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A NOTE ON DEGREES OF TWISTED ALEXANDER POLYNOMIALS

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ABSTRACT. In this short note we discuss degrees of twisted Alexander polynomials and demonstrate an explicit example for knots which is related to the degree formula due to Friedl, Kim and Kitayama.

1. INTRODUCTION

In this note we consider degrees of twisted Alexander polynomials. Recently Friedl, Kim and Kitayama showed the following theorem.

Theorem 1.1 (Friedl-Kim-Kitayama [1]). *Let N be an irreducible 3-manifold with empty or toroidal boundary such that $N \neq S^1 \times D^2$. Let $\rho : \pi_1 N \to GL(d, \mathbb{F})$ be a representation over a field $\mathbb{F}$ with involution and let $\alpha : \pi_1 N \to \mathbb{Z}$ be an admissible epimorphism (namely α is non-trivial if it is restricted to any boundary component). If ρ is conjugate to its dual and if the twisted Alexander polynomial $\tau(N, \alpha \otimes \rho) \in \mathbb{F}(t)$ does not vanish, then*

$$\deg \tau(N, \alpha \otimes \rho) \equiv d\|\alpha\| \mod 2$$

holds, where $\|\alpha\|$ denotes the Thurston norm of $\alpha \in H^1(N, \mathbb{Z}) = \mathrm{Hom}(\pi_1 N, \mathbb{Z})$.

Remark 1.2. When $d = 2$ and the image $\rho(\pi_1 N)$ is in $SL(2, \mathbb{C})$, the above theorem implies that $\tau(N, \alpha \otimes \rho) \in \mathbb{C}(t)$ is of even degree (see Remark 2.1 for the precise definition of degree of a rational function).

The purpose of this note is to give an example for the torus boundary case such that the highest and the next coefficients of the twisted Alexander polynomial of the exterior of a knot never vanish simultaneously as functions on the character variety of nonabelian $SL(2, \mathbb{C})$-representations. This means Theorem 1.1 is optimal in the sense that the formula holds modulo 2 but not modulo 4.

In the next section we quickly review the definition of the twisted Alexander polynomial, due to Wada [8] (so we will use different notations from those of Theorem 1.1). An explicit example for knots will be given in Section 3.

2. TWISTED ALEXANDER POLYNOMIALS

Let K be a knot in the 3-sphere S^3 and $N(K)$ be an open tubular neighborhood of K. For a knot group $G(K) = \pi_1 E(K)$, namely the fundamental group of the exterior $E(K) = S^3 - N(K)$ of K, we choose and fix a Wirtinger presentation

$$G(K) = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_{k-1} \rangle.$$

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Then the abelianization homomorphism

$$\alpha : G(K) \to H_1(E(K), \mathbb{Z}) \cong \mathbb{Z} = \langle t \rangle$$

is given by

$$\alpha(x_1) = \cdots = \alpha(x_k) = t.$$

Here we specify a generator t of $H_1(E(K), \mathbb{Z})$ and denote the sum in the infinite cyclic group $\mathbb{Z}$ multiplicatively.

Next we take a linear representation $\rho : G(K) \to SL(2, \mathbb{C})$. The tensor product of two representations α and ρ is defined by

$$(\alpha \otimes \rho)(x) = \alpha(x)\rho(x)$$

for $x \in G(K)$. These maps naturally induce two ring homomorphisms $\tilde{\alpha} : \mathbb{Z}[G(K)] \to \mathbb{Z}[t, t^{-1}]$ and $\tilde{\rho} : \mathbb{Z}[G(K)] \to M(2, \mathbb{C})$, where $\mathbb{Z}[G(K)]$ is the group ring of $G(K)$ over $\mathbb{Z}$ and $M(2, \mathbb{C})$ is the matrix algebra of degree 2 over $\mathbb{C}$. Combining them, we obtain a ring homomorphism

$$\tilde{\alpha} \otimes \tilde{\rho} : \mathbb{Z}[G(K)] \to M\left(2, \mathbb{C}[t, t^{-1}]\right).$$

Let F_k denote the free group on generators $x_1, \ldots, x_k$ and

$$\Phi : \mathbb{Z}[F_k] \to M\left(2, \mathbb{C}[t, t^{-1}]\right)$$

be the composition of the surjective homomorphism $\mathbb{Z}[F_k] \to \mathbb{Z}[G(K)]$ induced by the presentation of $G(K)$ and the tensor representation $\tilde{\alpha} \otimes \tilde{\rho}$.

