LENS SPACE SURGERIES ALONG TWO COMPONENT LINKS AND REIDEMEISTER-TURAEV TORSION

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1. INTRODUCTION

This article is a short survey of the core part of the authors' joint work "Lens space surgeries along certain 2-component links, Park's rational blow down, and Reidemeister–Turaev torsion" [KTYY, preprint]. In [KTYY], certain two families of 2-component links, denoted by $A_{m,n}$ and $B_{p,q}$ are focused, and the main result is the decision of the coefficient(s) of the knotted component yielding a lens space by Dehn surgery. The links are related to rational homology 4-ball used in J. Park's (generalized) rational blow down in 4-dimensional topology (see [3, 12]). Concrete calculus on the links $A_{m,n}$ and $B_{p,q}$ was important. The results made the contrast between $A_{m,n}$ (hyperbolic) and $B_{p,q}$ (Seifert) clear, which was one of the purpose of [KTYY] (see also [17]).

In this article, we focus another importance, the method itself to get some necessary conditions on the lens space surgery coefficients of a given link, by using Alexander polynomial and Reidemeister torsion. Our method satisfies that a result on a link $L$ always extends to the links whose Alexander polynomials are same with that of $L$.

We will compare the Reidemeister torsion of the result $M$ of Dehn surgery along a given link and that of a lens space $L(p, q)$ (in Example 3.4). Some necessary conditions are obtained from the value $\tau^\psi_d(M)$ of the Reidemeister torsion in the $d$-th cyclotomic field $\mathbb{Q}(\zeta_d)$ by $d$-norm, where $d\geq 2$ is a divisor of $p$. From the sequence of the equalities on $\tau^\psi_d(M)s$ in $\mathbb{Q}(\zeta_d)$ for all divisors $d$ of $p$ (with a fixed combinatorial Euler structure of $M$), we take an identity on symmetric Laurent polynomials, as a lift of the equalities. We regard the identity as an equation of the surgery coefficient for $M$ to be a lens space.

In the next section, we start with some definitions of the Reidemeister torsion. In Section 3, we review surgery formulae. In Section 4, we will study $d$-norms in the $d$-th cyclotomic field, and show a certain uniqueness of a symmetric polynomial as a lift of the sequence of the equalities in $\mathbb{Q}(\zeta_d)s$. In Section 5, we will explain the method to get some necessary condition of lens space surgery coefficients of a given link. In Section 6, as a demonstration, we will apply our method to Berge's link, which is one of the most famous targets in lens space surgery ([1]).

2. Reidemeister Torsion

For a precise definition of the Reidemeister torsion, the reader refer to V. Turaev [14, 15]. Let $X$ be a finite CW complex and $\pi : \tilde{X} \to X$ its maximal abelian covering. Then $\tilde{X}$ has a CW structure induced by that of $X$ and $\pi$, and the cell chain complex $C_\ast$ of $\tilde{X}$ has a

Received February 5, 2011.
$E_L$ the complement of $L$.
$m_i, l_i$ a meridian and a longitude of the $i$-th component $K_i$.
$[m_i], [l_i]$ their homology classes.
$\Delta_L(t_1, \ldots, t_\mu)$ the Alexander polynomial of $L$, where $t_i$ is represented by $[m_i]$.
$(L; r_1, \ldots, r_\mu)$ the result of Dehn surgery along $L$,
where $r_i \in \mathbb{Q} \cup \{\infty, 0\}$ is the surgery coefficient of $K_i$.
$V_i$ the solid torus attached along $K_i$ in the Dehn surgery.
$m_i', [m_i']$ a meridian of $V_i$, and its homology class.
l_i', [l_i'] an oriented core curve of $V_i$, and its homology class.

Table 1. Notations (for manifolds)

\[ \mathbb{Z}[H]\text{-module structure}, \text{ where } H = H_1(X; \mathbb{Z}) \text{ is the first homology of } X. \text{ For an integral domain } R \text{ and a ring homomorphism } \psi : \mathbb{Z}[H] \to R, \text{ "the chain complex of }\tilde{X} \text{ related with } \psi", \text{ denoted by } C^\psi_*, \text{ is constructed as } C_* \otimes_{\mathbb{Z}[H]} Q(R), \text{ where } Q(R) \text{ is the quotient field of } R. \text{ The Reidemeister torsion of } X \text{ related with } \psi, \text{ denoted by } \tau^\psi(X), \text{ is calculated from } C^\psi_*, \text{ and is an element of } Q(R) \text{ determined up to multiplication of } \pm \psi(h) \text{ (} h \in H). \]

If $R = \mathbb{Z}[H]$ and $\psi$ is the identity map, then we denote $\tau^\psi(X)$ by $\tau(X)$. We note that $\tau^\psi(X)$ is not zero if and only if $C^\psi_*$ is acyclic.

