

SOME PROBLEMS ON EPIMORPHISMS BETWEEN KNOT GROUPS

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1. INTRODUCTION

This is a survey article on epimorphisms between knot groups from a special point of views.

Let K be a knot in S^3 and $G(K) = \pi_1(S^3 - K)$ the knot group. For two knots K, K' , we write $K \geq K'$ if there exists an epimorphism $\varphi : G(K) \rightarrow G(K')$. This relation gives a partial order on the set of knot groups because any knot group is Hopfian. However we would like to treat it on the set of knots. Hence we have to put some assumption. It is known that this relation $K \geq K'$ is a partial order on the set of knots under some mild conditions.

For example, here we modify the definition to the following: we write $K \geq K'$ if there exists a peripheral structure preserving epimorphism $G(K) \rightarrow G(K')$. Then it is a partial order on the set of knots. See [14, 13] as a reference from this point of view.

In this article we do not assume it. Instead of the condition on the peripheral structure, we treat only prime knots. Then it gives also a partial order on the set of the prime knots because $K = K'$ if their knot groups are isomorphic for prime knots K, K' .

Some geometric reasons for the existence of an epimorphism are known. For example, any degree one map $f : E(K) \rightarrow E(K')$ induces an epimorphism $f_* : G(K) \rightarrow G(K')$. Here $E(K)$ is the exterior of K in S^3 . Secondly it is known that a quotient map of a periodic knot induces an epimorphism. Recently Ohtsuki-Riley-Sakuma gave a systematic construction of epimorphisms between 2-bridge link groups in [11]. They are sufficient conditions on the existence of epimorphisms.

On the other hand there are necessary conditions. First, some property on the Alexander polynomial gave us a necessary condition for the existence of an epimorphism between not only knot groups, but also finite presentable groups. It is an exercise in [3]. This necessary condition can also be extended to that of the twisted Alexander polynomial in [10], which is more effective for determining the non-existence of an epimorphism.

By using these above criteria and the computer, this partial order on the set of prime knots with up to 11 crossings, it was determined in [6, 4]. This is the direction to make a concrete and complete list with up to given crossing numbers.

On the other hand there are qualitative researches. In particular recently Agol and Liu [1] announced the positive answer for the Simon's conjecture. Namely, for any knot K , there exist at most finitely many knots which admit epimorphisms from $G(K)$.

In this article, we would like to discuss epimorphisms, which is related to (1) decomposability, and (2) degeneracy. We would like to pose some problems on the existence of epimorphisms.

2. DECOMPOSABILITY

The first problem is how to decompose a given epimorphism into more elementary epimorphisms. We give the following definition.

Definition 2.1. Let $\varphi : G(K) \rightarrow G(K')$ be an epimorphism.

- φ is called to be decomposable if there is another prime knot \tilde{K} such that φ is a composition of epimorphisms $G(K) \rightarrow G(\tilde{K})$ and $G(\tilde{K}) \rightarrow G(K')$.
- φ is called to be nondecomposable if it is not decomposable.

First we put the following problem.

Problem 2.2. For a given epimorphism between knot groups find a criterion to be decomposable.

Remark 2.3. It seems that there is no example of decomposable epimorphisms in the list of [4]. However we cannot see it directly. Because we have no information on the relation between the existence of epimorphisms and the crossing numbers.

For any knot K , there exists a prime knot \tilde{K} with an epimorphism $G(\tilde{K}) \rightarrow G(K)$. It is seen by applying Kawauchi's imitation theory [5]. It is also proved by Silver and Whitten in [13]. Then for a given any knot K , there are infinitely many knots $\{K_i\}$ with a sequence of epimorphisms

$$\cdots \rightarrow G(K_{i+1}) \rightarrow G(K_i) \rightarrow G(K_{i-1}) \rightarrow \cdots \rightarrow G(K_1) \rightarrow G(K).$$

Ohtsuki-Riley-Sakuma [11] mentioned the following problem.

Problem 2.4. For a given K characterize a knot \tilde{K} which admits an epimorphism $G(\tilde{K}) \rightarrow G(K)$.

Now we would like to do it as follows under the above situation.

Problem 2.5. For a given K characterize a knot \tilde{K} which admits a nondecomposable epimorphism $G(\tilde{K}) \rightarrow G(K)$.

