EPIMORPHISMS BETWEEN 2-BRIDGE LINK GROUPS: ESSENTIAL SIMPLE LOOPS ON 2-BRIDGE SPHERES

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1. INTRODUCTION

The purpose of this note is to explain some of the ideas in [4] which gives an answer to a question on certain word problems on 2-bridge link groups raised in [10]. The key tool used in the proof is small cancellation theory, applied to two-generator and one-relator presentations of 2-bridge link groups. We note that it has been proved by Weinbaum [16] and Appel and Schupp [2] that the word and conjugacy problems for prime alternating link groups are solvable, by using small cancellation theory (see also [3] and references in it). Moreover, it was also shown by Sela [14] and Préaux [11] that the word and conjugacy problems for any link group are solvable. A characteristic feature of [4] is that we give a complete answer to a special (but also natural) word problem for the groups of 2-bridge links, which form a special (but also important) family of prime alternating links. In the sequels [5, 6, 7] of [4], we give a complete answer to certain natural conjugacy problems, and the solutions will be used in [8] to establish a variation of McShane’s identity for 2-bridge link groups, which had been conjectured by [13].

2. MAIN RESULTS

Consider the discrete group, $H$, of isometries of the Euclidean plane $\mathbb{R}^2$ generated by the $\pi$-rotations around the points in the lattice $\mathbb{Z}^2$. Set $(S^2, P) := (\mathbb{R}^2, \mathbb{Z}^2)/H$ and call it the Conway sphere. Then $S^2$ is homeomorphic to the 2-sphere, and $P$ consists of four points in $S^2$. We also call $S^2$ the Conway sphere. Let $S := S^2 - P$ be the complementary 4-times punctured sphere. For each $r \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, let $\alpha_r$ be the simple loop in $S$ obtained as the projection of a line in $\mathbb{R}^2 - \mathbb{Z}^2$ of slope $r$. Then $\alpha_r$ is essential in $S$, i.e., it does not bound a disk in $S$ and is not homotopic to a loop around a puncture. Conversely, any essential simple loop in $S$ is isotopic to $\alpha_r$ for a unique $r \in \hat{\mathbb{Q}}$. Then $r$ is called the slope of the simple loop. Similarly, any simple arc $\delta$ in $S^2$ joining two different points in $P$ such that $\delta \cap P = \partial \delta$ is isotopic to the image of a line in $\mathbb{R}^2$ of some slope $r \in \mathbb{Q}$ which intersects $\mathbb{Z}^2$. We call $r$ the slope of $\delta$.

A trivial tangle is a pair $(B^3, t)$, where $B^3$ is a 3-ball and $t$ is a union of two arcs properly embedded in $B^3$ which is parallel to a union of two mutually disjoint arcs in $\partial B^3$. Let $\tau$ be the simple unknotted arc in $B^3$ joining the two components of $t$ as illustrated in Figure 1. We call it the core tunnel of the trivial tangle. Pick a base point $x_0$ in int $\tau$, and let $(\mu_1, \mu_2)$ be the generating pair of the fundamental group $\pi_1(B^3 - t, x_0)$ each of which is represented by a based loop consisting of a small peripheral simple loop around a component of $t$ and a subarc of $\tau$ joining the circle to $x_0$. For any base point $x \in B^3 - t$, the generating pair of $\pi_1(B^3 - t, x)$ corresponding to the generating pair $(\mu_1, \mu_2)$ of $\pi_1(B^3 - t, x_0)$ via

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path joining $x$ to $x_0$ is denoted by the same symbol. The pair $(\mu_1, \mu_2)$ is unique up to (i) reversal of the order, (ii) replacement of one of the members with its inverse, and (iii) simultaneous conjugation. We call the equivalence class of $(\mu_1, \mu_2)$ the meridian pair of the fundamental group $\pi_1(B^3 - t)$.

By a rational tangle, we mean a trivial tangle $(B^3, t)$ which is endowed with a homeomorphism from $\partial(B^3, t)$ to $(S^2, P)$. Through the homeomorphism we identify the boundary of a rational tangle with the Conway sphere. Thus the slope of an essential simple loop in $\partial B^3 - t$ is defined. We define the slope of a rational tangle to be the slope of an essential loop on $\partial B^3 - t$ which bounds a disk in $B^3$ separating the components of $t$. (Such a loop is unique up to isotopy on $\partial B^3 - t$ and is called a meridian of the rational tangle.) We denote a rational tangle of slope $r$ by $(B^3, t(r))$. By van Kampen’s theorem, the fundamental group $\pi_1(B^3 - t(r))$ is identified with the quotient $\pi_1(S)/\langle\langle \alpha_r \rangle \rangle$, where $\langle\langle \alpha_r \rangle \rangle$ denotes the normal closure.