Notation (for manifolds and homologies) Let $L = K_1 \cup \ldots \cup K_\mu$ be an oriented $\mu$-component link in $S^3$. We will use the notations in Table 1.

Notation (for algebra) For a pair of elements $A, B$ in $Q(R)$, if there exists an element $h \in H$ such that $A = \pm \psi(h)B$, then we denote the equality by $A \bowtie B$. We will often take a field $F$ and a ring homomorphism $\psi : \mathbb{Z}[H_1(M)] \to F$. We mainly use the $d$-th cyclotomic fields $\mathbb{Q}(<d>)$ as $F$, where $<d>$ is a primitive $d$-th root of unity.

3. Surgery Formulae

Let $E$ be a compact 3-manifold whose boundary $\partial E$ consists of tori ($E$ is possibly not $E_L$ for a link $L$). We study the 3-manifold $M = E \cup V_1 \cup \ldots \cup V_n$ obtained by attaching solid tori $V_i$s to $E$ by attaching maps $f_i : \partial V_i \to \partial E$ ($\text{Im}(f_i) \cap \text{Im}(f_j) = \emptyset$ for $i \neq j$). By $l_i'$ we denote the core of $V_i$. We let $\iota : E \to M$ denote the natural inclusion.

Lemma 3.1. (Surgery formula 1) If $\psi([l_i']) \neq 1$ for every $i = 1, \ldots, n$, then

$$\tau^\psi(M) \doteq \tau^\psi(E) \prod_{i=1}^n (\psi([l_i']) - 1)^{-1},$$

where $\psi' = \psi \circ \iota_* \ (\iota_* \text{ is a ring homomorphism induced by } \iota)$.

For the case of the complement $E_L$ of a $\mu$-component link $L$ in $S^3$ as in Table 1. The Reidemeister torsion is closely related with the Alexander polynomial.

Lemma 3.2. (Milnor [11]) Let $\Delta_L(t_1, \ldots, t_\mu)$ be the Alexander polynomial of a $\mu$-component link $L = K_1 \cup \ldots \cup K_\mu$ in $S^3$, where a variable $t_i$ is represented by the meridian
of $K_i$ ($i=1, \ldots, \mu$).

$$\tau(E_{I_{\lrcorner}}) = \left\{ \begin{array}{ll} \Delta_L(t_1)(t_1-1)^{-1} & (\mu=1), \\
\Delta_L(t_1, \ldots, t_\mu) & (\mu \geq 2). \end{array} \right.$$  

Next, we study the result of Dehn surgery $M = (L; p_1/q_1, \ldots, p_\mu/q_\mu)$ along $L$. We take integers $r_i$ and $s_i$ satisfying $p_ir_i - q_is_i = -1$.

Lemma 3.3. (Surgery formula II; T. Sakai [13], V. G. Turaev [14])

(1) In the case $M = (K; p/q)$ ($|p| \geq 2$), we have $H = H_1(M) \cong \langle T \mid T^p = 1 \rangle \cong \mathbb{Z}/|p|\mathbb{Z}$, where $T$ is represented by the meridian $[m]$. For a divisor $d \geq 2$ of $p$, we define a ring homomorphism $\psi_d : \mathbb{Z}[H] \rightarrow \mathbb{Q}($\(\zeta_d\)) by $\psi_d(T) = \zeta_d$. Then we have

$$\tau_{\psi_d}(M) = \Delta_K(\zeta_d)(\zeta_d-1)^{-1}(\zeta_d^{\overline{q}}-1)^{-1}$$

where $q\overline{q} \equiv 1 (\text{mod } p)$.

