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JOHNSON HOMOMORPHISMS AS FIBERING OBSTRUCTIONS OF HOMOLOGICALLY FIBERED KNOTS

HIROSHI GODA AND TAKUYA SAKASAI

1. INTRODUCTION

Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus $g \geq 1$ with one boundary component. We denote its mapping class group by $\mathcal{M}_{g,1}$. It is the group of all isotopy classes of diffeomorphisms of $\Sigma_{g,1}$ which fix the boundary pointwise. The action of $\mathcal{M}_{g,1}$ on $H_1(\Sigma_{g,1}) \cong \mathbb{Z}^{2g}$ gives a representation

$$\sigma_2 : \mathcal{M}_{g,1} \longrightarrow \mathrm{Sp}(2g;\mathbb{Z}),$$

which is the first step to investigate the structure of $\mathcal{M}_{g,1}$. The kernel $\mathcal{I}_{g,1}$ of this representation is called the Torelli group. In his study of $\mathcal{I}_{g,1}$, Johnson [13] defined a homomorphism

$$\tau_1 : \mathcal{I}_{g,1} \longrightarrow \wedge^3 H_1(\Sigma_{g,1})$$

and proved that it is surjective. Furthermore, Johnson [14] and Morita [18] generalized it to a series of homomorphisms $\{\tau_k\}_{k\geq 1}$ such that τ_k is defined on the kernel of τ_{k-1}, say $\mathcal{M}_{g,1}[k+1]$, and the target of τ_k is a finitely generated free abelian group for each k. Morita [19] introduced a submodule $\mathfrak{h}_{g,1}(k)$ of the target and showed that the image of τ_k is included in $\mathfrak{h}_{g,1}(k)$. The homomorphism

$$\tau_k : \mathcal{M}_{g,1}[k+1] \longrightarrow \mathfrak{h}_{g,1}(k)$$

is now called the *k-th Johnson homomorphism*. For $k \geq 2$, it is known that τ_k is *not* surjective [18, 19]. In the study of the mapping class group, it has been an important problem to determine the cokernel $\mathfrak{h}_{g,1}(k)/\mathrm{Image}\,\tau_k$ and its topological meaning.

On the other hand, the monoid $\mathcal{C}_{g,1}$ of homology cylinders is known to be an enlargement of $\mathcal{M}_{g,1}$. Garoufalidis-Levine [6] extended τ_k to $\mathcal{C}_{g,1}$ together with its filtration $\{\mathcal{C}_{g,1}[k]\}_{k\geq 1}$ and showed that the extended Johnson homomorphism

$$\widetilde{\tau}_k : \mathcal{C}_{g,1}[k+1] \longrightarrow \mathfrak{h}_{g,1}(k)$$

is surjective. For the detail, see Section 2.

In our previous papers [7, 8], we defined a class of knots called *homologically fibered knots*, in which fibered knots are included. This extension corresponds to that of $\mathcal{M}_{g,1}$ to $\mathcal{C}_{g,1}$. In fact, the complement of a homologically fibered knot includes a homology cylinder as a complementary sutured manifold, while the complement of a fibered knot includes a product sutured manifold.

In this paper, we use the cokernels of Johnson homomorphisms as fibering obstructions of homologically fibered knots. More precisely, we confirm that there exist totally 13 non-fibered homologically fibered knots with 12 crossings, where this fact was first shown by Friedl-Kim [4], by computing $\widetilde{\tau}_2$ in the setting mentioned in Section 3.

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2. Johnson homomorphisms and Homology cylinders

Take a basepoint p of $\Sigma_{g,1}$ on the boundary. The fundamental group $\pi_1(\Sigma_{g,1})$ of the surface $\Sigma_{g,1}$ is a free group of rank $2g$. We take a basis $\langle \gamma_1, \gamma_2, \ldots, \gamma_{2g}\rangle$ of $\pi_1(\Sigma_{g,1})$ as in Figure 1.

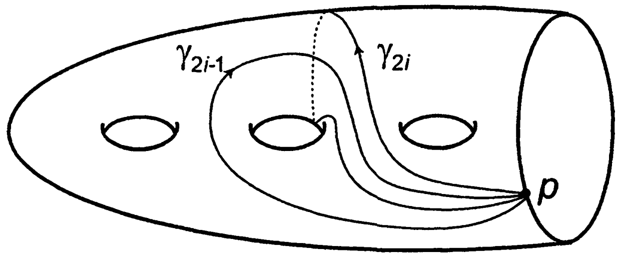


FIGURE 1. A basis of $\pi_1(\Sigma_{g,1})$

For a group G, the lower central series of G is defined by $\Gamma^1(G) := G$ and $\Gamma^k(G) = [G, \Gamma^{k-1}(G)]$ for $k \geq 2$. Here we use the notation $[a,b] := aba^{-1}b^{-1}$. For simplicity, we put $\Gamma^i = \Gamma^i(\pi_1(\Sigma_{g,1}))$.

