数理解析研究所講究録
第 1747 巻 2011 年 126-134

THE MAGNUS REPRESENTATION AND ABELIAN QUOTIENTS OF GROUPS OF HOMOLOGY CYLINDERS

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1. INTRODUCTION

The study of groups of homology cylinders over a surface was initiated by Goussarov [4] and Habiro [5] in their surgery theory and then deep relationships to mapping class groups and Johnson homomorphisms were given in Garoufalidis-Levine [2] and Levine [9]. Recently their structures are intensively studied by many people from various contexts.

Here we focus on abelian quotients of the homology cobordism group of homology cylinders. This group was shown to be infinitely generated by Cha-Friedl-Kim [1] by using the invariant which we call the *H-torsion* here. The purpose of this paper is to use another invariant called the *Magnus representation* to obtain abelian quotients of the same group and generalize it to higher dimensional cases.

All manifolds are assumed to be smooth throughout this paper, while similar statements hold for other categories.

This research was partially supported by KAKENHI (No. 21740044), Ministry of Education, Science, Sports and Technology, Japan.

2. HOMOLOGY CYLINDERS

Let $\Sigma_{g,1}$ be a compact oriented surface of genus $g \geq 1$ with one boundary component. We take a base point p of $\Sigma_{g,1}$ on the boundary $\partial\Sigma_{g,1}$ and $2g$ oriented loops $\gamma_1, \gamma_2, \cdots, \gamma_{2g}$ as in Figure 1. These loops form a spine R_{2g} of $\Sigma_{g,1}$ and therefore give a basis of $\pi_1\Sigma_{g,1}$ as a free group of rank $2g$. The boundary loop ζ is given by $\zeta = [\gamma_1, \gamma_{g+1}][\gamma_2, \gamma_{g+2}] \cdots [\gamma_g, \gamma_{2g}]$.

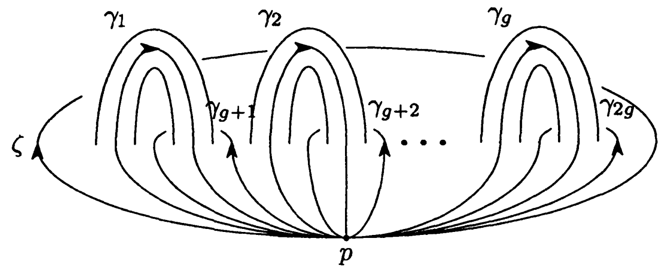

FIGURE 1. Our basis of $\pi_1\Sigma_{g,1}$

Put $H := H_1(\Sigma_{g,1})$. The group H can be identified with $\mathbb{Z}^{2g}$ by choosing $\{\gamma_1, \gamma_2, \cdots, \gamma_{2g}\}$ as a basis of H, where we write γ_j again for γ_j as an element of H. This basis is a symplectic basis with respect to the intersection pairing on H.

Received February 28, 2011.

Definition 2.1. A *homology cylinder over* $\Sigma_{g,1}$ consists of a compact oriented 3-manifold M with two embeddings $i_+, i_- : \Sigma_{g,1} \hookrightarrow \partial M$, called the *markings*, such that:

(i) i_+ is orientation-preserving and i_- is orientation-reversing;
(ii) $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})$ and $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$;
(iii) $i_+|_{\partial\Sigma_{g,1}} = i_-|_{\partial\Sigma_{g,1}}$;
(iv) $i_+, i_- : H_*(\Sigma_{g,1}) \to H_*(M)$ are isomorphisms, namely M is a *homology product* over $\Sigma_{g,1}$.

We denote a homology cylinder by (M, i_+, i_-) or simply M.

Two homology cylinders (M, i_+, i_-) and (N, j_+, j_-) over $\Sigma_{g,1}$ are said to be *isomorphic* if there exists an orientation-preserving diffeomorphism $f : M \xrightarrow{\cong} N$ satisfying $j_+ = f \circ i_+$ and $j_- = f \circ i_-$. We denote by $\mathcal{C}_{g,1}$ the set of all isomorphism classes of homology cylinders over $\Sigma_{g,1}$. We define a product operation on $\mathcal{C}_{g,1}$ by

$$(M, i_+, i_-) \cdot (N, j_+, j_-) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-)$$

for (M, i_+, i_-), $(N, j_+, j_-) \in \mathcal{C}_{g,1}$, which endows $\mathcal{C}_{g,1}$ with a monoid structure. The unit is $(\Sigma_{g,1} \times [0,1], \mathrm{id} \times 1, \mathrm{id} \times 0)$, where collars of $i_+(\Sigma_{g,1}) = (\mathrm{id} \times 1)(\Sigma_{g,1})$ and $i_-(\Sigma_{g,1}) = (\mathrm{id} \times 0)(\Sigma_{g,1})$ are stretched half-way along $(\partial\Sigma_{g,1}) \times [0,1]$ so that $i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$.