(2) In the case $M = (L; p_1/q_1, \ldots, p_\mu/q_\mu)$ ($\mu \geq 2$). Let $F$ be a field and $\psi : \mathbb{Z}[H_1(M)] \rightarrow F$ a ring homomorphism. If $\psi([m_i]^{r_i}[l_i]^{s_i}) \neq 1$ for every $i = 1, \ldots, \mu$, then we have

$$\tau^\psi(M) = \Delta_L(\psi([m_1]), \ldots, \psi([m_\mu])) \prod_{i=1}^\mu (\psi([m_i]^{r_i}[l_i]^{s_i}) - 1)^{-1}.$$  

Example 3.4. The lens space $L(p, q)$ is obtained as $-p/q$-surgery along the unknot. By Lemma 3.3 (1), for a divisor $d \geq 2$ of $p$, we have

$$\tau^\psi(L(p, q)) = (\zeta_d - 1)^{-1}(\zeta_d^{\overline{q}} - 1)^{-1},$$

where $q\overline{q} \equiv 1 (\text{mod } p)$.

4. CYCLOTONIC FIELD AND POLYNOMIAL

4.1. $d$-norm.

About algebraic fields, the reader refer to L. C. Washington [16] for example.

For an element $x$ in the $d$-th cyclotomic field $\mathbb{Q}(\zeta_d)$, the $d$-norm of $x$ is defined as

$$N_d(x) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})} \sigma(x),$$

where $\text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$ is the Galois group ($\cong (\mathbb{Z}/d\mathbb{Z})^\times$) related with a Galois extension $\mathbb{Q}(\zeta_d)$ over $\mathbb{Q}$. The following is well-known.

Proposition 4.1.

(1) If $x \in \mathbb{Q}(\zeta_d)$, then $N_d(x) \in \mathbb{Q}$. The map $N_d : \mathbb{Q}(\zeta_d) \setminus \{0\} \rightarrow \mathbb{Q} \setminus \{0\}$ is a group homomorphism.

(2) If $x \in \mathbb{Z}[\zeta_d]$, then $N_d(x) \in \mathbb{Z}$.

By easy calculations, we have the following.

Lemma 4.2.

(1) $N_d(\pm \zeta_d) = \left\{ \begin{array}{ll} \pm 1 & (d = 2), \\
1 & (d \geq 3). \end{array} \right.$

(2) $N_d(1 - \zeta_d) = \left\{ \begin{array}{ll} \ell & (d \text{ is a power of a prime } \ell \geq 2), \\
1 & \text{(otherwise)}. \end{array} \right.$

About applications of $d$-norms, for example, see [5, 6, 7, 8, 9, 10].
4.2. Reidemeister–Turaev torsion.

Let $M$ be a homology lens space with $H = H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ $(p \geq 2)$. Then the Reidemeister torsion $\tau^{\psi_d}(M)$ of $M$ related with $\psi_d$ is determined up to multiplication of $\pm \zeta_d^m$ $(m \in \mathbb{Z})$, where $d \geq 2$ is a divisor of $p$ and $\psi_d$ is the same ring homomorphism as in Lemma 3.3 (1). Once we fix a basis of a cell chain complex for the maximal abelian covering of $M$ as a $\mathbb{Z}[H] = \mathbb{Z}[t, t^{-1}] / (t^p - 1)$-module, the value $\tau^{\psi_d}(M)$ is uniquely determined as an element of $\mathbb{Q}(\zeta_d)$ for every $d$. The choice of the basis up to “base change equivalence” is called a combinatorial Euler structure of $M$ (cf. Turaev [15]). The Reidemeister torsion of a manifold with a fixed combinatorial Euler structure is said the Reidemeister–Turaev torsion.

We consider the sequence of the values $\tau^{\psi_d}(M)$ in $\mathbb{Q}(\zeta_d)$ of the Reidemeister–Turaev torsion for every divisor $d \geq 2$ of $p$, and regard them as a value sequence $\{\tau^{\psi_d}(M)\}_{d|p, d \geq 2}$ defined as below.