For each $r \in \hat{Q}$, the 2-bridge link $K(r)$ of slope $r$ is defined to be the sum of the rational tangles of slopes $\infty$ and $r$, namely, $(S^3, K(r))$ is obtained from $(B^3, t(\infty))$ and $(B^3, t(r))$ by identifying their boundaries through the identity map on the Conway sphere $(S^2, P)$. (Recall that the boundaries of rational tangles are identified with the Conway sphere.) $K(r)$ has one or two components according as the denominator of $r$ is odd or even. We call $(B^3, t(\infty))$ and $(B^3, t(r))$, respectively, the upper tangle and lower tangle of the 2-bridge link.

Let $D$ be the Farey tessellation, whose ideal vertex set is identified with $\hat{Q}$. For each $r \in \hat{Q}$, let $\Gamma_r$ be the group of automorphisms of $D$ generated by reflections in the edges of $D$ with an endpoint $r$, and let $\hat{\Gamma}_r$ be the group generated by $\Gamma_r$ and $\Gamma_\infty$. Then the region, $R$, bounded by a pair of Farey edges with an endpoint $\infty$ and a pair of Farey edges with an endpoint $r$ forms a fundamental domain of the action of $\hat{\Gamma}_r$ on $H^2$ (see Figure 2). Let $I_1$ and $I_2$ be the closed intervals in $\hat{\mathbb{R}}$ obtained as the intersection with $\hat{\mathbb{R}}$ of the closure of $R$. Suppose that $r$ is a rational number with $0 < r < 1$. (We may always assume this except when we treat the trivial knot and the trivial 2-component link.) Write

$$r = \frac{1}{m_1 + \frac{1}{m_2 + \cdots + \frac{1}{m_k}}} =: [m_1, m_2, \ldots, m_k],$$

**Figure 1.** A trivial tangle
where $k \geq 1$, $(m_1, \ldots, m_k) \in (\mathbb{Z}_+)^k$, and $m_k \geq 2$. Then the above intervals are given by $I_1 = [0, r_1]$ and $I_2 = [r_2, 1]$, where

$$r_1 = \begin{cases} [m_1, m_2, \ldots, m_{k-1}] & \text{if } k \text{ is odd,} \\ [m_1, m_2, \ldots, m_{k-1}, m_k] & \text{if } k \text{ is even,} \end{cases}$$

$$r_2 = \begin{cases} [m_1, m_2, \ldots, m_{k-1}, m_k] & \text{if } k \text{ is odd,} \\ [m_1, m_2, \ldots, m_{k-1}] & \text{if } k \text{ is even.} \end{cases}$$

![Figure 2](image-url)

**Figure 2.** A fundamental domain of $\hat{\Gamma}_r$ in the Farey tessellation (the shaded domain) for $r = 5/17 = \frac{1}{3 + \frac{1}{2 + \frac{1}{2}}} = : [3, 2, 2]$.

We recall the following fact ([10, Proposition 4.6 and Corollary 4.7] and [4, Lemma 7.1]) which describes the role of $\hat{\Gamma}_r$ in the study of 2-bridge link groups.

**Proposition 2.1.** (1) If two elements $s$ and $s'$ of $\hat{\mathbb{Q}}$ belong to the same orbit $\hat{\Gamma}_r$-orbit, then the unoriented loops $\alpha_s$ and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$.

(2) For any $s \in \hat{\mathbb{Q}}$, there is a unique rational number $s_0 \in I_1 \cup I_2 \cup \{\infty, r\}$ such that $s$ is contained in the $\hat{\Gamma}_r$-orbit of $s_0$. In particular, $\alpha_s$ is homotopic to $\alpha_{s_0}$ in $S^3 - K(r)$. Thus if $s_0 \in \{\infty, r\}$ then $\alpha_s$ is null-homotopic in $S^3 - K(r)$.

Thus the following question naturally arises (see [10, Question 9.1(2)]).

**Question 2.2.** (1) Which essential simple loops on $S$ are null-homotopic in $S^3 - K(r)$?

(2) For two distinct rational numbers $s, s' \in I_1 \cup I_2$, when are the unoriented loops $\alpha_s$ and $\alpha_{s'}$ homotopic in $S^3 - K(r)$?

A complete answer to Question 2.2(1) is given by [4, Main Theorem 2.3] as follows.

**Theorem 2.3.** The loop $\alpha_s$ is null-homotopic in $S^3 - K(r)$ if and only if $s$ belongs to the $\hat{\Gamma}_r$-orbit of $\infty$ or $r$. In other words, if $s \in I_1 \cup I_2$ then $\alpha_s$ is not null-homotopic in $S^3 - K(r)$.

This theorem implies the following theorem [4, Main Theorem 2.4], which gives a partial answer to [10, Question 9.1(1)].
Theorem 2.4. There is an upper-meridian-pair-preserving epimorphism from $G(K(s))$ to $G(K(r))$ if and only if $s$ or $s + 1$ belongs to the $\hat{\Gamma}_r$-orbit of $r$ or $\infty$.

The following theorem, established in the series of papers [5, 6, 7], gives a complete answer to Question 2.2(2).