Now let us consider the $(k-1) \times k$ matrix M whose (i, j)-component is the 2×2 matrix

$$\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M\left(2, \mathbb{C}[t, t^{-1}]\right),$$

where $\partial/\partial x$ denotes the free differential calculus. For $1 \leq j \leq k$, let us denote by M_j the $(k-1) \times (k-1)$ matrix obtained from M by removing the jth column. We regard M_j as a $2(k-1) \times 2(k-1)$ matrix with coefficients in $\mathbb{C}[t, t^{-1}]$. Then Wada's *twisted Alexander polynomial* of a knot K associated to a representation $\rho : G(K) \to SL(2, \mathbb{C})$ is defined to be a rational function

$$\Delta_{K,\rho}(t) = \frac{\det M_j}{\det \Phi(1 - x_j)}$$

and well-defined up to multiplication by t^{2n} ($n \in \mathbb{Z}$). Namely it is independent of the choice of the presentation of $G(K)$.

Remark 2.1. The *degree* of a rational function $f_1(t)/f_2(t) \in \mathbb{C}(t)$ is defined as follows. For a given $f(t) = \sum_{i=k}^{l} c_i t^i \in \mathbb{C}[t, t^{-1}]$ with $c_k \neq 0$ and $c_l \neq 0$, $\deg f(t)$ is defined to be $l - k$. For $f_1(t)/f_2(t)$ ($f_j(t) \in \mathbb{C}[t, t^{-1}]$), we define $\deg f_1(t)/f_2(t) = \deg f_1(t) - \deg f_2(t)$.

Remark 2.2. (1) It is known that the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ has a *reciprocal* property (see [2] for details). Namely $\Delta_{K,\rho}(t) = \Delta_{K,\rho}(t^{-1})$ holds up to multiplication by t^n ($n \in \mathbb{Z}$). Moreover it is shown in [1] that the equality holds up to multiplication by t^{2n} ($n \in \mathbb{Z}$).

(2) For a *nonabelian* representation $\rho : G(K) \to SL(2, \mathbb{C})$, namely the image $\rho(G(K))$ is a nonabelian subgroup of $SL(2, \mathbb{C})$, the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ associated to ρ is always a polynomial for any knot K (see [4]).

3. Example

Let us consider the knot $K = 8_4$ which is the 2-bridge knot $K(19,5)$. The Alexander polynomial of K is $\Delta_K(t) = 2t^4 - 5t^3 + 5t^2 - 5t + 2$ and thus the genus g_K of K is two. In particular the knot $K = 8_4$ is not fibered. The knot group $G(K)$ has a presentation

$$G(K) = \langle a, b \mid w^2 a = b w^2 \rangle, \quad w = (ba^{-1})^2 ba (b^{-1}a)^2.$$

Let $\rho : G(K) \to SL(2,\mathbb{C})$ be a map defined by

$$\rho(a) = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} s & 0 \\ 2-y & s^{-1} \end{pmatrix},$$

where $s \neq 0, y \in \mathbb{C}$. Here the entry $2-y$ is chosen so that the product of $\rho(a)$ and $\rho(b)^{-1}$ has trace y. Then ρ is a nonabelian representation of $G(K)$ if and only if a pair of complex numbers (s,y) satisfies $\phi(x,y) = 0$, where we put $x = s + s^{-1}$ (namely $\operatorname{tr}(\rho(a)) = x$) and the *Riley polynomial* $\phi(x,y)$ is given by an irreducible polynomial

$$\begin{aligned}\phi(x,y) = &-1 + x^2 - \left(5 - x^4\right) y + \left(10 - 13x^2 + 3x^4\right) y^2 + 10\left(2 - x^2\right) y^3 \\ &- \left(15 - 21x^2 + 5x^4\right) y^4 - \left(21 - 10x^2\right) y^5 + \left(7 - 12x^2 + 3x^4\right) y^6 \\ &+ \left(8 - 2x^2 - x^4\right) y^7 - \left(1 - 2x^2\right) y^8 - y^9.\end{aligned}$$

Remark 3.1. We refer to [5], [7] for the definition of the Riley polynomial. Roughly speaking, the Riley polynomial gives a defining equation of the nonabelian part of the space of conjugacy classes of $SL(2,\mathbb{C})$-representations of a 2-bridge knot. See [3], [6] for twisted Alexander polynomials and character varieties of 2-bridge knot groups.