**Definition 4.3.** We define that a sequence of values $x = \{x_d\}_{d|p, d \geq 2}$ is a value sequence (of degree $p$) if $x_d \in \mathbb{Q}(\zeta_d)$ for every $d$. Two value sequences $x = \{x_d\}_{d|p, d \geq 2}$ and $y = \{y_d\}_{d|p, d \geq 2}$ are equal ($x = y$) if $x_d = y_d$ for every $d$. We are mainly concerned with the value sequence of type $x = \{F(\zeta_d)\}_{d|p, d \geq 2}$ for a rational function $F(t) \in \mathbb{Q}(t)$. In such a case, we say that $x$ is induced by $F(t)$ and that $F(t)$ is a lift of $x$. A control of $x = \{x_d\}_{d|p, d \geq 2}$ by a trivial unit $u = \eta t^m \in \mathbb{Z}[t, t^{-1}] / (t^p - 1)$ is defined by

$$ux = \{\eta \zeta_d^m x_d\}_{d|p, d \geq 2},$$

where $\eta = 1$ or $-1$ (constant) and $m \in \mathbb{Z}$. Two value sequences $x = \{x_d\}_{d|p, d \geq 2}$ and $y = \{y_d\}_{d|p, d \geq 2}$ are control equivalent if there is a trivial unit $u \in \mathbb{Z}[t, t^{-1}] / (t^p - 1)$ such that $y = ux$. A value sequence $x = \{x_d\}_{d|p, d \geq 2}$ is a real value sequence if $x_d$ is a real number for every $d$.

**Example 4.4.** A value sequence $x$ of degree 12 is in the form $x = \{x_2, x_3, x_4, x_6, x_{12}\}$. The following two value sequences $x, y$ of degree 12 are not equal, but control equivalent for $u = t^6$.

$$x = \{2, -1, -2, -1, 1\}, \quad y = \{2, -1, 2, 1, -1\}.$$

In fact, $x$ and $y$ is induced by $t^2 + t^{-2}$ and $t^4 + t^{-4}$, respectively.

Let $M$ be a homology lens space with $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ $(p \geq 2)$. Then a sequence $\{\tau^{\psi_d}(M)\}_{d|p, d \geq 2}$ of the Reidemeister torsions of $M$ with a combinatorial Euler structure is a value sequence of degree $p$. We say the value sequence a torsion sequence of $M$.

**Lemma 4.5.**

(1) Let $M$ and $M'$ be homeomorphic homology lens spaces with $H_1(M) \cong H_1(M') \cong \mathbb{Z}/p\mathbb{Z}$ $(p \geq 2)$. Then torsion sequences $\{\tau^{\psi_d}(M)\}_{d|p, d \geq 2}$ and $\{\tau^{\psi_d'}(M')\}_{d|p, d \geq 2}$ related with the corresponding ring homomorphisms $\psi_d$ and $\psi'_d$ (i.e., $\psi_d = \psi'_d \circ h_*$, where $h_*$ is the induced homomorphism of the homeomorphism) are control equivalent.

(2) Let $M$ be a homology lens space with $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ $(p \geq 2)$. Then we can control a torsion sequence of $M$ into a real value sequence.

**Proof.** (1) It is easy to see. 

(2) Here we let $\zeta$ denote any $d$-th primitive root ($\zeta_d$) of unity. Since $M$ is obtained by $p/q$-surgery along a knot $K$ in a homology 3-sphere for some $q$ (cf. [2]), and we can also
apply Lemma 2.5 (1) for the case, we have
\[ \tau_{\psi}(M) = \Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^q - 1)^{-1} \]
where \( q \equiv 1 \pmod{p} \). By the duality of the Alexander polynomial (cf. [11, 14, 15]), we may assume
\[ \Delta_K(t) = \Delta_K(t^{-1}). \]
This is also a control of the combinatorial Euler structure of the exterior of \( K \), which induces a control of a torsion sequence of \( M \). We take an odd integer lift of \( q \). Then
\[ \zeta^{\frac{1+q}{2}} \Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^q - 1)^{-1} \]
is a real number for every \( d \).

Lemma 4.6. If two real value sequences \( x = \{x_d\}_{d|p,d \geq 2} \) and \( y = \{y_d\}_{d|p,d \geq 2} \) of degree \( p \) are control equivalent satisfying \( y = ux \) for a trivial unit \( u = \eta t^m \in \mathbb{Z}[t^{-1}]/(t^p - 1) \), where \( \eta = \pm 1 \) and \( m \in \mathbb{Z} \), then the possibility of \( u \) is restricted as follows:
(i) If \( p \) is odd, then \( u = 1 \) or \(-1\).
(ii) If \( p \) is even, then \( u = 1, -1, t^{p/2} \) or \(-t^{p/2}\).