Theorem 2.5. (1) Suppose $r = 1/p$, where $p \geq 2$ is an integer. Then, for any two distinct $s, s' \in I_1 \cup I_2$, the unoriented loops $\alpha_s$ and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$ if and only if $s = q_1/p_1$ and $s' = q_2/p_2$ satisfy $q_1 = q_2$ and $q_1/(p_1 + p_2) = 1/p$, where $(p_1, q_1)$ is a pair of relatively prime positive integers.

(2) Suppose $r = 3/8$, namely $K(r)$ is the Whitehead link. Then, for any two distinct $s, s' \in I_1 \cup I_2$, the unoriented loops $\alpha_s$ and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$ if and only if the set $\{s, s'\}$ equals either $\{1/6, 3/10\}$ or $\{3/4, 5/12\}$.

(3) Suppose $r \neq 1/p$ and $r \neq 3/8$. Then, for any two distinct $s, s' \in I_1 \cup I_2$, the unoriented loops $\alpha_s$ and $\alpha_{s'}$ are never homotopic in $S^3 - K(r)$.

These results will be used in [8] to prove the following variation of McShane's identity, which had been conjectured in [13].

Theorem 2.6. Suppose $r = q/p$ satisfies the condition $q \not\equiv \pm 1 \pmod{p}$, and let $\rho$ be the holonomy representation of the complete hyperbolic structure of $S^3 - K(r)$. Then the following identity holds:

$$2 \sum_{s \in \text{int}I_1} \frac{1}{1 + e^{l_\rho(\alpha_s)}} + 2 \sum_{s \in \text{int}I_2} \frac{1}{1 + e^{l_\rho(\alpha_s)}} + \sum_{s \in \partial I_1 \cup \partial I_2} \frac{1}{1 + e^{l_\rho(\alpha_s)}} = -1.$$ 

Further the modulus $\lambda(L(r))$ of the cusp torus of the cusped hyperbolic manifold $S^3 - K(r)$ with respect to a suitable choice of a longitude is given by the following formula:

$$\lambda(K(r)) = 2 \sum_{s \in \text{int}I_1} \frac{1}{1 + e^{l_\rho(\alpha_s)}} + \sum_{r \in \partial I_1} \frac{1}{1 + e^{l_\rho(\alpha_s)}}.$$ 

In the above theorem, $l_\rho(\alpha_s)$ is an element of $\mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$ defined as follows. The $PSL(2, \mathbb{C})$-representation of $\pi_1(S)$ induced by $\rho$ extends to a representation, denoted by the same symbol $\rho$, of the orbifold fundamental group of the $(2, 2, 2, \infty)$-orbifold, $\mathcal{O}$, obtained as the quotient of $S$ by the natural $\mathbb{Z}/(2\mathbb{Z}) \oplus \mathbb{Z}/(2\mathbb{Z})$-action (see e.g., [1, Proposition 2.2.2]). Each simple loop $\alpha_s$ in $S$ doubly covers a simple loop in $\mathcal{O}$. Let $\sqrt{u_s}$ be (a conjugacy class of) an element of $\pi_1(\mathcal{O})$ represented by the simple loop. Then $l_\rho(\alpha_s)$ denotes the complex translation length of the hyperbolic isometry $\rho(\sqrt{u_s}) \in PSL(2, \mathbb{C}) \cong \text{Isom}(\mathbb{H}^3)$.

We also obtain the following theorem concerning the set of end invariants $E(\rho)$, defined by Tan, Wong and Zhang [15], of the $PSL(2, \mathbb{C})$-representation of $\pi_1(T)$ induced by the representation $\rho$ in Theorem 2.6, where $T$ is the once-punctured torus obtained as the double covering of the orbifold $\mathcal{O}$.

Theorem 2.7. Let $r = q/p$ be a rational number. If $q \not\equiv \pm 1 \pmod{p}$, then let $\rho$ be the holonomy representation of the complete hyperbolic structure of $S^3 - K(r)$. If $q \equiv \pm 1 \pmod{p}$, then let $\rho$ be the faithful discrete $PSL(2, \mathbb{R})$-representation of the quotient of $G(K(r))$ by the infinite cyclic center. In both cases, we continue to denote by the same symbol $\rho$ the $PSL(2, \mathbb{C})$-representation of $\pi_1(T)$ induced by $\rho$. Then the set of end invariants $E(\rho)$ of $\rho$ is equal to the limit set $\Lambda(\hat{\Gamma}_r)$ of $\hat{\Gamma}_r$. 

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3. Presentations of 2-Bridge Link Groups

In this section, we introduce the upper presentation of a 2-bridge link group which we shall use throughout this paper. By van Kampen’s theorem, the link group $G(K(r)) = \pi_1(S^3 - K(r))$ is identified with $\pi_1(S)/\langle \langle \alpha_\infty, \alpha_r \rangle \rangle$. We call the image in the link group of the meridian pair of the fundamental group $\pi_1(B^3 - t(\infty))$ (resp. $\pi_1(B^3 - t(r))$) the upper meridian pair (resp. lower meridian pair). The link group is regarded as the quotient of the rank 2 free group, $\pi_1(B^3 - t(\infty)) \cong \pi_1(S)/\langle \langle \alpha_\infty \rangle \rangle$, by the normal closure of $\alpha_r$. This gives a one-relator presentation of the link group.