3. EPIMORPHISMS BETWEEN UNTWISTED DOUBLED KNOTS

We have no example of an epimorphism between satellite knots because there are no satellite knots with up to 11-crossings. In this section we give such an epimorphism induced by a degree one map. This is a special construction for untwisted doubles starting from a degree one map.

Recall the method of constructing untwisted doubled knots. Let V be a standard solid torus in S^3 . Here K is a geometrically essential knot in V , but

- (1) K is trivial in S^3 ,
- (2) the linking number of K and the meridian curve of V is zero.

More precisely the union of K and the standard meridian curve of V is Whitehead link in S^3 .

Let $K_1 \subset S^3$ be another knot and let V_1 be a tubular neighborhood of K_1 in S^3 . We take a homeomorphism $h : V \rightarrow V_1$ and assume h maps the standard longitude and the meridian of V respectively to the preferred longitude and the meridian of V_1 . By using this homeomorphism h , we obtain a new knot $K_2 = h(K) \subset V_1 \subset S^3$. Any knot K_2

obtained in this construction is said to be the untwisted double of K_1 . Untwisted doubles are special cases of satellite knots. We write simply $utd(K)$ to the untwisted double of K here.

Remark 3.1. If we take the untwisted double $utd(K)$ of K , then $G(utd(K))$ contains a subgroup isomorphic with $G(K)$.

Now let K, K' be prime knots with an epimorphism $\varphi : G(K) \rightarrow G(K')$. Here we assume that this φ is induced by a degree one map $f : E(K) \rightarrow E(K')$. Because the map on the boundary torus is also a degree one map between tori, then we further assume that the meridian goes to the meridian and the longitude does to the longitude. Then it can be a homeomorphism on the torus boundary by deforming the map if we need.

We take untwisted doubles $utd(K)$ and $utd(K')$ of K and K' respectively. Naturally f can be extended to the degree one map

$$\bar{f} : E(utd(K)) \rightarrow E(utd(K'))$$

by attaching $E(K)$ and V , and $E(utd(K'))$ and V . This \bar{f} induces an epimorphism between $G(utd(K))$ and $G(utd(K'))$. In this case, both the Alexander polynomials of $utd(K), utd(K')$ are trivial. We put some examples.

Example 3.2. It holds $8_{18} \geq 3_1$, as it is shown in [9]. A degree one map that maps the meridian to the meridian, and the preferred longitude to the preferred longitude can realize this epimorphism. Here we take the untwisted doubles $utd(8_{18}), utd(3_1)$. Then it is seen $utd(8_{18}) \geq utd(3_1)$.

Similarly the following relations

$$10_{23}, 10_{103}, 10_{159} \geq 3_1, 9_{40} \geq 4_1, 10_{42} \geq 5_1$$

are realized by degree one maps. See also [9]. Then we can apply the above construction and obtain epimorphisms between untwisted doubled knots as follows.

Proposition 3.3.

$$\begin{aligned} utd(8_{18}), utd(10_{23}), utd(10_{103}), utd(10_{159}) &\geq utd(3_1), \\ utd(9_{40}) &\geq utd(4_1), utd(10_{42}) \geq utd(5_1). \end{aligned}$$

The following problem appears naturally.

Problem 3.4. Let $\varphi : G(K) \rightarrow G(K')$ be a nondecomposable epimorphism induced by a degree one map. Then $\bar{\varphi} : G(utd(K)) \rightarrow G(utd(K'))$ is nondecomposable ?

Remark 3.5. Let K_2 be a satellite knot with pattern K . Then we can see easily $K_2 \geq K$. See [14].

4. DEGENERATE EPIMORPHISMS

A degree one map induces an epimorphism as we mentioned before. More generally the following proposition holds. Let $f : E(K) \rightarrow E(K')$ be a nonzero degree map.

Proposition 4.1. The index $[G(K'), f_*(G(K))]$ divides the degree of f . In particular the index is trivial if the degree is one.

Remark 4.2. An epimorphism $\varphi : G(K) \rightarrow G(K')$ induced by a nonzero degree map is called a virtual epimorphism. Such a map can be lifted to a degree one map from $E(K)$ onto a finite covering of $E(K')$.

Definition 4.3. An epimorphism $\varphi : G(K) \rightarrow G(K')$ is called a degenerate epimorphism if it is induced from a degree zero map $E(K) \rightarrow E(K')$.