The mapping class group $\mathcal{M}_{g,1}$ acts naturally on $\Gamma^1 = \pi_1(\Sigma_{g,1})$. By a theorem of Dehn-Nielsen, the induced representation $\mathcal{M}_{g,1} \to \operatorname{Aut}\Gamma^1$ is injective. Through this embedding, $\mathcal{M}_{g,1}$ acts on the k-th nilpotent quotient $N_k := \Gamma^1/\Gamma^k$ of Γ^1 and we have a representation

$$\sigma_k : \mathcal{M}_{g,1} \longrightarrow \operatorname{Aut} N_k \qquad (k = 2, 3, \ldots).$$

When $k = 2$, we have $N_2 = H_1(\Sigma_{g,1}) \cong \mathbb{Z}^{2g}$, so that $\sigma_2 : \mathcal{M}_{g,1} \to \mathrm{GL}(2g, \mathbb{Z})$. It is known that $\operatorname{Image}\sigma_2 = \mathrm{Sp}(2g, \mathbb{Z})$. These representations yield a filtration of $\mathcal{M}_{g,1}$ defined by $\mathcal{M}_{g,1}[1] = \mathcal{M}_{g,1}$ and $\mathcal{M}_{g,1}[k] := \operatorname{Ker}\sigma_k$ for $k \geq 2$. By definition, $\mathcal{M}_{g,1}[2]$ is the Torelli group $\mathcal{I}_{g,1}$.

Let us recall the definition of Johnson homomorphisms. We simply write H for $H_1(\Sigma_{g,1})$. Andreadakis [1] showed that there exists an exact sequence

$$1 \longrightarrow \operatorname{Hom}(H, \mathcal{L}_k) \longrightarrow \operatorname{Aut} N_{k+1} \longrightarrow \operatorname{Aut} N_k \longrightarrow 1,$$

where $\mathcal{L}_k$ is the degree k part of the free Lie algebra generated by H. Therefore if we restrict σ_{k+2} to $\mathcal{M}_{g,1}[k+1]$, we obtain a homomorphism

$$\tau_k := \sigma_{k+2}|_{\mathcal{M}_{g,1}[k+1]} : \mathcal{M}_{g,1}[k+1] \longrightarrow \operatorname{Hom}(H, \mathcal{L}_{k+1}) \qquad (k = 1, 2, \ldots).$$

More specifically, the homomorphism τ_k is given as follows. Let $f \in \mathcal{M}_{g,1}[k+1]$. We write f_m for the automorphism of N_m induced by f. Since $f_{k+1} = \mathrm{id}$ by definition, we have $f_{k+2}(x)x^{-1} \in \Gamma^{k+1}/\Gamma^{k+2}$ for each $x \in N_{k+2}$. It is known that $\Gamma^{k+1}/\Gamma^{k+2}$ is naturally isomorphic to $\mathcal{L}_{k+1}$. Hence we can define a map $\varphi_f : N_{k+2} \to \mathcal{L}_{k+1}$ by $\varphi_f(x) = f_{k+2}(x)x^{-1}$. By the centrality of $\mathcal{L}_{k+1}$, we can see that φ_f is a homomorphism. Since $\mathcal{L}_{k+1}$ is abelian, φ_f induces a homomorphism $\overline{\varphi}_f : H \to \mathcal{L}_{k+1}$. Then we define $\tau_k : \mathcal{M}_{g,1}[k+1] \to \operatorname{Hom}(H, \mathcal{L}_{k+1}) \cong H^* \otimes \mathcal{L}_{k+1} \cong H \otimes \mathcal{L}_{k+1}$ by $\tau_k(f) = \overline{\varphi}_f$, where we use the natural isomorphism $H \cong H^*$ by Poincaré duality (we identify $\gamma_{2i-1} \in H$ with $-\gamma_{2i}^* \in H^*$ and $\gamma_{2i} \in H$ with $\gamma_{2i-1}^* \in H^*$). The map τ_k becomes a homomorphism and we call it the *k-th Johnson homomorphism*. Morita studied this homomorphism in [19] and proved the following.

Theorem 2.1 (Morita [19, Corollary 3.2]). *Let $\mathfrak{h}_{g,1}(k)$ be the kernel of the bracket map $H \otimes \mathcal{L}_{k+1} \to \mathcal{L}_{k+2}$ given by $(w, \xi) \mapsto [w, \xi]$ for $w \in H$ and $\xi \in \mathcal{L}_{k+1}$. Then the image of $\tau_k : \mathcal{M}_{g,1}[k+1] \to H \otimes \mathcal{L}_{k+1}$ is included in $\mathfrak{h}_{g,1}(k)$, so that we may write $\tau_k : \mathcal{M}_{g,1}[k+1] \to \mathfrak{h}_{g,1}(k)$.*

Johnson [13] showed that $\tau_1 : \mathcal{M}_{g,1}[2] \to \mathfrak{h}_{g,1}(1) \cong \wedge^3 H$ is surjective, where $\mathfrak{h}_{g,1}(1) \cong \wedge^3 H$ is embedded in $H \otimes \mathcal{L}_2 = H \otimes \wedge^2 H$ by

$$x \wedge y \wedge z \longmapsto x \otimes (y \wedge z) + y \otimes (z \wedge x) + z \wedge (x \wedge y)$$

for $x, y, z \in H$. However, it is known that $\tau_k : \mathcal{M}_{g,1}[k+1] \to \mathfrak{h}_{g,1}(k)$ is not surjective for $k \geq 2$. (see Morita [18, 19]). In Section 4, we will discuss more about the homomorphism τ_2.

Next, we recall the definition of homology cylinders. We refer to Goussarov [9], Habiro [11], Garoufalidis-Levine [6] and Levine [15] for their origin.

Definition 2.2. A *homology cylinder* (M, i_+, i_-) *over* $\Sigma_{g,1}$ consists of a compact oriented 3-manifold M with two embeddings $i_+, i_- : \Sigma_{g,1} \hookrightarrow \partial M$ such that:

(i) i_+ is orientation-preserving and i_- is orientation-reversing;
(ii) $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})$ and $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$;
(iii) $i_+|_{\partial\Sigma_{g,1}} = i_-|_{\partial\Sigma_{g,1}}$; and
(iv) $i_+, i_- : H_*(\Sigma_{g,1}) \to H_*(M)$ are isomorphisms.