To find the presentation of $G(K(r))$ explicitly, let $a$ and $b$, respectively, be the elements of $\pi_1(B^3 - t(\infty), x_0)$ represented by the oriented loops $\mu_1$ and $\mu_2$ based on $x_0$ as illustrated in Figure 3. Then $\{a, b\}$ forms the meridian pair of $\pi_1(B^3 - t(\infty))$, which is identified with the free group $F(a, b)$. Note that $\mu_i$ intersects the disk, $\delta_i$, in $B^3$ bounded by a component of $t(\infty)$ and the essential arc, $\gamma_i$, on $\partial(B^3, t(\infty)) = (S^2, P)$ of slope $1/0$, in Figure 3. Obtain a word $u_r$ in $\{a, b\}$ by reading the intersection of the (suitably oriented) loop $\alpha_r$ with $\gamma_1 \cup \gamma_2$, where a positive intersection with $\gamma_1$ (resp. $\gamma_2$) corresponds to $a$ (resp. $b$). Then the word $u_r$ represents the free homotopy class of $\alpha_r$. It then follows that

$$G(K(r)) = \pi_1(S^3 - K(r)) \cong \pi_1(B^3 - t(\infty)) / \langle \langle \alpha_r \rangle \rangle \cong F(a, b) / \langle \langle u_r \rangle \rangle \cong \langle a, b \mid u_r \rangle.$$

If $r \neq \infty$, then $\alpha_r$ intersects $\gamma_1$ and $\gamma_2$ alternately, and hence $a$ and $b$ appear in $(u_r)$ alternately.

By using the universal abelian covering $\mathbb{R}^2 - \mathbb{Z}^2 \to S$, we can write down the word $u_r$ explicitly. Note that the inverse image of $\gamma_1$ (resp. $\gamma_2$) in $\mathbb{R}^2 - \mathbb{Z}^2$ is the union of the single arrowed (resp. double arrowed) vertical edges in Figure 4. Let $L(r)$ be the line in $\mathbb{R}^2$ of slope $r$ passing through the origin, and let $L^+(r)$ be the line obtained by translating $L(r)$ by the vector $(0, \eta)$ for sufficiently small positive real number $\eta$. Then $L^+(r)$ lies in $\mathbb{R}^2 - \mathbb{Z}^2$ and projects to the simple loop $\alpha_r$. Pick a base point, $z$, from the intersection of $L^+(r)$ with the second quadrant, and consider the sub-line-segment of $L^+(r)$ bounded by $z$ and $z + (2p, 2q)$. Then it forms a fundamental domain of the covering $L^+(r) \to \alpha_r$, and the word $u_r$ is obtained by reading the intersection of the line-segment with the vertical lattice lines. To be precise, for each integer $0 \leq i \leq 2p - 1$, let $P_i^+$ be the intersection of
the line-segment with the vertical lattice line $x = i$. We define the letter at $P_{i}^{+}$ to be $a$ or $b$ according as $P_{i}^{+}$ lies on a vertical edge with a single arrow or double arrow in Figure 4, namely according as $i$ is even or odd. We define the sign of $P_{i}^{+}$ to be $+1$ or $-1$ according as the corresponding arrow is upward or downward. Then the letter and the sign of $P_{i}^{+}$, respectively, give the letter and the exponent of the $(i+1)$-th term of the word $u_r$ for each $0 \leq i \leq 2p - 1$. This gives the following formula for the word $u_r$ (see Figure 4).

$$u_r = a^{\epsilon_1}b^{\epsilon_2}\cdots a^{\epsilon_{2p-1}}b^{\epsilon_{2p}},$$

where $\epsilon_i = (-1)^{[i(q/p)]^*+1}$. Here $[t]$* denotes the smallest integer greater than $t$.

In order to simplify this formula, let $\hat{u}_r$ be the subword of $u_r$ corresponding to the set \(\{P_i^+ | 1 \leq i \leq p-1\}\). Then $\hat{u}_r$ is obtained from the open interval in $L(r)$ bounded by $(0,0)$ and $(p,q)$ by reading its intersection with the vertical lattice lines, and so we obtain the following formula.

$$\hat{u}_r = \begin{cases} b^{\epsilon_1}a^{\epsilon_2}\cdots b^{\epsilon_{p-2}}a^{\epsilon_{p-1}} & \text{if } p \text{ is odd,} \\ b^{\epsilon_1}a^{\epsilon_2}\cdots a^{\epsilon_{p-2}}b^{\epsilon_{p-1}} & \text{if } p \text{ is even,} \end{cases}$$

where $\epsilon_i = (-1)^{[i(q/p)]}$. By using the symmetry around $(p,q)$ of $\mathbb{R}^2 - \mathbb{Z}^2$, we can observe that the subword of $u_r$ corresponding to the set \(\{P_i^+ | p+1 \leq i \leq 2p-1\}\) is equal to $\hat{u}_r^{-1}$. Hence we obtain the following formula (see [12, Proposition 1]).