Example 2.2. For each diffeomorphism φ of $\Sigma_{g,1}$ which fixes $\partial\Sigma_{g,1}$ pointwise, we can construct a homology cylinder by setting

$$(\Sigma_{g,1} \times [0,1], \mathrm{id} \times 1, \varphi \times 0)$$

with the same treatment of the boundary as above. It is easily checked that the isomorphism class of $(\Sigma_{g,1} \times [0,1], \mathrm{id} \times 1, \varphi \times 0)$ depends only on the (boundary fixing) isotopy class of φ and that this construction gives a monoid homomorphism from the mapping class group $\mathcal{M}_{g,1}$ to $\mathcal{C}_{g,1}$. In fact, it is an injective homomorphism (see Garoufalidis-Levine [2, Section 2.4] and Levine [9, Section 2.1]). We may regard $\mathcal{C}_{g,1}$ as an *enlargement* of $\mathcal{M}_{g,1}$.

Example 2.3 (Levine [9]). Let L be a pure string link of g strings. We can embed a g-holed disk D^2_g into $\Sigma_{g,1}$ as a closed regular neighborhood of the union of the loops $\gamma_{g+1}, \gamma_{g+2}, \ldots, \gamma_{2g}$ in Figure 1. Let C be the complement of an open tubular neighborhood of L in $D^2 \times [0,1]$. By choosing a framing of L, we can fix a diffeomorphism $h : \partial C \xrightarrow{\cong} \partial(D^2_g \times [0,1])$. Then the manifold M_L obtained from $\Sigma_{g,1} \times [0,1]$ by removing $D^2_g \times [0,1]$ and regluing C by h becomes a homology cylinder with the same marking as the trivial homology cylinder.

In [2], Garoufalidis-Levine further introduced the following equivalence relation among homology cylinders.

Definition 2.4. Two homology cylinders (M, i_+, i_-) and (N, i_+, i_-) over $\Sigma_{g,1}$ are said to be *homology cobordant* if there exists a compact oriented smooth 4-manifold W such that:

(1) $\partial W = M \cup (-N)/(i_+(x) = j_+(x),\ i_-(x) = j_-(x)) \quad x \in \Sigma_{g,1}$;
(2) The inclusions $M \hookrightarrow W$, $N \hookrightarrow W$ induce isomorphisms on the integral homology.

We denote by $\mathcal{H}_{g,1}$ the quotient set of $\mathcal{C}_{g,1}$ with respect to the equivalence relation of homology cobordism. The monoid structure of $\mathcal{C}_{g,1}$ induces a group structure of $\mathcal{H}_{g,1}$.

We call $\mathcal{H}_{g,1}$ the *homology cobordism group* of homology cylinders. It is known that the composition $\mathcal{M}_{g,1} \hookrightarrow \mathcal{C}_{g,1} \twoheadrightarrow \mathcal{H}_{g,1}$ is an injective group homomorphism.

The group $\mathcal{M}_{g,1}$ and the monoid $\mathcal{C}_{g,1}$ share many properties. The most fundamental one is given by the action on H. We can define a map

$$\sigma : \mathcal{C}_{g,1} \longrightarrow \operatorname{Aut} H$$

by assigning to $(M, i_+, i_-) \in \mathcal{C}_{g,1}$ an automorphism $i_+^{-1} \circ i_-$ of H. This map extends the natural action of $\mathcal{M}_{g,1}$ on H and it is a monoid homomorphism. We can check that the image consists of the automorphisms of H preserving the intersection pairing. Therefore, under the identification $H \cong \mathbb{Z}^{2g}$ mentioned above, we have an epimorphism

$$\sigma : \mathcal{C}_{g,1} \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}).$$

We put $\mathcal{IC}_{g,1} := \operatorname{Ker} \sigma$, which is an analogue of the Torelli group $\mathcal{I}_{g,1} = \operatorname{Ker}(\sigma : \mathcal{M}_{g,1} \to \mathrm{Sp}(2g, \mathbb{Z}))$. We can see that σ induces a group homomorphism $\sigma : \mathcal{H}_{g,1} \to \mathrm{Sp}(2g, \mathbb{Z})$ and we denote its kernel by $\mathcal{IH}_{g,1}$.

3. Magnus representation and H-torsion for homology cylinders

Here, we recall two invariants for homology cylinders from [15]. For our purpose, it suffices to consider a simplified version corresponding to commutative rings.

Since $H = H_1(\Sigma_{g,1})$ is a free abelian group, its group ring $\mathbb{Z}[H]$ can be embedded in the fractional field $\mathcal{K}_H := \mathbb{Z}[H](\mathbb{Z}[H] - \{0\})^{-1}$. Let $(M, i_+, i_-) \in \mathcal{C}_{g,1}$ be a homology cylinder. Since $H_1(M) \cong H_1(\Sigma_{g,1})$, the field $\mathcal{K}_{H_1(M)} := \mathbb{Z}[H_1(M)](\mathbb{Z}[H_1(M)] - \{0\})^{-1}$ is defined. We regard $\mathcal{K}_H$ and $\mathcal{K}_{H_1(M)}$ as local coefficient systems on $\Sigma_{g,1}$ and M respectively. By an argument using covering spaces, we have the following.