Example 4.4. It holds $8_{20} \geq 3_1$. It can be realized by a degenerate epimorphism as it is shown in [9].

In particular, there does not exist any nondegenerate epimorphism from 8_{20} to 3_1 , because the epimorphism between Alexander modules over the rational field does not split. Here we mention the following proposition. See [15] as a reference of this splitting property.

Proposition 4.5. *Let $\varphi : G(K) \rightarrow G(K')$ be an epimorphism induced by a degree one map. It induces an epimorphism between Alexander modules over the integers. Then it has a split section. If φ is induced by a nonzero degree map, then there is a split section over the rationals.*

The following example is a degenerate epimorphism. Although the source is not a prime knot, this gives a fundamental example for degenerate epimorphisms between prime knots.

Example 4.6. Let $K \# \bar{K}$ be a connected sum of a knot K and its mirror image \bar{K} . Here there is a natural orientation reversing involution of $(S^3, K \# \bar{K})$. Then its quotient space by this involution is (S^3, K) . It is easy to see that this quotient map is a degree one map.

We put the following problem.

Problem 4.7. *For a given K characterize a knot \tilde{K} which admits a degenerate and nondecomposable epimorphism $G(\tilde{K}) \rightarrow G(K)$.*

In [11], the following construction is given. Let $\varphi : G(K) \rightarrow G(K')$ be an epimorphism. First we take $[\gamma] \in \text{Ker}(\varphi)$ and γ its representative simple closed curve. Further we assume that γ is a trivial loop in S^3 . Then replace (S^3, K) with a new knot (S^3, \tilde{K}) obtained from (S^3, K) by surgery along γ . Then $\varphi : G(K) \rightarrow G(K')$ induces $\tilde{\varphi} : G(\tilde{K}) \rightarrow G(K')$ because $[\gamma]$ belongs to $\text{Ker}(\varphi)$.

Remark 4.8. Because γ is a trivial loop in S^3 , then we obtain a knot in S^3 after surgery. If it is not trivial, we do a new knot in some 3-manifold. In this case there also exists an epimorphism.

It is proved that any epimorphism between hyperbolic 2-bridge knot groups is not degenerate in [2]. Namely any epimorphism between them is induced from a nonzero degree map.

We put some examples of degenerate epimorphism between hyperbolic knots.

Example 4.9. It holds that $10_{59}, 10_{137} \geq 4_1$. Here $10_{59}, 10_{137}$ are 3-bridge hyperbolic knots and have structures of a Montesinos knot as follows:

- $10_{59} = M(-1; (5, 2), (5, -2), (2, 1))$
- $10_{137} = M(0; (5, 2), (5, -2), (2, 1))$

We can apply this construction to $4_1 \# \bar{4}_1 = 4_1 \# 4_1$. First we recall that there exists an epimorphism

$$G(4_1 \# \bar{4}_1) \rightarrow G(4_1)$$

which is induced from a quotient map of a reflection. The rational tangle $(5, 2)$ is corresponding to 4_1 , and $(5, -2)$ is doing to $\bar{4}_1 = 4_1$. By surgery along some simple closed curve in $S^3 \setminus 4_1 \# \bar{4}_1$, we get both of $G(10_{59}) \rightarrow G(4_1)$, and $G(10_{137}) \rightarrow G(4_1)$. Here we can take γ as follows.

- (1) γ is trivial in S^3 .
- (2) For the standard 2-disk D bounded by γ the geometric intersection number of D and γ is two.
- (3) The algebraic intersection number of D and γ is zero.

Generally we can obtain the following by applying this construction to a 2-bridge knot K , which is given by a rational tangle. Namely we can do for $K \# \bar{K}$.

Proposition 4.10. *For any 2-bridge knot K , there exist infinitely many Montesinos knots $\{\tilde{K}_i\}$ with degenerate epimorphisms $\{G(\tilde{K}_i) \rightarrow G(K)\}$.*

Finally we put the following problem.

Problem 4.11. *Can any degenerate nondecomposable epimorphism $G(\tilde{K}) \rightarrow G(K)$ be obtained from a fundamental degenerate epimorphism $G(K \# \bar{K}) \rightarrow G(K)$ by applying the above construction for a given knot K ?*

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