Two homology cylinders (M, i_+, i_-) and (N, j_+, j_-) over $\Sigma_{g,1}$ are said to be *isomorphic* if there exists an orientation-preserving diffeomorphism $f : M \xrightarrow{\cong} N$ satisfying $j_+ = f \circ i_+$ and $j_- = f \circ i_-$. We denote by $\mathcal{C}_{g,1}$ the set of all isomorphism classes of homology cylinders over $\Sigma_{g,1}$. We define a product operation on $\mathcal{C}_{g,1}$ by

$$(M, i_+, i_-) \cdot (N, j_+, j_-) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-)$$

for (M, i_+, i_-), $(N, j_+, j_-) \in \mathcal{C}_{g,1}$, so that $\mathcal{C}_{g,1}$ becomes a monoid. The unit is given by $(\Sigma_{g,1} \times [0,1], \mathrm{id} \times 1, \mathrm{id} \times 0)$, where collars of $i_+(\Sigma_{g,1}) = (\mathrm{id} \times 1)(\Sigma_{g,1})$ and $i_-(\Sigma_{g,1}) = (\mathrm{id} \times 0)(\Sigma_{g,1})$ are stretched half-way along $(\partial\Sigma_{g,1}) \times [0,1]$ so that $i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$.

Example 2.3. The mapping class group $\mathcal{M}_{g,1}$ can be embedded in $\mathcal{C}_{g,1}$ by assigning to $[\varphi] \in \mathcal{M}_{g,1}$ a homology cylinder

$$(\Sigma_{g,1} \times [0,1], \mathrm{id} \times 1, \varphi \times 0)$$

with the same treatment of the boundary as above.

Johnson homomorphisms were extended by Garoufalidis-Levine [6] as follows. Given $(M, i_+, i_-) \in \mathcal{C}_{g,1}$, we consider the homomorphisms $i_{\pm,*} : \pi_1(\Sigma_{g,1}) \to \pi_1(M)$, where we share a basepoint taken on $\partial i_+(\Sigma_{g,1}) = \partial i_-(\Sigma_{g,1})$. Since $i_\pm$ induce homology isomorphisms, it follows from Stallings [21] that they induce isomorphisms $i_{\pm,k} : N_k \to \pi_1(M)/\Gamma^k(\pi_1(M))$. Define a map $\widetilde{\sigma}_k : \mathcal{C}_{g,1} \to \mathrm{Aut}\, N_k$ by $\widetilde{\sigma}_k(M, i_+, i_-) = (i_{+,k})^{-1} \circ i_{-,k} \in \mathrm{Aut}\, N_k$ and it turns out to be a homomorphism. The restriction of $\widetilde{\sigma}_k$ to $\mathcal{M}_{g,1}$ coincides with the homomorphism σ_k defined before. When $k = 2$, we can check that the image of $\sigma_2 : \mathcal{C}_{g,1} \to \mathrm{Aut}\, N_2 \cong GL(2g, \mathbb{Z})$ is $\mathrm{Sp}(2g, \mathbb{Z})$, the same as σ_2.

Set $\mathcal{C}_{g,1}[1] = \mathcal{C}_{g,1}$ and $\mathcal{C}_{g,1}[k] = \operatorname{Ker} \widetilde{\sigma}_k$ for $k \geq 2$. By the same construction as $\mathcal{M}_{g,1}$, we obtain a homomorphism

$$\widetilde{\tau}_k : \mathcal{C}_{g,1}[k+1] \longrightarrow H \otimes \mathcal{L}_{k+1},$$

which extends τ_k. It can be shown that $\operatorname{Image} \widetilde{\tau}_k \subset \mathfrak{h}_{g,1}(k)$. Moreover the following holds:

Theorem 2.4 (Garoufalidis-Levine [6, Proposition 2.5], Habegger [10]). *For any $k \geq 2$,*

$$\operatorname{Image} \widetilde{\tau}_k = \mathfrak{h}_{g,1}(k).$$

Consequently, we see that the cokernel $\mathfrak{h}_{g,1}(k)/\operatorname{Image} \tau_k$ for $k \geq 2$ is a product obstruction for homology cylinders. Note that the first Johnson homomorphism $\widetilde{\tau}_1 : \mathcal{C}_{g,1}[2] \to \mathfrak{h}_{g,1}(1) \cong \wedge^3 H$ has the same image as τ_1.

3. Homologically fibered knots

Here we recall the definition of homologically fibered knots, which enable us to encode the theory of homology cylinders to knot theory.

For a knot K in S^3 and a Seifert surface $\bar{R}$ of K, we set $R := \bar{R} \cap E(K)$, where $E(K) = \overline{S^3 - N(K)}$ is the complement of a regular neighborhood $N(K)$ of K. Then $(M_R, \gamma) := (\overline{E(K) - N(R)}, \overline{\partial E(K) - N(\partial R)})$ defines a sutured manifold [5]. We call it the *complementary sutured manifold* for R. The boundary of M_R is divided into two parts along K, so that we may regard M_R as a cobordism between two copies of R.

Definition 3.1 ([7]). A knot K in S^3 is called a *homologically fibered knot* if it has the following properties which are equivalent to each other:

(a) The Alexander polynomial $\Delta_K(t)$ of K is monic (i.e. the leading coefficient is ± 1) and its degree is equal to twice the genus $g = g(K)$ of K;
(b) For any minimal genus Seifert surface R of K, its Seifert matrix is invertible over $\mathbb{Z}$; and
(c) The complementary sutured manifold (M_R, γ) is a homology cobordism over R.