Lemma 3.1. *$i_\pm : H_*(\Sigma_{g,1}, p; i_\pm^* \mathcal{K}_{H_1(M)}) \to H_*(M, p; \mathcal{K}_{H_1(M)})$ are isomorphisms as right $\mathcal{K}_{H_1(M)}$-vector spaces.*

This lemma plays an important role in defining our invariants below.

(I) Magnus representation

By using the spine R_{2g} taken in the previous section, we identify $\pi_1 \Sigma_{g,1} = \langle \gamma_1, \ldots, \gamma_{2g} \rangle$ with a free group F_{2g} of rank $2g$. Since $R_{2g} \subset \Sigma_{g,1}$ is a deformation retract, we have

$$\begin{aligned} H_1(\Sigma_{g,1}, p; i_\pm^* \mathcal{K}_{H_1(M)}) &\cong H_1(R_{2g}, p; i_\pm^* \mathcal{K}_{H_1(M)}) \\ &= C_1(\widetilde{R_{2g}}) \otimes_{F_{2g}} i_\pm^* \mathcal{K}_{H_1(M)} \cong \mathcal{K}_{H_1(M)}^{2g} \end{aligned}$$

with a basis $\{\widetilde{\gamma_1} \otimes 1, \ldots, \widetilde{\gamma_{2g}} \otimes 1\} \subset C_1(\widetilde{R_{2g}}) \otimes_{F_{2g}} i_\pm^* \mathcal{K}_{H_1(M)}$ as a right free $\mathcal{K}_{H_1(M)}$-module, where $\widetilde{\gamma_i}$ is a lift of γ_i on the universal covering $\widetilde{R_{2g}}$ of R_{2g}. We denote by $\mathcal{K}_{H_1(M)}^{2g}$ the space of column vectors with n entries in $\mathcal{K}_{H_1(M)}$.

Definition 3.2. (1) For $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$, we denote by $r'(M) \in \mathrm{GL}(2g, \mathcal{K}_{H_1(M)})$ the representation matrix of the right $\mathcal{K}_{H_1(M)}$-isomorphism

$$\mathcal{K}_{H_1(M)}^{2g} \cong H_1(\Sigma_{g,1}, p; i_-^* \mathcal{K}_{H_1(M)}) \xrightarrow[i_+^{-1} \circ i_-]{\cong} H_1(\Sigma_{g,1}, p; i_+^* \mathcal{K}_{H_1(M)}) \cong \mathcal{K}_{H_1(M)}^{2g}$$

(2) The *Magnus representation* for $\mathcal{C}_{g,1}$ is the map $r : \mathcal{C}_{g,1} \to \mathrm{GL}(2g, \mathcal{K}_H)$ which assigns to $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$ the matrix $r(M) := {}^{i_+^{-1}} r'(M)$ obtained from $r'(M)$ by applying i_+^{-1} to each entry.

We call $r(M)$ the *Magnus matrix* for M. The map r has the following properties:

Theorem 3.3 ([15, 14]). (1) (*Crossed homomorphism*) *For $M_1, M_2 \in \mathcal{C}_{g,1}$, we have*

$$r(M_1 \cdot M_2) = r(M_1) \cdot {}^{\sigma(M_1)}r(M_2).$$

In particular, the restriction of r to $\mathcal{IC}_{g,1}$ is a homomorphism.
(2) (*Symplecticity*) *For any $M \in \mathcal{C}_{g,1}$, we have the equality*

$$\overline{r(M)^T}\ \widetilde{J}\ r(M) = {}^{\sigma(M)}\widetilde{J},$$

where $\overline{r(M)^T}$ is obtained from $r(M)$ by taking the transpose and applying the involution induced from the map $(H \ni x \mapsto x^{-1} \in H)$ to each entry, and $\widetilde{J} \in \mathrm{GL}(2g, \mathbb{Z}[H])$ is the matrix which appeared in Papakyriakopoulos' paper [12] and it is mapped to the usual symplectic matrix by applying the trivializer $\mathbb{Z}[H] \to \mathbb{Z}$ to each entry.
(3) (*Homology cobordism invariance*) *The map $r : \mathcal{C}_{g,1} \to \mathrm{GL}(2g, \mathcal{K}_H)$ induces a crossed homomorphism $r : \mathcal{H}_{g,1} \to \mathrm{GL}(2g, \mathcal{K}_H)$ and its restriction to $\mathcal{IH}_{g,1}$ is a homomorphism.*

Remark 3.4. Definition 3.2 and Theorem 3.3 (1), (2) are extensions of those for the mapping class group $\mathcal{M}_{g,1}$ (see Morita [10] and Suzuki [16]). By a theorem of Dehn-Nielsen, the group $\mathcal{M}_{g,1}$ is naturally embedded in the automorphism group $\mathrm{Aut}\, F_{2g}$ of $F_{2g} \cong \pi_1\Sigma_{g,1}$ as the subgroup consisting of automorphisms which fix ζ. The Magnus representation for $\mathcal{M}_{g,1}$ is a restriction of that for $\mathrm{Aut}\, F_{2g}$ using Fox derivations. In particular, we see that $r(\mathcal{M}_{g,1}) \subset \mathrm{GL}(2g, \mathbb{Z}[H])$. Note that there exists a homology cylinder $M \in \mathcal{C}_{g,1}$ satisfying $r(M) \notin \mathrm{GL}(2g, \mathbb{Z}[H])$ (see Example 3.7).