Proof. Since the ratio \( \zeta^m_p = \pm y_p/x_p \) is a real number, we have (i) \( m \equiv 0 \pmod{p} \) if \( p \) is odd, and (ii) \( m \equiv 0 \) or \( p/2 \pmod{p} \) if \( p \) is even.

Definition 4.7. (Symmetric Laurent polynomial) A Laurent polynomial \( F(t) \in \mathbb{Z}[t, t^{-1}] \) is symmetric if it is of the form
\[ F(t) = a_0 + \sum_{i=1}^{\infty} a_i(t^i + t^{-i}), \]
where \( a_i \) is an integer for all \( i = 1, 2, \ldots \) and \( a_i = 0 \) for every sufficiently large \( i \). Note that, if \( F(t) \) is a symmetric Laurent polynomial, the induced value sequence \( \{F(\zeta_d)\}_{d|p,d \geq 2} \) is a real value sequence. We are concerned with symmetric Laurent polynomials that are lifts (in \( \mathbb{Z}[t, t^{-1}] \)) of a polynomial in the quotient ring \( \mathbb{Z}[t^{-1}]/(t^p - 1) \). We say that \( F(t) \) (as above) is reduced if \( a_i = 0 \) for all \( i > [p/2] \). We often reduce the symmetric polynomials by using \( t^i + t^{-i} = t^{p+i} + t^{-(p+i)} \) modulo \( (t^p - 1) \). We let \( \text{red}(F(t)) \) denote the reduction of \( F(t) \) (i.e., \( \text{red}(F(t)) \) is reduced and \( \text{red}(F(t)) = F(t) \) in \( \mathbb{Z}[t^{-1}]/(t^p - 1) \)). We will use a notation \( \langle t^i \rangle = t^i + t^{-i} \), for short.

For a Laurent polynomial \( F(t) \in \mathbb{Z}[t, t^{-1}] \), the span of \( F(t) \) is the difference of the maximal degree of \( F(t) \) and the minimal degree of \( F(t) \), and we denote it by \( \text{span}(F(t)) \).

Lemma 4.8. Let \( N \geq 2 \) be an integer. Let \( F(t), G(t) \) be symmetric Laurent polynomials and \( x = \{F(\zeta_d)\}_{d|N,d \geq 2}, y = \{G(\zeta_d)\}_{d|N,d \geq 2} \) the induced real value sequences, respectively. If \( x \) and \( y \) are control equivalent, i.e., \( ux = y \) for a trivial unit \( u \) (here, \( u = 1 \) or \(-1 \) if \( N \) is odd, \( u = 1, -1, t^{N/2} \) or \(-t^{N/2} \) if \( N \) is even, by Lemma 4.6), and \( F(1) = G(1) = 0 \), then we have a congruence
\[ uF(t) \equiv G(t) \pmod{t^N - 1} \]
Furthermore, assuming \( \text{span}(G(t)) \leq 2[N/2] \),
(i) In the case that \( u = 1 \) or \(-1 \) and \( \text{span}(F(t)) \leq N - 1 \), we have an identity
\[ uF(t) = G(t) \] in \( \mathbb{Z}[t, t^{-1}] \).
(ii) Otherwise (in the case that $N$ is even and $u = \eta t^{N/2}$ with $\eta = 1$ or $-1$), we have $\text{red}(t^{N/2}F(t)) = \eta G(t)$ in $\mathbb{Z}[t, t^{-1}]$.

Proof. By Chinese Remainder Theorem, we have a ring isomorphism:

$$\mathbb{Q}[t, t^{-1}]/(t^{N} - 1) \cong \bigoplus_{d|N, d \geq 1} \mathbb{Q}(\zeta_d),$$

where $f(t)$ in the left-hand side maps to the value sequences $\{f(\zeta_d)\}_{d|N, d \geq 2}$ in the right-hand side. The isomorphism implies the required congruence. \hfill \Box

Note that $F(t)$ and $t^{N/2}F(t)$ induce the control equivalent real value sequences by $u = t^{N/2}$, but $\text{red}(t^{N/2}F(t)) \neq F(t)$ in general, see Example 4.4. Thus we have to care the case (ii) in the lemma. Here, we study relation between the coefficients of $F(t)$ and those of $\text{red}(t^{N/2}F(t))$.