Therefore, if we fix an identification $i : \Sigma_{g,1} \xrightarrow{\cong} R$ of $\Sigma_{g,1}$ with a minimal genus Seifert surface R of a homogically fibered knot, we obtain a homology cylinder (M_R, i_+, i_-). It is well known that fibered knots satisfy the above conditions. They define homology cylinders with the product cobordism $\Sigma_{g,1} \times [0,1]$. The following proposition is an analogue of the well-known fact that a fibered knot determines a mapping class, called the *monodromy*, uniquely up to conjugation.

Proposition 3.2 ([8]). *Let R_1 and R_2 be (maybe parallel) minimal genus Seifert surfaces of a homologically fibered knot of genus g. For any identification i and j of $\Sigma_{g,1}$ with R_1 and R_2, there exists another homology cylinder $N \in \mathcal{C}_{g,1}$ such that*

$$(M_{R_1}, i_+, i_-) \cdot N = N \cdot (M_{R_2}, j_+, j_-)$$

holds as elements of $\mathcal{C}_{g,1}$.

Now let us discuss how we can apply Johnson homomorphisms to homologically fibered knots. We here focus on (non-)fiberedness of homologically fibered knots. Let K be a homologically fibered knot of genus g with a minimal genus Seifert surface R. We fix an identification $i : \Sigma_{g,1} \xrightarrow{\cong} R$, so that we obtain a homology cylinder $M_R = (M_R, i_+, i_-) \in \mathcal{C}_{g,1}$. If K is a fibered knot, then there exists a mapping class $[\varphi] \in \mathcal{M}_{g,1}$ such that

$M_R = [\varphi] \in \mathcal{M}_{g,1}$. In particular, $M_R \cdot [\varphi]^{-1}$ is in the kernel of all $\widetilde{\tau}_k$. In the case where K is not fibered, there *may* exist some $k \geq 2$ such that no mapping classes $[\psi] \in \mathcal{M}_{g,1}$ satisfy $M_R \cdot [\psi]^{-1} \in \operatorname{Ker} \widetilde{\tau}_k$.

Remark 3.3. Since $\operatorname{Image} \widetilde{\sigma}_2 = \operatorname{Image} \sigma_2 = \mathrm{Sp}(2g, \mathbb{Z})$ and $\operatorname{Image} \widetilde{\tau}_1 = \operatorname{Image} \tau_1 = \wedge^3 H$ as mentioned in Section 2, there always exists a mapping class $[\varphi] \in \mathcal{M}_{g,1}$ for a homology cylinder $M \in \mathcal{C}_{g,1}$ such that $M \cdot [\varphi]^{-1} \in \mathcal{C}_{g,1}[3]$.

Consequently we may regard the cokernels of the Johnson homomorphisms $\widetilde{\tau}_k$ $(k \geq 2)$ as *step-by-step fibering obstructions* for homologically fibered knots.

In the following sections, we exhibit some computations. Recall that all homologically fibered knots are fibered among prime knots with ≤ 11 crossings. On the other hand, it was first shown by Friedl-Kim [4] that there are 13 non-fibered homologically fibered knots with 12 crossings. They are $12_n P$ with

$$P = 0057, 0210, 0214, 0258, 0279, 0382, 0394, 0464, 0483, 0535, 0650, 0801, 0815,$$

where we follow the notation of [3]. The knots $12_n 0210$ and $12_n 0214$ are of genus 3, and the others are of genus 2. We can find several data of the homology cylinders corresponding to the above 13 knots in [8]. We will see that non-fiberedness of all the 13 knots can be detected by $\widetilde{\tau}_2$.

4. The second Johnson homomorphism

In this section, we recall a useful graphical description of the module $\mathfrak{h}_{g,1}(k)$ due to Levine and use it to describe the cokernel of $\tau_2 : \mathcal{M}_{g,1}[3] \to \mathfrak{h}_{g,1}(2)$.

Let $\mathcal{A}_k^t$ be the abelian group generated by unitrivalent trees with

- $k+2$ univalent vertices labeled by elements of H,
- a cyclic order of edges around each trivalent vertex

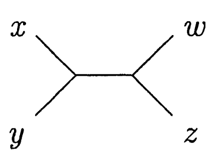
modulo AS-, IHX-relations and linearity of labels. We define a map $\eta_k : \mathcal{A}_k^t \to H \otimes \mathcal{L}_{k+1}$ by

$$\eta_k(T) := \sum_v c_v \otimes T_v$$

for a labeled tree T and extend it linearly to the whole of $\mathcal{A}_k^t$, where the sum is taken over all univalent vertices of T, and for each univalent vertex v, c_v denotes the label of v and T_v denotes the rooted labeled planar binary tree obtained from T by removing the label c_v and considering v to be an unlabeled root, which can be regarded as an element of $\mathcal{L}_{k+1}$ by a standard method. It is easily checked that $\operatorname{Image} \eta_k \subset \mathfrak{h}_{g,1}(k)$. Moreover $\eta_k \otimes \mathbb{Q} : \mathcal{A}_k^t \otimes \mathbb{Q} \to \mathfrak{h}_{g,1}(k) \otimes \mathbb{Q}$ is an isomorphism (see Garoufalidis-Levine [6] and Levine [15] for example). To obtain a graphical description of $\mathfrak{h}_{g,1}(k)$ as a $\mathbb{Z}$-module, we need more information. Levine [17] gave a complete description of $\mathfrak{h}_{g,1}(2)$ and a number of observations for higher degrees. Here we only recall his description of $\mathfrak{h}_{g,1}(2)$.