At present, no embedding of $\mathcal{C}_{g,1}$ or $\mathcal{H}_{g,1}$ to the automorphism group of some group is known. However, by using a completion F_{2g}^{acy} of F_{2g} called the *acyclic closure* (or HE-closure), which was defined by Levine [7, 8], we can define a homomorphism

$$\mathrm{Acy} : \mathcal{C}_{g,1} \longrightarrow \mathrm{Aut}\, F_{2g}^{\mathrm{acy}}$$

and it factors through $\mathcal{H}_{g,1}$. This extends the embedding $\mathcal{M}_{g,1} \hookrightarrow \mathrm{Aut}\, F_{2g}$ and the Magnus representation for homology cylinders is derived from that for $\mathrm{Aut}\, F_{2g}^{\mathrm{acy}}$ (see [13] for details). In this paper, we only use the following properties of F_n^{acy} $(n \geq 2)$:

(i) F_n^{acy} includes F_n and it is strictly bigger than F_n.
(ii) The embedding $F_n \hookrightarrow F_n^{\mathrm{acy}}$ induces an isomorphism $H_1(F_n) \xrightarrow{\cong} H_1(F_n^{\mathrm{acy}})$.
(iii) Any endomorphism of F_n which induces an isomorphism on $H_1(F_n)$ is extended to an isomorphism of F_n^{acy}.
(iv) $\mathrm{Aut}\, F_n^{\mathrm{acy}}$ includes $\mathrm{Aut}\, F_n$ and the action of $\mathrm{Aut}\, F_n$ on $H_1(F_n) \cong \mathbb{Z}^n$ is extended to that of $\mathrm{Aut}\, F_n^{\mathrm{acy}}$ on $H_1(F_n^{\mathrm{acy}}) \cong H_1(F_n)$, which is expressed by an epimorphism $\sigma : \mathrm{Aut}\, F_n^{\mathrm{acy}} \twoheadrightarrow \mathrm{GL}(n, \mathbb{Z})$.

(II) H-torsion

Since the relative complex $C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{H_1(M)})$ obtained from any smooth triangulation of $(M, i_+(\Sigma_{g,1}))$ is acyclic by Lemma 3.1, we can consider its Reidemeister torsion $\tau(C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{H_1(M)})) \in \mathcal{K}_{H_1(M)}^{\times}/(\pm H_1(M))$, where $\mathcal{K}_{H_1(M)}^{\times} := \mathcal{K}_{H_1(M)} - \{0\}$ is the unit group of $\mathcal{K}_{H_1(M)}$. We refer to Turaev [17] for generalities of Reidemeister torsions.

Definition 3.5. The *H-torsion* $\tau(H)$ of a homology cylinder $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$ is defined by

$$\tau(M) := {}^{i_+^{-1}}\tau(C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{H_1(M)})) \in \mathcal{K}_H^\times/(\pm H),$$

where $\mathcal{K}_H^\times = \mathcal{K}_H - \{0\}$ is the unit group of $\mathcal{K}_H$.

The map $\tau : \mathcal{C}_{g,1} \to \mathcal{K}_H^\times/(\pm H)$ has the following properties:

Theorem 3.6. (1) (*Crossed homomorphism* [15]) *For $M_1, M_2 \in \mathcal{C}_{g,1}$, we have*

$$\tau(M_1 \cdot M_2) = \tau(M_1) \cdot {}^{\sigma(M_1)}\tau(M_2).$$

In particular, the restriction of τ to $\mathcal{IC}_{g,1}$ is a homomorphism.

(2) (Cha-Friedl-Kim [1, Theorem 3.10]) *If $(M, i_+, i_-), (N, j_+, j_-) \in \mathcal{C}_{g,1}$ are homology cobordant, then there exists $q \in \mathcal{K}_H^\times$ such that*

$$\tau(M) = \tau(N) \cdot q \cdot \overline{q} \in \mathcal{K}_H^\times/(\pm H).$$

Note that the restriction of τ to $\mathcal{M}_{g,1}$ is trivial since $\Sigma_{g,1} \times [0,1]$ is simple homotopy equivalent to $\Sigma_{g,1} \times \{1\}$.

Example 3.7. Let L be the string link of 2 strings depicted in Figure 2. We can construct a homology cylinder $(M_L, i_+, i_-) \in \mathcal{C}_{2,1}$ as mentioned in Example 2.3.