Lemma 4.9. Let $N$ be an even integer.

If $F(t) = a_0 + \sum_{i=1}^{N/2} a_i(t^i + t^{-i})$, then $\text{red}(t^{N/2}F(t)) = b_0 + \sum_{i=1}^{N/2} b_i(t^i + t^{-i})$

with

$$b_0 = 2a_{N/2}, \quad b_{N/2} = a_0/2 \quad \text{and} \quad b_j = a_{N/2-j} \quad (j = 1, 2, \ldots, N/2 - 1).$$

Proof. It is because

$$t^{N/2}(t^j + t^{-j}) = t^{N/2+j} + t^{N/2-j} \equiv t^{(N/2-j)} + t^{-(N/2-j)} \quad \text{mod} \ t^N - 1.$$

\hfill \Box

5. Method

Let $L = K_1 \cup K_2 \cup \cdots \cup K_\mu$ be a link. We let $M$ simply denote the result $(L; r_1, \ldots, r_\mu)$ of the Dehn surgery. We use the notations in Table 1.

**Step 1** Study the first homologies (the generators and relations), from the exterior $E_L$ of $L$ (Of course, $H_1(E_L; \mathbb{Z}) \cong \bigoplus_{i=1}^\mu \mathbb{Z}[m_i]$) to the result $M$, by attaching solid tori $V_i$ one by one.

The first (obvious) necessary condition for the result $M$ of Dehn surgery to be a lens space $L(p, q)$ is

$$H_1(M; \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}.$$  

**Step 2** Calculate the Alexander polynomial $\Delta_L(t_1, \ldots, t_\mu)$ of $L$. Using Lemma 3.2 and Lemma 3.3, calculate the Reidemeister torsion $\tau^\psi(M)$ related with a ring homomorphism $\psi : \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$, where $d (\geq 2)$ is a divisor of $p$.

If $M$ is homeomorphic to a lens space $L(p, q)$ (with undecided $q$), then their Reidemeister torsions are to each other. By Example 3.4, there exists integers $i, j$ coprime to $p$ with $0 < i, j < p$ (they are lifts of $(\mathbb{Z}/p\mathbb{Z})^\times/\{\pm 1\}$) such that

$$\tau^\psi(M) = \frac{1}{(\zeta_d^i - 1)(\zeta_d^j - 1)} \quad \text{in} \ \mathbb{Q}(\zeta_d),$$

for each divisor $d (\geq 2)$ of $p$. We can assume $i + j$ is even by retaking $p - j$ instead of $j$. 72
Step 3 Using $d$-norm in $\mathbb{Q}(\zeta_d)$, studied in Subsection 4.1, to the equality (1), we have a necessary condition on the coefficient of lens space surgery.

We fix a combinatorial Euler structure (multiple of trivial unit $\pm \zeta_d^k$), deform both hand-sides of the equality (1) into real values by Lemma 4.5(2). If $M$ is homeomorphic to $L(p, q)$, we have a control equivalence between the real value sequence:

$$\{\tau^\psi(M)\}_{d|p, d\geq 2} = u\{\zeta_d^{i+1}(\zeta_d^i-1)^{-1}(\zeta_d^j-1)^{-1}\}_{d|p, d\geq 2},$$

where $u$ is a trivial unit $\pm 1$, or $\pm t^{p/2}$ (only in the case $p$ is even). By Lemma 4.8, we have, via a congruence $\mod (t^p-1)$, an identity between symmetric Laurent polynomials. We regard the identity as an equation (on $(i, j)$) of the coefficients of lens space surgery.

Step 4 By the equation, we have a necessary condition on the coefficient(s) of lens space surgery.

6. DEMONSTRATION

We call the link in Figure 1 Berge’s link $BL$. The compliment is a hyperbolic 3-manifold, known as Berge’s manifold in [1]. The component $K_1$ is the famous pretzel knot $P(-2, 3, 7)$. The link, regarded as a knot in a solid torus (the exterior of the component $K_2$), admits two surgery coefficients yielding solid torus itself, and it is proved that such a hyperbolic link is unique [1]. We demonstrate our method in Section 5 to Berge’s link,

![Figure 1. Berge link BL](image)

...to study lens space surgeries $M := (BL; r, 0)$, where $r = \alpha/\beta$ ($\alpha, \beta \in \mathbb{Z}, \gcd(\alpha, \beta) = 1$). We assume that $\beta \geq 1$.