$$u_r = \begin{cases} a\hat{u}_{q/p}b^{(-1)^q}\hat{u}_{q/p}^{-1} & \text{if } p \text{ is odd,} \\ a\hat{u}_{q/p}a^{-1}\hat{u}_{q/p}^{-1} & \text{if } p \text{ is even,} \end{cases}$$

We now describe a natural decomposition of the word $u_r$, which plays a key role in this paper. Let $r_i = q_i/p_i$ ($i = 1, 2$) be the rational number introduced in Section 2. Consider the infinite broken line, $B$, obtained by joining the lattice points

$$\cdots, (0, 0), (p_2, q_2), (p, q), (p + p_2, q + q_2), (2p, 2q), \cdots$$

FIGURE 4. The line of slope 5/7 gives $\alpha_{5/7} = ba^{-1}bab^{-1}a$, so the free homotopy class of $\alpha_{5/7}$ is represented by the cyclic word $(u_{5/7}) = (a\hat{u}_{5/7}b^{-1}\hat{u}_{5/7}^{-1}) = (aba^{-1}bab^{-1}a^{-1}ba^{-1}b^{-1}ab^{-1})$. Since the inverse image of $\gamma_1$ (resp. $\gamma_2$) in $\mathbb{R}^2$ is the union of the single arrowed (resp. double arrowed) vertical edges, a positive intersection with a single arrowed (resp. double arrowed) edge corresponds to $a$ (resp. $b$).
Canonical decomposition of the relator $u = \alpha_{r}$

\[ \frac{v_{i}}{p_{i}} \text{, } (i=1,2) : \text{Farey neighbors } v_{i} \frac{p_{i}}{q_{i}} \text{, so } \frac{p_{i}}{q_{i}} < \frac{r}{q_{i}} < \frac{p_{i}+1}{q_{i}} \]

\[ u = v_{1} v_{2} v_{3} v_{4} \quad |v_{1}| = |v_{3}| = p_{i}+1 \]
\[ |v_{2}| = |v_{4}| = p_{i}-1 \]

**Figure 5.** The decomposition of the relator $u_{r} = v_{1} v_{2} v_{3} v_{4}$

which is invariant by the translation $(x, y) \mapsto (x + p, y + q)$. By slightly modifying $B$ near the lattice points, we obtain a (topological) line, $B^{+}$, in $\mathbb{R}^2 - Z^2$, invariant by the translation, which is homotopic to the line $L^{+}(r)$. Pick a point, $z_{0} \in B^{+}$ in the second quadrant, and consider the sub-path of $B^{+}$ bounded by $z_{0}$ and $z_{4} := z_{0} + (2p, 2q)$. Then the word $u_{r}$ is also obtained by reading the intersection of the sub-path with the vertical lattice lines. Pick a point $z_{i} \in B^{+}$ whose $x$-coordinate is $p_{2} +$ (small positive number), and set $z_{2} := z_{0} + (p, q)$ and $z_{3} := z_{1} + (p, q)$. Let $B_{i}^{+}$ be the sub-path of $B^{+}$ joining $z_{i-1}$ with $z_{i} (i = 1, 2, 3, 4)$. Let $v_{i}$ be the subword of $u_{r}$ corresponding to $B_{i}^{+}$. Then we have the decomposition

\[ u_{r} = v_{1} v_{2} v_{3} v_{4} \]

The subword $v_{i}$ is non-empty except when $r = 1/p (p \in N)$ and $i \in \{1, 3\}$. The importance of this decomposition is described in the following section.

4. **Sequences associated with the simple loop $\alpha_{r}$**

In this section, we define a sequence $S(r)$ of slope $r$ and a cyclic sequence $CS(r)$ of slope $r$ all of which arise from the single word $u_{r}$ representing the simple loop $\alpha_{r}$, and observe several important properties of these sequences, so that we can adopt small cancellation theory in the succeeding sections.

We fix some definitions and notation. Let $X$ be a set. By a *word* in $X$, we mean a finite sequence $x_{1}^{c_{1}} x_{2}^{c_{2}} \cdots x_{n}^{c_{n}}$ where $x_{i} \in X$ and $c_{i} = \pm 1$. Here we call $x_{i}^{c_{i}}$ the $i$-*th letter* of the word. For two words $u, v$ in $X$, by $u \equiv v$ we denote the visual equality of $u$ and $v$, meaning
that if \( u = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n} \) and \( v = y_1^{\delta_1} \cdots y_m^{\delta_m} \) \((x_i, y_j \in X; \, \epsilon_i, \delta_j = \pm 1)\), then \( n = m \) and \( x_i = y_i \) and \( \epsilon_i = \delta_i \) for each \( i = 1, \ldots, n \). The length of a word \( v \) is denoted by \(|v|\). A word \( v \) in \( X \) is said to be reduced if \( v \) does not contain \( xx^{-1} \) or \( x^{-1}x \) for any \( x \in X \). A word is said to be cyclically reduced if all its cyclic permutations are reduced. A cyclic word is defined to be the set of all cyclic permutations of a cyclically reduced word. By \( (v) \) we denote the cyclic word associated with a cyclically reduced word \( v \). Also by \((u) \equiv (v)\) we mean the visual equality of two cyclic words \((u)\) and \((v)\). In fact, \((u) \equiv (v)\) if and only if \( v \) is visually a cyclic shift of \( u \).