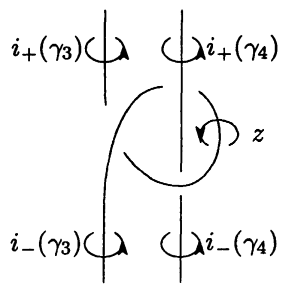

FIGURE 2. String link L

A presentation of $\pi_1 M_L$ is given by

$$\left\langle \begin{array}{c} i_-(\gamma_1), \ldots, i_-(\gamma_4) \\ z \\ i_+(\gamma_1), \ldots, i_+(\gamma_4) \end{array} \middle| \begin{array}{l} i_+(\gamma_1) i_-(\gamma_3)^{-1} i_+(\gamma_4) i_-(\gamma_1)^{-1}, \\ [i_+(\gamma_1), i_+(\gamma_3)] i_+(\gamma_2) z i_-(\gamma_2)^{-1} [i_-(\gamma_3), i_-(\gamma_1)], \\ i_+(\gamma_4) i_-(\gamma_3) i_+(\gamma_4)^{-1} z^{-1}, \\ i_-(\gamma_3) i_+(\gamma_3)^{-1} i_-(\gamma_3)^{-1} z, \ i_-(\gamma_4) z^{-1} i_+(\gamma_4)^{-1} z \end{array} \right\rangle.$$

The Magnus matrix and H-torsion are computed from this presentation by using Fox derivations and they are given by

$$r(M_L) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-\gamma_1^{-1}}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1}-\gamma_4^{-1}+1}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_3^{-1}}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_4^{-1}(\gamma_4^{-1}-1)}{\gamma_3^{-1}+\gamma_4^{-1}-1} \\ \frac{\gamma_1^{-1}\gamma_3\gamma_4^{-1}}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{(1-\gamma_3^{-1})(\gamma_2^{-1}\gamma_3^{-1}-\gamma_2^{-1}-1)}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_3^{-1}-1}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{-\gamma_3^{-1}\gamma_4^{-1}+\gamma_3^{-1}+2\gamma_4^{-1}-1}{\gamma_3^{-1}+\gamma_4^{-1}-1} \end{pmatrix},$$

$$\tau(M_L) = -1 + \gamma_3 - \gamma_3\gamma_4^{-1}.$$

Note that

$$\det(r(M_L)) = \gamma_3^{-1}\gamma_4^{-1}\frac{\gamma_3+\gamma_4-1}{\gamma_3^{-1}+\gamma_4^{-1}-1}.$$

4. Abelian quotients

Abelian quotients of a monoid or group are helpful in extracting information on the structure of the monoid or group. Here, we focus on abelian quotients of $\mathcal{C}_{g,1}$ and $\mathcal{H}_{g,1}$ and compare them to the corresponding result for $\mathcal{M}_{g,1}$.

Before discussing, as commented in [3], we point out that $\mathcal{C}_{g,1}$ has the monoid $\theta^3_{\mathbb{Z}}$ of homology 3-spheres as a big abelian quotient. In fact, we have a *forgetful* homomorphism $F: \mathcal{C}_{g,1} \to \theta^3_{\mathbb{Z}}$ defined by $F(M, i_+, i_-) = S^3 \sharp X_1 \sharp X_2 \sharp \cdots \sharp X_n$ for the prime decomposition $M = M_0 \sharp X_1 \sharp X_2 \sharp \cdots \sharp X_n$ of M where M_0 is the unique factor having non-empty boundary and $X_i \in \theta^3_{\mathbb{Z}}$ $(1 \leq i \leq n)$. The map F owes its well-definedness to the uniqueness of the prime decomposition of 3-manifolds and it is a monoid epimorphism.

The underlying 3-manifolds of homology cylinders obtained from $\mathcal{M}_{g,1}$ are all $\Sigma_{g,1} \times [0,1]$ and, in particular, irreducible. Therefore it seems more reasonable to compare $\mathcal{M}_{g,1}$ with the submodule $\mathcal{C}^{\mathrm{irr}}_{g,1}$ of $\mathcal{C}_{g,1}$ consisting of all (M, i_+, i_-) with M irreducible.

In contrast with the fact that $\mathcal{M}_{g,1}$ is a perfect group for $g \geq 3$ (see Harer [6]), many infinitely generated abelian quotients for monoids and homology cobordism groups of irreducible homology cylinders have been found until now. For example, we have the following results:

- In [15, Corollary 6.16], we showed the submonoids $\mathcal{C}^{\mathrm{irr}}_{g,1} \cap \mathcal{IC}_{g,1}$ and $\mathrm{Ker}\,(\mathcal{C}^{\mathrm{irr}}_{g,1} \to \mathcal{H}_{g,1})$ have abelian quotients isomorphic to $(\mathbb{Z}_{\geq 0})^\infty$. The proof uses the H-torsion τ and its non-commutative generalization.
- Morita [11, Corollary 5.2] used what is called the trace maps to show that $\mathcal{IH}_{g,1}$ has an abelian quotient isomorphic to $\mathbb{Z}^\infty$.
- In [3, Theorem 2.6], we showed that $\mathcal{C}^{\mathrm{irr}}_{g,1}$ has an abelian quotient isomorphic to $(\mathbb{Z}_{\geq 0})^\infty$ by using sutured Floer homology (a variant of Heegaard Floer homology). However, this abelian quotient does not induces that of $\mathcal{H}_{g,1}$.