(Step 1)

$$H_1(M) \cong \langle [m_1], [m_2] | [l_1] = [m_2]^7, [l_2] = [m_1]^7, [m_1]^\alpha[l_1]^\beta = 1, [m_1]^7 = 1 \rangle.$$ 

It is finite cyclic $\mathbb{Z}/p\mathbb{Z}$ if and only if $\gcd(\alpha, 7) = 1$, and then we have $p = 7^2\beta = 49\beta$. An element $T = [m_1]^\gamma [m_2]^\delta$ with $\alpha\delta' - 7\beta\gamma' = -1$ is a generator: $T^{49\beta} = 1$. We also have $[l_1'] = [m_1]^\gamma[l_1]^\delta$ with $\alpha\delta - \beta\gamma = -1$, and

$$[m_1] = T^{\gamma\beta}, \quad [m_2] = [l_2'] = T^{-\alpha}, \quad [l_1'] = T^7.$$
The Alexander polynomial of Berge’s link is
\[ \Delta_{BL}(t, x) = 1 + t^3x + t^5x^2 + t^8x^3 + t^{11}x^4 + t^{13}x^5 + t^{16}x^6 = \sum_{i=0}^{6} t^{s_i}x^i, \]
where we define a sequence \((s_0, s_1, \ldots, s_6) = (0, 3, 5, 8, 11, 13, 16)\). This is not periodic, but we regard it as “Periodicity is broken a little”. We let \(M_1 = E_{BL} \cup V_1 = (BL; \alpha/\beta, -)\).
We have, up to the ambiguity (multiplication \(\pm T^k\)),
\[ \tau(M_1) \doteq \Delta_{BL}(T^{7\beta}, T^{-\alpha})(T^7 - 1)^{-1} = \left(\sum_{i=0}^{6} T^{7\beta s_i - \alpha i}\right)(T^7 - 1)^{-1}. \]
We take a divisor \(d = 7\) of \(p = 49\beta\) and let \(\zeta\) denote a primitive 7-th root of unity. We use deformations
\[ T^{7\beta s_i - \alpha i} = T^{-\alpha i}(T^{7\beta s_i} - 1) + T^{-\alpha i}, \quad \frac{T^{7\beta s_i} - 1}{T^7 - 1} = 1 + T^7 + T^{14} + \cdots + T^{7(\beta s_i - 1)}. \]
For a ring homomorphism \(\psi\) satisfying \(\psi(T) = \zeta\) with \(\xi = \zeta^{-\alpha}\) (then \(\xi\) is still a primitive unity, since \(\gcd(\alpha, 7) = 1\)),
\[ \tau^\psi(M) \doteq \left\{ \beta(\xi^{2} + \xi^{-2}) - (\alpha - 19\beta) \right\}(\xi - 1)^{-2}. \]
In the 7-th cyclotomic field \(\mathbb{Q}(\zeta_7)\), using the equalities \(\xi^7 = 1\) and \(1 + \xi + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \xi^6 = 0\),
\[ (\xi - 1) \sum_{i=0}^{6} s_i \xi^i = -3\xi - 2\xi^2 - 3\xi^3 - 3\xi^4 - 2\xi^5 - 3\xi^6 + 16 \]
\[ = -3\xi - 2\xi^2 - 3\xi^3 - 3\xi^4 - 2\xi^5 - 3\xi^6 + 16 \]
\[ + 3(1 + \xi + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \xi^6) \]
\[ = 19 + \xi^2 + \xi^5 \]
\[ = 19 + \xi^2 + \xi^{-2}. \]
The Reidemeister–Turaev torsion of Dehn surgery \(M = (BL; \alpha/\beta, 0)\) is
\[ (2) \quad \tau^\psi(M) \doteq \left\{ \beta(\xi^2 + \xi^{-2}) - (\alpha - 19\beta) \right\}(\xi - 1)^{-2}. \]
Now, suppose that \(M\) is a lens space \(L(p, q)\) with \(p = 49\beta\) (by Step 1) and undecided \(q\). Then there exist integers \(i, j\) coprime to \(p\) with \(0 < i, j < p\) such that
\[ (3) \quad \tau^\psi(M) \doteq (\xi^i - 1)^{-1}(\xi^j - 1)^{-1}. \]
We can assume \(i + j\) is even. We treat with \(i, j\) mod 7 \((i, j \in \{1, 2, 3, 4, 5, 6\})\), since \(d = 7\).
(Step 3) Using Lemma 4.2 on \(d\)-norm with \(d = 7\) on (2) and (3), we have a necessary condition for the Dehn surgery \(M = (BL; \alpha/\beta, 0)\) to be a lens space:
\[ N_d \left( \beta(\xi^2 + \xi^{-2}) - (\alpha - 19\beta) \right) = 1. \]
Roughly, it means \(r = \alpha / \beta\) is near 19.
We set $\alpha' = \alpha - 19\beta$. By (2) and (3), we have
\[
\xi \{ \beta(t^2 + t^{-2}) - \alpha' \} (\xi - 1)^{-2} = \pm \xi^{(i+j)/2}(\xi^i-1)^{-1}(\xi^j-1)^{-1}.
\]
We regard it as an equality between real value sequence. Without loss of generality, we assume $0 < i < d/2$ (i.e., $i = 1, 2$ or $3$), $i \leq j$, and define $f = (i+j)/2, e = (j-i)/2$. The equality lifts as an identity of symmetric Laurent polynomial
\[
(\beta(t^2) - \alpha')(t^f) - (\xi^{i+j}/2)(\xi^{i}-1)^{-1}(\xi^{j}-1)^{-1}.
\]
in $\mathbb{Z}[t, t^{-1}]/(t^7-1)$, where $\langle t^i \rangle = t^i + t^{-i}$, $\langle t^0 \rangle = 2$, $\langle t^4 \rangle = \langle t^3 \rangle$, $\langle t^5 \rangle = \langle t^2 \rangle$ mod $(t^7-1)$.