**Definition 4.1.** (1) Let \( v \) be a nonempty reduced word in \( \{a, b\} \). Decompose \( v \) into

\[
v \equiv v_1 v_2 \cdots v_t,
\]

where, for each \( i = 1, \ldots, t - 1 \), all letters in \( v_i \) have positive (resp. negative) exponents, and all letters in \( v_{i+1} \) have negative (resp. positive) exponents. Then the sequence of positive integers \( S(v) := (|v_1|, |v_2|, \ldots, |v_t|) \) is called the \( S \)-sequence of \( v \).

(2) Let \( (v) \) be a nonempty reduced cyclic word in \( \{a, b\} \) represented by a word \( v \). Decompose \((v)\) into

\[
(v) \equiv (v_1 v_2 \cdots v_t),
\]

where all letters in \( v_i \) have positive (resp. negative) exponents, and all letters in \( v_{i+1} \) have negative (resp. positive) exponents (taking subindices modulo \( t \)). Then the cyclic sequence of positive integers \( CS(v) := ((|v_1|, |v_2|, \ldots, |v_t|) \) is called the cyclic \( S \)-sequence of \( (v) \). Here the double parentheses denote that the sequence is considered modulo cyclic permutations.

**Definition 4.2.** For a rational number \( r \) with \( 0 < r \leq 1 \), let \( u_r \) be the word in \( \{a, b\} \) defined in Section 3. Then the symbol \( S(r) \) (resp. \( CS(r) \)) denotes the \( S \)-sequence \( S(u_r) \) of \( u_r \) (resp. cyclic \( S \)-sequence \( CS(u_r) \) of \( (u_r) \)), which is called the \( S \)-sequence of slope \( r \) (resp. the cyclic \( S \)-sequence of slope \( r \)).

In the remainder of this paper unless specified otherwise, we suppose that \( r \) is a rational number with \( 0 < r \leq 1 \), and write \( r \) as a continued fraction:

\[
r = [m_1, m_2, \ldots, m_k],
\]

where \( k \geq 1, (m_1, \ldots, m_k) \in (\mathbb{Z}_+)^k \) and \( m_k \geq 2 \) unless \( k = 1 \). For brevity, we write \( m \) for \( m_1 \).

The following proposition plays a key role in the proof of Lemma 5.4 and Theorem 5.2.

**Proposition 4.3** ([4, Proposition 4.10]). The sequence \( S(r) \) has a decomposition \((S_1, S_2, S_1, S_2)\) which satisfies the following.

1. Each \( S_i \) is symmetric, i.e., the sequence obtained from \( S_i \) by reversing the order is equal to \( S_i \). (Here, \( S_1 \) is empty if \( k = 1 \).)
2. Each \( S_i \) occurs only twice in the cyclic sequence \( CS(r) \).
3. \( S_1 \) begins and ends with \( m + 1 \).
4. \( S_2 \) begins and ends with \( m \).

The above decomposition corresponds to the decomposition \( u_r = v_1 v_2 v_3 v_4 \) introduced in Section 3. To be precise, we have \( S_1 = S(v_1) = S(v_3) \) and \( S_2 = S(v_2) = S(v_4) \). The following proposition plays a key role in the proof of the main theorem.
Proposition 4.4. Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Proposition 4.3. For a rational number $s$ with $0 < s \leq 1$, suppose that the cyclic $S$-sequence $CS(s)$ contains both $S_1$ and $S_2$ as a subsequence. Then $s \notin I_1 \cup I_2$.

5. SMALL CANCELLATION CONDITIONS FOR 2-BRIDGE LINK GROUPS

Let $F(X)$ be the free group with basis $X$. A subset $R$ of $F(X)$ is called symmetrized, if all elements of $R$ are cyclically reduced and, for each $w \in R$, all cyclic permutations of $w$ and $w^{-1}$ also belong to $R$.

Definition 5.1. Suppose that $R$ is a symmetrized subset of $F(X)$. A nonempty word $b$ is called a piece if there exist distinct $w_1, w_2 \in R$ such that $w_1 \equiv bc_1$ and $w_2 \equiv bc_2$. Small cancellation conditions $C(p)$ and $T(q)$, where $p$ and $q$ are integers such that $p \geq 2$ and $q \geq 3$, are defined as follows (see [9]).