The module $\mathcal{A}_2^t$ is generated by graphs of the form

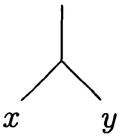
$T(x, y, z, w) :=$



with $x, y, z, w \in H$. We have

$$\eta_2(T(x,y,z,w)) = x \otimes [y,[z,w]] + y \otimes [[z,w],x] + z \otimes [w,[x,y]] + w \otimes [[x,y],z].$$

Let $\widetilde{\mathcal{A}}_2^t$ be the module obtained from $\mathcal{A}_2^t$ by adding generators

$Y(x,y) :=$ 

with $x, y \in H$ and relations

$$Y(x,y) = -Y(y,x), \qquad T(x,y,x,y) = 2Y(x,y).$$

We can extend the homomorphism η_2 to $\widetilde{\eta}_2 : \widetilde{\mathcal{A}}_2^t \to \mathfrak{h}_{g,1}(2)$ by setting

$$\widetilde{\eta}_2(Y(x,y)) = x \otimes [y,[x,y]] + y \otimes [x,[y,x]].$$

Levine [17, Section 2] showed that $\widetilde{\eta}_2$ is an isomorphism. Hereafter we identify $\widetilde{\mathcal{A}}_2^t$ with $\mathfrak{h}_{g,1}(2)$ by $\widetilde{\eta}_2$.

Example 4.1. Let $T_c \in \mathcal{M}_{g,1}[3]$ be the (right-handed) Dehn-twist map along the bounding simple closed curve c of genus h as depicted in Figure 2.

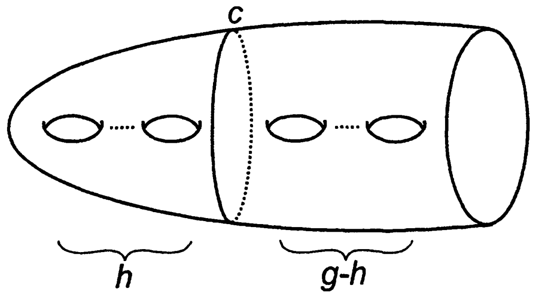


FIGURE 2. Bounding simple closed curve c of genus h

Then $\tau_2(T_c)$, which was originally computed by Morita [18], is given in terms of graphs by

$$\tau_2(T_c) = -\sum_{1 \leq i < j \leq h} T(\gamma_{2i-1}, \gamma_{2i}, \gamma_{2j-1}, \gamma_{2j}) - \sum_{k=1}^{h} Y(\gamma_{2k-1}, \gamma_{2k}).$$

By using the above computational result, Morita showed that the cokernel $\mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2$ is a 2-torsion group. After that, Yokomizo determined the cokernel as follows.

Theorem 4.2 (Yokomizo [22]). *The cokernel $\mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2$ is a $(g-1)(2g+1)$-dimensional $(\mathbb{Z}/2\mathbb{Z})$-vector space and a basis is given by*

$$Y(\gamma_{2i-1}, \gamma_{2j-1}), \quad Y(\gamma_{2i-1}, \gamma_{2j}), \quad Y(\gamma_{2i}, \gamma_{2j-1}), \quad Y(\gamma_{2i}, \gamma_{2j}) \quad (1 \leq i < j \leq g),$$
$$T(\gamma_{2k-1}, \gamma_{2k+1}, \gamma_{2k}, \gamma_{2k+2}) \quad (1 \leq k \leq g-1).$$

Remark 4.3. (1) Yokomizo also gave an explicit description of $\mathrm{Image}\,\tau_2$, which we omit here, in the same paper.

(2) In the above cited papers of Morita and Yokomizo, they use a group $\overline{T} \subset H \otimes \mathcal{L}_3$ as a target of τ_2. We can check that $\overline{T} = \mathfrak{h}_{g,1}(2)$.

5. Recipe for computation

The following is the recipe of our computation for 13 non-fibered homologically fibered knots with 12 crossings. We also give some technical comments.

Let K be the knot 12_nP with $P = 0057, 0210, \ldots, 0815$. It is known that K has a unique minimal genus Seifert surface R_P. In [8], we fixed an identification $i : \Sigma_{g,1} \xrightarrow{\cong} R_P$ and gave a presentation (called an *admissible* presentation) for $\pi_1(M_{R_P})$ of the homology cylinder $M_P := (M_{R_P}, i_+, i_-) \in \mathcal{C}_{g,1}$. Our computation starts from here.

1. Compute $\widetilde{\sigma}_4(M_P) \in \operatorname{Aut} N_4$. Recall that an element of N_4 is written in a normal form using the Hall basis (see Sims [20] for example). We associate a normal form with variables to each generator of the admissible presentation and substitute it to the relations of the presentation, which yields an algebraic equation. Stallings' theorem says that this equation has a unique solution, from which $\widetilde{\sigma}_4(M_P)$ is obtained.
2. Find a mapping class $f \in \mathcal{M}_{g,1}$ such that $M_P \cdot f \in \mathcal{C}_{g,1}[3]$. For that, we first find a mapping class $f_1 \in \mathcal{M}_{g,1}$ such that $M_P \cdot f_1 \in \mathcal{C}_{g,1}[2]$, which is done by finding a lift of $\widetilde{\sigma}_2(M_P) \in \mathrm{Sp}(2g, \mathbb{Z})$ to $\mathcal{M}_{g,1}$. We can use arguments of Birman [2] and Hua-Reiner [12] to find such a lift. Next we find a mapping class $f_2 \in \mathcal{I}_{g,1}$ such that $M_P \cdot f_1 \cdot f_2 \in \mathcal{C}_{g,1}[3]$, which is not difficult (use Johnson's computation in [13]). Put $f = f_1 \cdot f_2$.
3. Compute $\widetilde{\tau}_2(M_P \cdot f) \in \mathfrak{h}_{g,1}(2)$ and project it onto $\mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2$. While the resulting value itself depends on a choice of f, whether the value is projected trivially on $\mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2$ or not is independent of the choice. If this value is non-trivial, we can conclude that K is not fibered.