Let us focus on abelian quotients of $\mathcal{H}_{g,1}$. By taking into account the similarity between the two groups $\mathcal{M}_{g,1}$ and $\mathcal{H}_{g,1}$, it had been conjectured that $\mathcal{H}_{g,1}$ was perfect. However, Cha-Friedl-Kim [1] found a method for extracting homology cobordant invariants of homology cylinders from the H-torsion $\tau : \mathcal{C}_{g,1} \longrightarrow \mathcal{K}^\times_H/(\pm H)$, which is a crossed homomorphism, as follows.

First they consider the subgroup $A \subset \mathcal{K}^\times_H$ defined by

$$A := \{f^{-1} \cdot \varphi(f) \mid f \in \mathcal{K}^\times_H,\ \varphi \in \mathrm{Sp}(2g, \mathbb{Z})\},$$

by which we can obtain a *homomorphism*

$$\tau : \mathcal{C}_{g,1} \longrightarrow \mathcal{K}^\times_H/(\pm H \cdot A).$$

Note that $f = \overline{f}$ holds in $\mathcal{K}^\times_H/(\pm H \cdot A)$ since $-I_{2g} \in \mathrm{Sp}(2g, \mathbb{Z})$. Second, they used the equality mentioned in Theorem 3.6 (2). Namely, if we put

$$N := \{f \cdot \overline{f} \mid f \in \mathcal{K}^\times_H\},$$

then we obtain a homomorphism

$$\widetilde{\tau} : \mathcal{H}_{g,1} \longrightarrow \mathcal{K}^\times_H/(\pm H \cdot A \cdot N).$$

Note that $f^2 = 1$ holds for any $f \in \mathcal{K}_H^\times/(\pm H \cdot A \cdot N)$.

The structure of $\mathcal{K}_H^\times/(\pm H \cdot A \cdot N)$ is given as follows. Recall that $\mathcal{K}_H = \mathbb{Z}[H](\mathbb{Z}[H] - \{0\})^{-1}$. The ring $\mathbb{Z}[H]$ is a Laurent polynomial ring of $2g$ variables and it is a unique factorization domain. Thus every Laurent polynomial f is factorized into irreducible polynomials uniquely up to multiplication by a unit in $\mathbb{Z}[H]$. Therefore, for every irreducible polynomial λ, we can count the exponent of λ in the factorization of f. This counting naturally extends to that for elements in $\mathcal{K}_H^\times$. Under the identification by $\pm H \cdot A \cdot N$, an element in $\mathcal{K}_H^\times/(\pm H \cdot A \cdot N)$ is determined by the exponents of all $\mathrm{Sp}(2g, \mathbb{Z})$-orbits of irreducible polynomials (up to multiplication by a unit in $\mathbb{Z}[H]$) modulo 2. Hence $\mathcal{K}_H^\times/(\pm H \cdot A \cdot N)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\infty$. Finally by using $(\mathbb{Z}/2\mathbb{Z})$-torsion of the knot concordance group, they show the following:

Theorem 4.1 (Cha-Friedl-Kim [1]). *The homomorphism*

$$\widetilde{\tau} : \mathcal{H}_{g,1} \longrightarrow \mathcal{K}_H^\times/(\pm H \cdot A \cdot N)$$

is not surjective but its image is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\infty$.

Now we try to investigate abelian quotients of $\mathcal{H}_{g,1}$ by using the Magnus representation r. It looks easier to extract information of $\mathcal{H}_{g,1}$ from the representation r together with Cha-Friedl-Kim's idea, since r itself is an homology cobordism invariant as mentioned in Theorem 3.3 (3). Consider two maps

$$\widehat{r} : \mathcal{H}_{g,1} \xrightarrow{r} \mathrm{GL}(2g, \mathcal{K}_H) \xrightarrow{\det} \mathcal{K}_H^\times \longrightarrow \mathcal{K}_H^\times/(\pm H),$$

$$\widetilde{r} : \mathcal{H}_{g,1} \xrightarrow{\widehat{r}} \mathcal{K}_H^\times/(\pm H) \longrightarrow \mathcal{K}_H^\times/(\pm H \cdot A).$$

While $\widehat{r}$ is a crossed homomorphism, its restriction to $\mathcal{IH}_{g,1}$ and $\widetilde{r}$ are homomorphisms. Note that both $\mathcal{K}_H^\times/(\pm H)$ and $\mathcal{K}_H^\times/(\pm H \cdot A)$ are isomorphic to $\mathbb{Z}^\infty$.

Theorem 4.2. (1) *For $(M, i_+, i_-) \in \mathcal{C}_{g,1}$, the equality*

$$\widehat{r}(M) = \overline{\tau(M)} \cdot (\tau(M))^{-1} \quad \in \mathcal{K}_H^\times/(\pm H)$$

holds.
(2) *For $g \geq 1$, the homomorphism $\widetilde{r} : \mathcal{H}_{g,1} \to \mathcal{K}_H^\times/(\pm H \cdot A)$ is trivial.*
(3) *For $g \geq 2$, the homomorphism $\widehat{r} : \mathcal{IH}_{g,1} \to \mathcal{K}_H^\times/(\pm H)$ is not surjective but its image is isomorphic to $\mathbb{Z}^\infty$.*

Sketch of Proof. (1) follows from the definitions of the invariants and and torsion duality. We omit the details.