1. Condition $C(p)$: If $w \in R$ is a product of $n$ pieces, then $n \geq p$.
2. Condition $T(q)$: For $w_1, \ldots, w_n \in R$ with no successive elements $w_i, w_{i+1}$ an inverse pair $(i \mod n)$, if $n < q$, then at least one of the products $w_1w_2, \ldots, w_{n-1}w_n, w_nw_1$ is freely reduced without cancellation.

The following key theorem enables us to apply small cancellation theory to the groups presentation $\langle a, b \mid u_r \rangle$ of $G(K(r))$.

Theorem 5.2. Let $r$ be a rational number such that $0 < r < 1$. Recall the presentation $\langle a, b \mid u_r \rangle$ of $G(K(r))$ given in Section 3, and let $R$ be the symmetrized subset of $F(a, b)$ generated by the single relator $u_r$. Then $R$ satisfies $C(4)$ and $T(4)$.

Definition 5.3. For a positive integer $n$, a non-empty subword $w$ of the cyclic word $(u_r)$ is called a maximal $n$-piece if $w$ is a product of $n$ pieces and if any subword $w'$ of $u_r$ which properly contains $w$ as an initial subword is not a product of $n$-pieces.

Theorem 5.2 actually follows from the following complete characterizations of the maximal $n$-pieces for $n = 1, 2, 3$. (For simplicity, we describe the result only for generic case.)

Lemma 5.4. Suppose that $r$ is a rational number such that $0 < r < 1$ and $r \neq 1/p$ for any integer $p \geq 2$. Let $v_{ib}^*$ be the maximal proper initial subword of $v_i$, i.e., the initial subword of $v_i$ such that $|v_{ib}^*| = |v_i| - 1$ ($i = 1, 2, 3, 4$). Then the following hold, where $v_{ib}$ and $v_{ie}$ are nonempty initial and terminal subwords of $v_i$ with $|v_{ib}|, |v_{ie}| \leq |v_i| - 1$, respectively.

1. The following is the list of all maximal 1-pieces of $(u_r)$, arranged in the order of the position of the initial letter:

$$v_{1b}^*, v_{1e}v_2, v_2v_{3b}^*, v_2v_3v_{4b}, v_3v_4, v_4v_1v_{1b}^*, v_4v_1v_{2b}^*.$$  

2. The following is the list of all maximal 2-pieces of $(u_r)$, arranged in the order of the position of the initial letter:

$$v_1v_2, v_1e v_2v_{3b}^*, v_2v_3v_4, v_2v_3v_4v_{1b}^*, v_3v_4, v_4v_1v_2, v_4v_1v_{2b}^*.$$  

3. The following is the list of all maximal 3-pieces of $(u_r)$, arranged in the order of the position of the initial letter:

$$v_1v_2v_{3b}^*, v_1e v_2v_3v_4, v_2v_3v_4v_{1b}^*, v_2v_3v_4v_{1b}^*, v_3v_4v_1v_2, v_4v_1v_2v_{3b}^*, v_4v_1v_2v_{3b}^*.$$
Corollary 5.5. (1) A subword $w$ of the cyclic word $(u_{1}^{\pm 1})$ is a piece if and only if $S(w)$ does not contain $S_{1}$ as a subsequence and does not contain $S_{2}$ in its interior, i.e., $S(w)$ does not contain a subsequence $(\ell_{1}, S_{2}, \ell_{2})$ for some $\ell_{1}, \ell_{2} \in \mathbb{Z}_{+}$.

(2) For a subword $w$ of the cyclic word $(u_{1}^{\pm 1})$, $w$ is not a product of two pieces if and only if $S(w)$ either contains $(S_{1}, S_{2})$ as a proper initial subsequence or contains $(S_{2}, S_{1})$ as a proper terminal subsequence.

6. Outline of the proof of Theorem 2.3

Let $R$ be the symmetrized subset of $F(a, b)$ generated by the single relator $u_{r}$ of the group presentation $G(K(r)) = \langle a, b \mid u_{r} \rangle$. Suppose on the contrary that $a_{r}$ is null-homotopic in $S^{3} - K(r)$, i.e., $u_{r} = 1$ in $G(K(r))$, for some $s \in I_{1} \cup I_{2}$. Then there is a van Kampen diagram $M$ over $G(K(r)) = \langle a, b \mid R \rangle$ such that the boundary label is $u_{r}$. Here $M$ is a simply connected 2-dimensional complex embedded in $\mathbb{R}^{2}$, together with a function $\phi$ assigning to each oriented edge $e$ of $M$, a label, a reduced word $\phi(e)$ in \{a, b\} such that the following hold.