6. Computational results

Here we exhibit our computational results, following the recipe in Section 5. For each knot we are considering, we give an admissible presentation of $\pi_1(M_P)$, the action $\widetilde{\sigma}_4(M_P) \in \operatorname{Aut} N_4$ of M_P on N_4 and an example of a pair $f_1 \in \mathcal{M}_{g,1}$ and $f_2 \in \mathcal{I}_{g,1}$ such that $M_P \cdot f_1 \in \mathcal{C}_{g,1}[2]$ and $M_P \cdot f_1 \cdot f_2 \in \mathcal{C}_{g,1}[3]$. Then we give the value of $\widetilde{\tau}_2(M_P \cdot f_1 \cdot f_2)$ in $\mathfrak{h}_{g,1}(2)$ and project it on $\mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2$.

Let T_i $(1 \leq i \leq 8)$ and S_j $(1 \leq j \leq 16)$ denote the Dehn twist maps along the simple closed curves c_i and d_j in Figure 3 and 4 respectively, where we regard $\Sigma_{2,1}$ as a subsurface of $\Sigma_{3,1}$. The mapping classes E_{12} and E_{23} exchange the handles by rotating clockwise as in Figure 5. We also use a mapping class U whose action on $\pi_1(\Sigma_{3,1})$ is given by

$$U: \begin{array}{lll} \gamma_1 \longmapsto \gamma_1\gamma_5\gamma_1^{-1}, & \gamma_2 \longmapsto \gamma_1\gamma_6\gamma_1^{-1}, & \gamma_3 \longmapsto \gamma_1[\gamma_6,\gamma_5]\gamma_3[\gamma_5,\gamma_6]\gamma_1^{-1}, \\ \gamma_4 \longmapsto \gamma_1[\gamma_6,\gamma_5]\gamma_4[\gamma_5,\gamma_6]\gamma_1^{-1}, & \gamma_5 \longmapsto \gamma_1, & \gamma_6 \longmapsto [\gamma_6,\gamma_5][\gamma_4,\gamma_3]\gamma_2. \end{array}$$

Figure 6 defines the mapping classes I_{123}, I_{124}, I_{134} and I_{234} in $\mathcal{I}_{2,1}$, where $+$ in the figure means a positive (right-handed) Dehn twist and $-$ means a negative one. I_{ijk} is chosen so that it satisfies $\tau_1(I_{ijk}) = \gamma_i \wedge \gamma_j \wedge \gamma_k \in \Lambda^3 H$. Similarly, we use $I_{125}, I_{246}, \ldots \in \mathcal{I}_{3,1}$ in the case when $P = 0210, 0214$, although the precise definition is omitted. We can easily calculate the value $\widetilde{\tau}_1(M_P \cdot f_1)$ from $f_2 \in \mathcal{I}_{g,1}$.

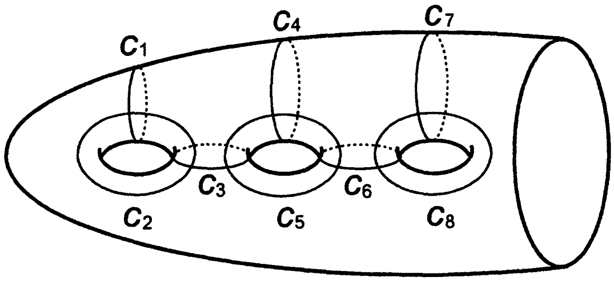


FIGURE 3. Simple closed curves c_i

12_n0057

An admissible presentation of $\pi_1(M_{0258})$:

$$
\begin{gathered}
z_1z_5z_6^{-1}, \quad z_1z_2z_3z_4, \quad z_3z_9^{-1}z_5^{-1}, \quad z_7z_4z_8^{-1}, \quad z_8z_{10}z_6, \quad z_2z_5z_7^{-1}z_5^{-1},\\
i_-(\gamma_1)z_1^{-1}z_5^{-1}, \quad i_-(\gamma_2)z_2, \quad i_-(\gamma_3)z_4z_8z_7z_5^{-1}, \quad i_-(\gamma_4)z_4,\\
i_+(\gamma_1)z_5^{-1}, \quad i_+(\gamma_2)z_9^{-1}z_6^{-1}, \quad i_+(\gamma_3)z_6z_4z_7z_5^{-1}z_3^{-1}z_5z_6^{-1}, \quad i_+(\gamma_4)z_1z_2^{-1}z_1^{-1}.
\end{gathered}
$$

The action of M_{0057} on N_4:

$$
\begin{aligned}
\gamma_1 \mapsto\ & [\gamma_4,[\gamma_3,\gamma_4]]^2[\gamma_4,[\gamma_2,\gamma_4]]^{-4}[\gamma_4,[\gamma_1,\gamma_4]]^{-2}[\gamma_3,[\gamma_3,\gamma_4]][\gamma_3,[\gamma_2,\gamma_4]]^{-2}[\gamma_3,[\gamma_1,\gamma_4]]^{-1}\\
& [\gamma_3,[\gamma_1,\gamma_3]][\gamma_2,[\gamma_3,\gamma_4]][\gamma_2,[\gamma_2,\gamma_4]]^{-1}[\gamma_2,[\gamma_1,\gamma_3]]^{-3}[\gamma_2,[\gamma_1,\gamma_2]]^2[\gamma_1,[\gamma_3,\gamma_4]]^{-1}\\
& [\gamma_1,[\gamma_2,\gamma_4]]^3[\gamma_1,[\gamma_2,\gamma_3]]^3[\gamma_1,[\gamma_1,\gamma_4]][\gamma_1,[\gamma_1,\gamma_2]]\\
& [\gamma_3,\gamma_4]^{-1}[\gamma_2,\gamma_4]^2[\gamma_1,\gamma_4][\gamma_1,\gamma_3]^{-1}[\gamma_1,\gamma_2]^3\gamma_3^{-1}\gamma_2^2\gamma_1,
\end{aligned}
$$