As mentioned above, the action of $\mathrm{Sp}(2g, \mathbb{Z})$ implies that $f = \overline{f}$ for any $f \in \mathcal{K}_H^\times/(\pm H \cdot A)$. Then our claim (2) immediately follows from (1). (We can also use the symplecticity (Theorem 3.3 (2)) of r to show (2).)

To show (3), we use the homology cylinder $M_L \in \mathcal{C}_{2,1}$ in Example 3.7. While $M_L \notin \mathcal{IC}_{2,1}$, we can adjust it by some $g_1 \in \mathcal{M}_{2,1}$ so that $M_L \cdot g_1 \in \mathcal{IC}_{2,1}$. Since $\widehat{r}$ is trivial on $\mathcal{M}_{2,1}$, we have

$$\widehat{r}(M_L \cdot g_1) = \widehat{r}(M_L) = \frac{\gamma_3 + \gamma_4 - 1}{\gamma_3^{-1} + \gamma_4^{-1} - 1} \in \mathcal{K}_H^\times/(\pm H).$$

Take $f \in \mathcal{M}_{2,1}$ such that $\sigma(f) \in \mathrm{Sp}(4, \mathbb{Z})$ maps

$$\gamma_1 \longmapsto \gamma_1 + \gamma_4, \quad \gamma_2 \longmapsto \gamma_2, \quad \gamma_3 \longmapsto \gamma_2 + \gamma_3, \quad \gamma_4 \longmapsto \gamma_4.$$

Consider $f^m \cdot M_L \in \mathcal{C}_{2,1}$ and adjust it by some $g_m \in \mathcal{M}_{2,1}$ so that $f^m \cdot M_L \cdot g_m \in \mathcal{IC}_{2,1}$. Then we have

$$\widetilde{r}(f^m \cdot M_L \cdot g_m) = {}^{\sigma(f^m)}\widetilde{r}(M_L) = \frac{\gamma_2^m\gamma_3 + \gamma_4 - 1}{\gamma_2^{-m}\gamma_3^{-1} + \gamma_4^{-1} - 1} \in \mathcal{K}_H^\times/(\pm H).$$

We can check that the values $\left\{\dfrac{\gamma_2^m\gamma_3 + \gamma_4 - 1}{\gamma_2^{-m}\gamma_3^{-1} + \gamma_4^{-1} - 1}\right\}_{m=0}^{\infty}$ generate an infinitely generated subgroup of $\mathcal{K}_H^\times/(\pm H)$. This completes the proof when $g = 2$. We can use the above computation for $g \geq 3$ by a stabilization. □

Consequently, we have obtained a result similar to Morita [11, Corollary 5.2] and Cha-Friedl-Kim [1, Theorem 7.2 (2)].

5. Generalization to higher-dimensional cases

We can consider homology cylinders over X for any compact oriented connected k-dimensional manifold X with $k \geq 3$ by rewriting Definition 2.1 word-by-word. Let $\mathcal{M}(X)$, $\mathcal{C}(X)$ and $\mathcal{H}(X)$ denote the corresponding diffeotopy group, monoid of homology cylinders and homology cobordism group of homology cylinders. We have natural homomorphisms

$$\mathcal{M}(X) \longrightarrow \mathcal{C}(X) \longrightarrow \mathcal{H}(X).$$

Remark 5.1. In contrast with the case of surfaces, the homomorphism $\mathcal{M}(X) \to \mathcal{C}(X)$ is *not* necessarily injective for a general manifold X. In fact, if $[\varphi] \in \operatorname{Ker}(\mathcal{M}(X) \to \mathcal{C}(X))$, the definition of the homomorphism only says that φ is a *pseudo isotopy* over X.

For $k \geq 2$ and $n \geq 1$, we put

$$X_n^k := \mathop{\#}_{n}(S^1 \times S^{k-1}).$$

The manifold X_n^k may be regarded as a generalization of a closed surface since $X_n^2 = \Sigma_{n,0}$.

Suppose $k \geq 3$. Then $\pi_1 X_n^k \cong \pi_1(X_n^k - \operatorname{Int} D^k) \cong F_n$, where $\operatorname{Int} D^k$ is an open k-ball, and $H_1 := H_1(F_n) \cong \mathbb{Z}^n$. Let $\langle x_1, x_2, \ldots, x_n\rangle$ be a basis of F_n (and H_1). We have a monoid homomorphism

$$\mathrm{Acy} : \mathcal{C}(X_n^k - \operatorname{Int} D^k) \longrightarrow \operatorname{Aut}(F_n^{\mathrm{acy}})$$

and it induces a group homomorphism

$$\mathrm{Acy} : \mathcal{H}(X_n^k - \operatorname{Int} D^k) \longrightarrow \operatorname{Aut}(F_n^{\mathrm{acy}})$$