Next let us calculate the twisted Alexander polynomial of K. Put $r = w^2 a w^{-2} b^{-1}$ and take the free differential by the generator a:

$$\frac{\partial r}{\partial a} = w^2 \left(1 + (1-a)(w^{-1} + w^{-2})\frac{\partial w}{\partial a}\right),$$

where

$$\frac{\partial w}{\partial a} = -ba^{-1} - ba^{-1}ba^{-1} + ba^{-1}ba^{-1}b + ba^{-1}ba^{-1}bab^{-1} + ba^{-1}ba^{-1}bab^{-1}ab^{-1}.$$

Let $\rho(a) = A,\ \rho(b) = B$ and $\rho(w) = W$. For a matrix V defined by

$$\begin{aligned}V = &-BA^{-1} - BA^{-1}BA^{-1} + tBA^{-1}BA^{-1}B + tBA^{-1}BA^{-1}BAB^{-1} \\ &+ tBA^{-1}BA^{-1}BAB^{-1}AB^{-1},\end{aligned}$$

the numerator of the twisted Alexander polynomial is given by

$$\begin{aligned}\det \Phi\left(\frac{\partial r}{\partial a}\right) &= t^8 \cdot \det\left(I + (I - tA)(t^{-2}W^{-1} + t^{-4}W^{-2})V\right) \\ &= (2+y)t^8 - x(4+3y)t^7 + (\text{lower terms in } t),\end{aligned}$$

where I denotes the identity matrix. On the other hand the denominator of $\Delta_{K,\rho}(t)$ is

$$\det \Phi(1-y) = t^2 - xt + 1.$$

Therefore the twisted Alexander polynomial of $K = 8_4$ is given by

$$\Delta_{K,\rho}(t) = (2+y)t^6 - 2x(1+y)t^5 + (\text{lower terms in } t).$$

Remark 3.2. We see that each coefficient of the twisted Alexander polynomial can be regarded as a function on the *character variety*

$$X^{\mathrm{nab}}(K) = \{(x,y) \in \mathbb{C}^2 \mid \phi(x,y) = 0\}.$$

More precisely they are polynomial functions on $X^{\mathrm{nab}}(K)$.

Now let us assume that the highest coefficient function of $\Delta_{K,\rho}(t)$ is zero. Then we obtain $y = -2$. Moreover if the next coefficient function is zero (in other words if degree of $\Delta_{K,\rho}(t)$ drops by 4), then we have $x = 0$. However we easily see that $\phi(0,-2) = 1 \neq 0$. It means that there is no character such that the highest and the next coefficient functions of the twisted Alexander polynomial vanish simultaneously.

As was shown in [3, Section 4], for every 2-bridge knot, there is an irreducible curve component X_1 in the character variety $X^{\mathrm{nab}}(K)$ such that

$$\deg \Delta_{K,\rho}(t) = 4g_K - 2$$

for all but finitely many characters. In this example, the character variety $X^{\mathrm{nab}}(8_4)$ is irreducible (namely $X^{\mathrm{nab}}(8_4) = X_1$), and if the highest coefficient function of $\Delta_{K,\rho}(t)$ vanishes (namely $y = -2$), the equation

$$\phi(x,-2) = 1 + 45x^2 + 250x^4 = 0$$

has four roots

$$x = \frac{\sqrt{-9 \pm \sqrt{41}}}{10}, \quad -\frac{\sqrt{-9 \pm \sqrt{41}}}{10}.$$

Then we can easily check that the corresponding twisted Alexander polynomials are of degree four (because of symmetry of coefficients).

Remark 3.3. It is easy to see that the above argument can be applied to any nonfibered 2-bridge knot K with $g_K \geq 2$. However, at the present moment, we do not have this kind of example for closed 3-manifolds.

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