We regard the identity (4) as an equation on $(f, e)$: It is a necessary condition on $(\alpha', \beta)$ for the equation to have a solution $(f, e)$. Since $f \neq e$ is obvious and $\langle t^4 \rangle = \langle t^3 \rangle = \langle t^2 \rangle$ mod $(t^7-1)$, we only have to consider six cases
\[(f, e) = (1,0), (2,0), (3,0), (2,1), (3,1), (3,2).
\]

Note that $\langle t^{-x} \rangle = \langle t^x \rangle$ and $\langle t^0 \rangle = 2$.

<table>
<thead>
<tr>
<th>$(f, e)$</th>
<th>$(\alpha', \beta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,0)</td>
<td>$\beta(t^2) - 2\beta(t^2) - (\alpha' - \beta)(t^1) + 2\alpha'$</td>
</tr>
<tr>
<td>(2,0)</td>
<td>$-\alpha'(t^3) - (\alpha' + 2\beta)(t^2) + 2(\alpha' + \beta)$</td>
</tr>
<tr>
<td>(3,0)</td>
<td>$-\alpha'(t^3) - (\alpha' + 2\beta)(t^2) + \beta(t^1) + 2\alpha'$</td>
</tr>
<tr>
<td>(2,1)</td>
<td>$-\alpha'(t^3) + (\alpha' - \beta)(t^1) + 2\beta$</td>
</tr>
<tr>
<td>(3,1)</td>
<td>$-(\alpha' + \beta)(t^3) + \beta(t^2) + \alpha'(t^1)$</td>
</tr>
<tr>
<td>(3,2)</td>
<td>$-(\alpha' + \beta)(t^3) + (\alpha' + \beta)(t^2) + \beta(t^1) - 2\beta$</td>
</tr>
</tbody>
</table>

Since $\alpha' = \alpha - 19\beta$, $(\alpha', \beta) = (0, 1)$ (and $(-1, 1)$, respectively) corresponds to $\alpha/\beta = 19$ (and 18). We have the required conclusion (pointed out in [1]):

Berge’s link $BL$ yields a lens space as $(BL; r, 0)$ only if $r = 19$ or $r = 18$.

Acknowledgement The authors would like to express their sincere gratitude to the organizers of the fruitful seminar “Twisted topological invariants and topology of low-dimensional manifolds”. The first author was supported by a grant (No.10801021/a010402) of NSFC. The second author was supported by KAKENHI (Grant-in-Aid for Scientific Research) No.21540072.

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