Consider the composition

$$\widetilde{r} : \operatorname{Aut}(F_n^{\mathrm{acy}}) \xrightarrow{r} \mathrm{GL}(n, \mathcal{K}_{H_1}) \xrightarrow{\det} \mathcal{K}_{H_1}^\times \longrightarrow \mathcal{K}_{H_1}^\times/(\pm H_1 \cdot A') \cong \mathbb{Z}^\infty,$$

where $\mathcal{K}_{H_1} := \mathbb{Z}[H_1](\mathbb{Z}[H_1] - \{0\})^{-1}$ and $A' := \{f^{-1} \cdot \varphi(f) \mid f \in \mathcal{K}_{H_1}^\times,\ \varphi \in \mathrm{GL}(n, \mathbb{Z})\}$. The map $\widetilde{r}$ is a homomorphism for the same reason mentioned in the previous section.

Theorem 5.2. *For any $k \geq 3$ and $n \geq 2$, we have:*

1. *The homomorphism* $\mathrm{Acy} : \mathcal{H}(X_n^k - \operatorname{Int} D^k) \to \operatorname{Aut}(F_n^{\mathrm{acy}})$ *is surjective.*
2. *The image of $\widetilde{r}$ is an infinitely generated subgroup of $\mathbb{Z}^\infty$. In particular, the abelian groups $H_1(\operatorname{Aut}(F_n^{\mathrm{acy}}))$ and $H_1(\mathcal{H}(X_n^k - \operatorname{Int} D^k))$ have infinite rank.*

Sketch of Proof. (1) follows from a construction similar to the one used in the proof of [13, Theorem 6.1]. To show (2), consider a homomorphism $f_m : F_n \to F_n$ defined by

$$f_m(\gamma_1) = (\gamma_1\gamma_2^{-1}\gamma_1^{-1}\gamma_2^{-1})^m\gamma_1\gamma_2^{2m}, \qquad f_m(\gamma_i) = \gamma_i \ (2 \leq i \leq n)$$

for each $m \geq 1$. The homomorphism f_m induces an isomorphism on $H_1(F_n)$ and therefore it extends to an automorphism (denoted also by f_m) of F_n^{acy}. By the Fox calculus, we can easily check that

$$\widetilde{r}(f_m) = 1 - \gamma_2 + \gamma_2^2 - \gamma_2^3 + \cdots + \gamma_2^{2m}.$$

Then (2) follows from the irreducibility of these polynomials when $2m+1$ is prime by a well known fact on the cyclotomic polynomials. □

References

[1] J. C. Cha, S. Friedl, T. Kim, *The cobordism group of homology cylinders*, preprint (2009), arXiv:0909.5580.

[2] S. Garoufalidis, J. Levine, *Tree-level invariants of three-manifolds, Massey products and the Johnson homomorphism*, Graphs and patterns in mathematics and theorical physics, Proc. Sympos. Pure Math. **73** (2005), 173–205.

[3] H. Goda, T. Sakasai, *Abelian quotients of monoids of homology cylinders*, preprint (2009), arXiv:0905.4775, to appear in Geom. Dedicata.

[4] M. Goussarov, *Finite type invariants and n-equivalence of 3-manifolds*, C. R. Math. Acad. Sci. Paris **329** (1999), 517–522.

[5] K. Habiro, *Claspers and finite type invariants of links*, Geom. Topol. **4** (2000), 1–83.

[6] J. Harer, *The second homology group of the mapping class group of an orientable surface*, Invent. Math. **72** (1983), 221–239.

[7] J. Levine, *Link concordance and algebraic closure, II*, Invent. Math. **96** (1989), 571–592.

[8] J. Levine, *Algebraic closure of groups*, Contemp. Math. **109** (1990), 99–105.

[9] J. Levine, *Homology cylinders: an enlargement of the mapping class group*, Algebr. Geom. Topol. **1** (2001), 243–270.

[10] S. Morita, *Abelian quotients of subgroups of the mapping class group of surfaces*, Duke Math. J. **70** (1993), 699–726.

[11] S. Morita, *Symplectic automorphism groups of nilpotent quotients of fundamental groups of surfaces*, Adv. Stud. Pure Math. **52** (2008), 443–468.

[12] C. D. Papakyriakopoulos, *Planar regular coverings of orientable closed surfaces*, Ann. of Math. Stud. **84**, Princeton Univ. Press (1975), 261–292.

[13] T. Sakasai, *Homology cylinders and the acyclic closure of a free group*, Algebr. Geom. Topol. **6** (2006), 603–631.

[14] T. Sakasai, *The symplecticity of the Magnus representation for homology cobordisms of surfaces*, Bull. Austral. Math. Soc. **76** (2007), 421–431.

[15] T. Sakasai, *The Magnus representation and higher-order Alexander invariants for homology cobordisms of surfaces*, Algebr. Geom. Topol. **8** (2008), 803–848.

[16] M. Suzuki, *Geometric interpretation of the Magnus representation of the mapping class group*, Kobe J. Math. **22** (2005), 39–47.

[17] V. Turaev, *Introduction to combinatorial torsions*, Lectures Math. ETH Zürich, Birkhäuser (2001).

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