. KORTWEG: *Sur la forme que prenent les équations du mouvements des fluides si l'on tient compte des forces capilaires causées par des variations de densité considérables mains continues et sur la théorie de la capollarité dans*

l'hypothése d'une varation continue de la densité. Arch. Néerl. Sci. Exactes Nat. Ser. II, **6**, 1-24 (1901).

[13] J. MÁLEK AND K. R. RAJAGOPAL: *On the modeling of inhomogeneous incompressible fluid-like bodies.* Mech. Mat. **38**, 233-242 (2006).

[14] J. MÁLEK AND K. R. RAJAGOPAL: *Incompressible rate type fluids with pressure and shear-rate dependent material moduli.* Nonlinear Analysis: Real World Applications **8**, 156-164 (2007).

[15] N. NAKANO AND A. TANI: *An initial-boundary value problem for motion of incompressible inhomogeneous fluid-like bodies.* Adv. Math. Sci. Appl. **20**, (2010), 11-35.

[16] N. NAKANO AND A. TANI: *Navier's slip problem for motion of incompressible inhomogeneous fluid-like bodies.* J. Math. Fluid Mech., accepted (Published online: 19th Sep. 2009).

[17] C. L. M. H. NAVIER, *Mémoir sur les lois du mouvement des fluides*, Mem. Acad. R. Sci. Paris **6**, (1823), 389-416.

[18] K. R. RAJAGOPAL: *Multiple natural configurations in continuum mechanics.* Technical Report 6, Institute for Computational and Applied Mechanics, University of Pittsburgh, 1995.

[19] K. R. RAJAGOPAL, W. C. TROY AND M. MASSOUDI: *Existence of solutions to the equations governing the flow of granular materials.* Eur. J. Mech. B/Fluids **11**, 265-276 (1992).

[20] K. R. RAJAGOPAL AND M. MASSOUDI: *A method for measuring material moduli of granular materials: flow in an orthogonal rheomer.* Topical Report U.S. Department of Energy DOE/PETC/TR-90/3 (1990).

[21] C. TRUESDELL AND W. NOLL: The non-linear field theories of mechanics. *Handbuch der Physik III/3*, Springer, Berlin 1965.

[22] R. C. YALAMANCHILI, R. GUDHE AND K. R. RAJAGOPAL: *Flow of granular materials in a vertical channel under the action of gravity.* Powder Tech. 81, 65-73 (1994).