1. If $e$ is an oriented edge of $M$ and $e^{-1}$ is the oppositely oriented edge, then $\phi(e^{-1}) = \phi(e)^{-1}$.

2. For any boundary cycle $\delta$ of any face of $M$, $\phi(\delta)$ is a cyclically reduced word representing an element of $R$. (If $\alpha = e_{1}, \ldots, e_{n}$ is a path in $M$, we define $\phi(\alpha) \equiv \phi(e_{1}) \cdots \phi(e_{n}).$)

We may assume $M$ is reduced, namely it satisfies the following condition: Let $D_{1}$ and $D_{2}$ be faces (not necessarily distinct) of $M$ with an edge $e \subseteq \partial D_{1} \cap \partial D_{2}$, and let $e\delta_{1}$ and $\delta_{2}e^{-1}$ be boundary cycles of $D_{1}$ and $D_{2}$, respectively. Set $\phi(\delta_{1}) = f_{1}$ and $\phi(\delta_{2}) = f_{2}$. Then we have $f_{2} \neq f_{1}^{-1}$.

Moreover, we may assume the following conditions:

1. $d_{M}(v) \geq 3$ for every vertex $v \in M - \partial M$.
2. For every edge $e$ of $\partial M$, the label $\phi(e)$ is a piece.
3. For a path $e_{1}, \ldots, e_{n}$ in $\partial M$ of length $n \geq 2$ such that the vertex $\partial_{i} \cap \partial_{i+1}$ has degree 2 for $i = 1, 2, \ldots, n - 1$, $\phi(e_{1})\phi(e_{2}) \cdots \phi(e_{n})$ cannot be expressed as a product of less than $n$ pieces.

Since $R$ satisfies the conditions $C(4)$ and $T(4)$ by Theorem 5.2, $M$ is a $[4, 4]$-map, i.e.,

1. $d_{M}(v) \geq 4$ for every vertex $v \in M - \partial M$.
2. $d_{M}(D) \geq 4$ for every face $D \in M$.

Here, $d_{M}(v)$, the degree of $v$, denotes the number of oriented edges in $M$ having $v$ as initial vertex, and $d_{M}(D)$, the degree of $D$, denotes the number of oriented edges in a boundary cycle of $D$.

Now, for simplicity, assume that $M$ is homeomorphic to a disk. (In general, we need to consider an extremal disk of $M$.) Then by the Curvature Formula of Lyndon and Schupp (see [9, Corollary V.3.4]), we have

$$\sum_{v \in \partial M} (3 - d_{M}(v)) \geq 4.$$ 

By using this formula, we see that there are three edges $e_{1}$, $e_{2}$ and $e_{3}$ in $\partial M$ such that $e_{1} \cap e_{2} = \{v_{1}\}$ and $e_{2} \cap e_{3} = \{v_{2}\}$, where $d_{M}(v_{i}) = 2$ for each $i = 1, 2$. Since $\phi(e_{1})\phi(e_{2})\phi(e_{3})$ is not expressed as a product of two pieces, we see by Corollary 5.5 that the boundary label
of $M$ contains a subword, $w$, with $S(w) = (S_1, S_2, \ell)$ or $(\ell, S_2, S_1)$. This in turn implies that the $S$-sequence of the boundary label contains both $S_1$ and $S_2$ as subsequences. Hence, by Proposition 4.4, we have $s \not\in I_1 \cup I_2$, a contradiction.

7. Outline of the proof of Theorem 2.5

Suppose, for two distinct $s, s' \in I_1 \cup I_2$, the unoriented loops $\alpha_s$ and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$. Then there is a reduced annular $R$-diagram, such that $u_s$ is an outer boundary label and $u_s^{\pm 1}$ is an inner boundary label of $M$. Again we can see that $M$ is a $[4,4]$-map and hence we have the following curvature formula.

$$0 \leq \sum_{v \in \partial M} (3 - d_M(v)).$$

By using this formula, we obtain the following very strong structure theorem for $M$, which plays key roles throughout the series of papers [5, 6, 7].

**Theorem 7.1.** Figure 6(a) illustrates the only possible type of the outer boundary layer of $M$, while Figure 6(b) illustrates the only possible type of whole $M$. (The number of faces per layer and the number of layers are variable.)

In the above theorem, the outer boundary layer of the annular map $M$ is a submap of $M$ consisting of all faces $D$ such that the intersection of $\partial D$ with the outer boundary of $M$ contains an edge, together with the edges and vertices contained in $\partial D$.

![Figure 6](image-url)

The first paper [5] of the series treats the case when the 2-bridge link is a $(2,p)$-torus link, the second paper [6], treats the case of 2-bridge links of slope $n/(2n+1)$ and $(n+1)/(3n+2)$, where $n \geq 2$ is an arbitrary integer, and the third paper [7] treats the general case. The two families treated in the second paper play special roles in the project in the sense that the treatment of these links form a base step of an inductive proof of the theorem for general 2-bridge links. We note that both a 2-bridge link of slope $n/(2n+1)$ with $n = 2$ and a 2-bridge link of slope $(n+1)/(3n+2)$ with $n = 1$ are the figure-eight knot. It is a bit surprising that the treatment of the figure-eight knot is the most complicated. This reminds us of the phenomenon in the theory of exceptional Dehn filling that the figure-eight knot attains the maximal number of exceptional Dehn fillings.
REFERENCES


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