$$
\begin{aligned}
\gamma_2 \mapsto\ & [\gamma_4,[\gamma_3,\gamma_4]]^2[\gamma_4,[\gamma_2,\gamma_4]]^{-4}[\gamma_4,[\gamma_1,\gamma_4]]^{-1}[\gamma_2,[\gamma_3,\gamma_4]]^2\\
& [\gamma_2,[\gamma_2,\gamma_4]]^{-3}[\gamma_2,[\gamma_1,\gamma_4]]^{-1}[\gamma_1,[\gamma_2,\gamma_4]][\gamma_3,\gamma_4]^{-1}[\gamma_2,\gamma_4]^2\gamma_4^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\gamma_3 \mapsto\ & [\gamma_4,[\gamma_3,\gamma_4]]^4[\gamma_4,[\gamma_2,\gamma_4]]^{-8}[\gamma_4,[\gamma_1,\gamma_4]]^{-4}[\gamma_2,[\gamma_3,\gamma_4]]^3[\gamma_2,[\gamma_2,\gamma_4]]^{-4}[\gamma_2,[\gamma_1,\gamma_4]]^{-1}\\
& [\gamma_2,[\gamma_1,\gamma_2]][\gamma_1,[\gamma_2,\gamma_4]]^2[\gamma_3,\gamma_4]^{-2}[\gamma_2,\gamma_4]^4[\gamma_1,\gamma_4]^2[\gamma_1,\gamma_2]^2\gamma_2^2\gamma_1,
\end{aligned}
$$

$$
\gamma_4 \mapsto \gamma_4\gamma_2.
$$

$$
\begin{aligned}
f_1 &= T_4 \cdot T_2 \cdot S_1^{-1} \cdot T_5^{-1} \cdot S_2 \cdot T_1\\
f_2 &= I_{234} \cdot I_{124}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\tau}_2(M_{0057} \cdot f_1 \cdot f_2) = & -T(\gamma_1,\gamma_4,\gamma_3,\gamma_4) + T(\gamma_1,\gamma_4,\gamma_2,\gamma_4) + T(\gamma_1,\gamma_2,\gamma_2,\gamma_4)\\
& + Y(\gamma_1,\gamma_2) + Y(\gamma_1,\gamma_4)\\
\mapsto\ & Y(\gamma_2,\gamma_4) \neq 0 \in \mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2.
\end{aligned}
$$

12_n0258

An admissible presentation of $\pi_1(M_{0258})$:

$$
\begin{gathered}
z_1z_2z_3z_4, \quad z_1z_2z_4z_6^{-1}z_7^{-1}, \quad z_7z_6z_5,\\
i_-(\gamma_1)z_7z_6z_7^{-1}, \quad i_-(\gamma_2)z_7z_6z_5^{-1}z_4z_6^{-1}z_7^{-1}, \quad i_-(\gamma_3)z_1z_2^2z_4^2z_6^{-1}z_7^{-1}, \quad i_-(\gamma_4)z_1z_2^2z_1^{-2},\\
i_+(\gamma_1)z_7^{-1}, \quad i_+(\gamma_2)z_6z_4, \quad i_+(\gamma_3)z_2z_1^{-1}z_4, \quad i_+(\gamma_4)z_2z_1^{-2}.
\end{gathered}
$$

The action of M_{0258} on N_4:

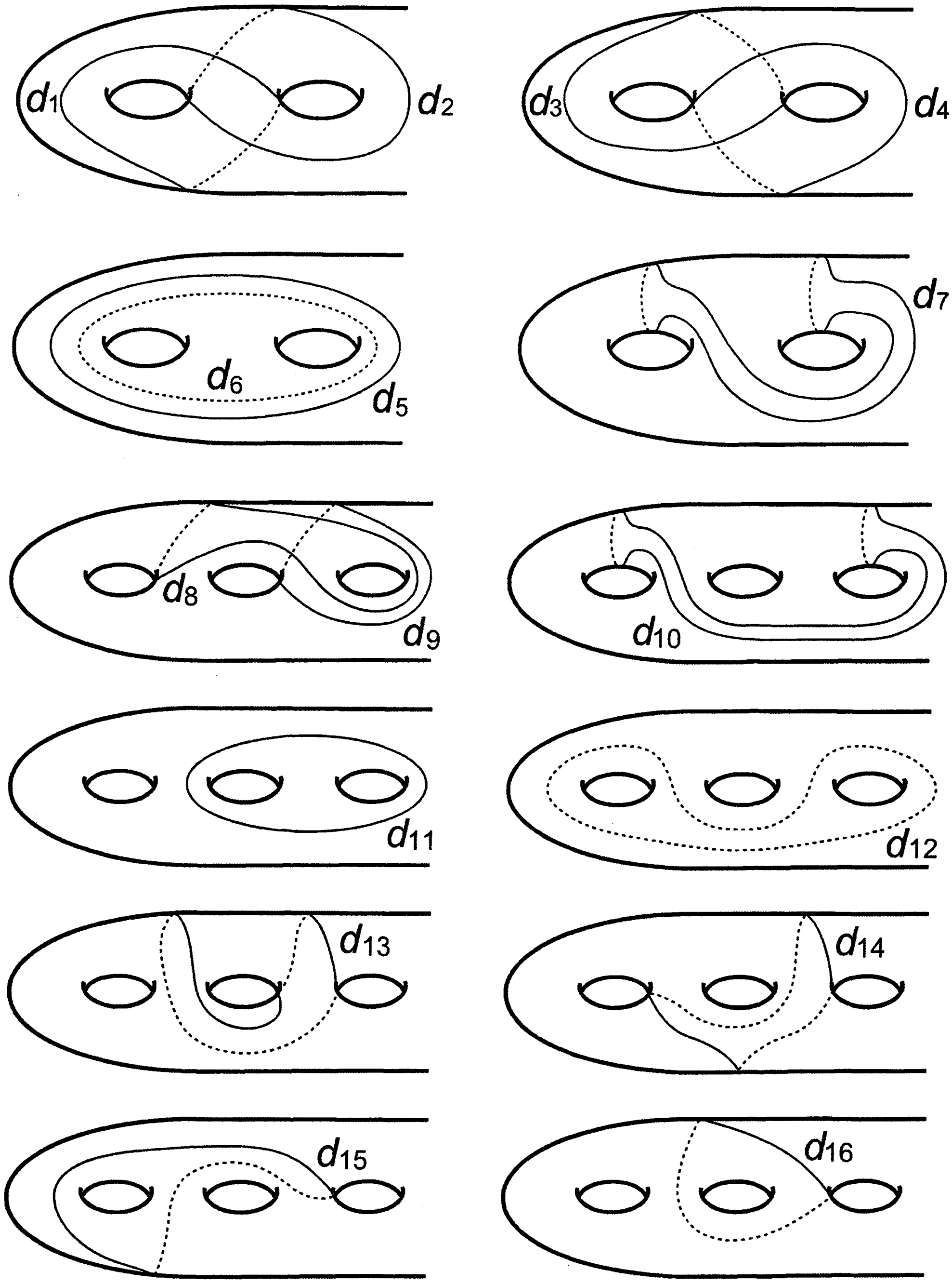


FIGURE 4. Simple closed curves d_j

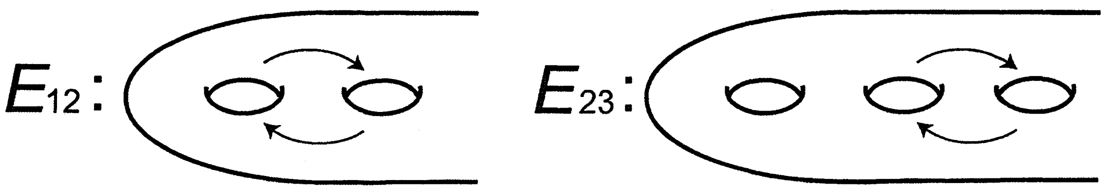


FIGURE 5. E_{12} and E_{23}

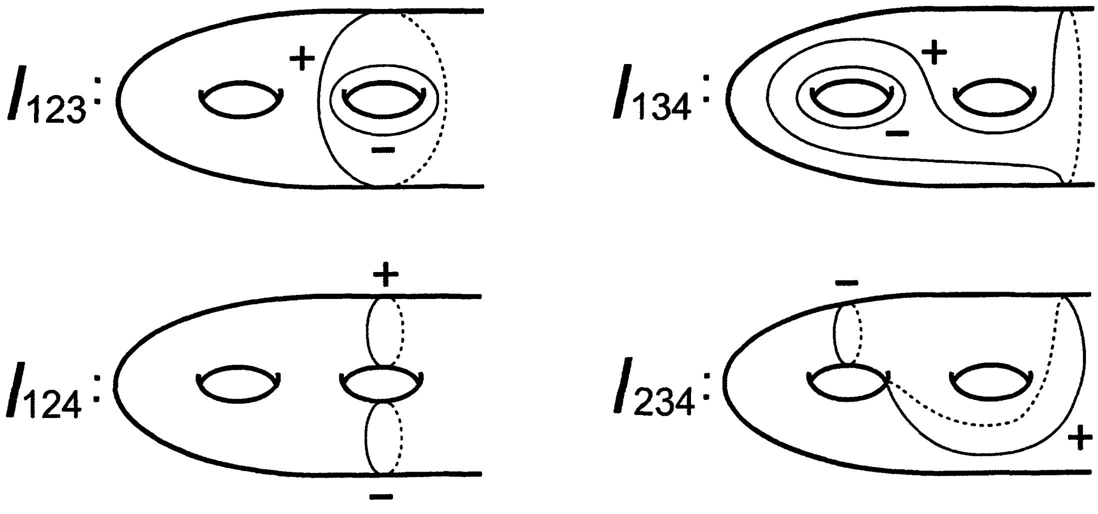


FIGURE 6. Mapping classes I_{ijk} in $\mathcal{I}_{2,1}$

$$\begin{aligned}
\gamma_1 \mapsto\ & [\gamma_4,[\gamma_3,\gamma_4]]^{-3}[\gamma_4,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_3,\gamma_4]]^{4}[\gamma_3,[\gamma_2,\gamma_4]]^{-2}[\gamma_3,[\gamma_2,\gamma_3]][\gamma_3,[\gamma_1,\gamma_4]]^{-4}\\
& [\gamma_3,[\gamma_1,\gamma_3]]^{5}[\gamma_2,[\gamma_2,\gamma_4]][\gamma_2,[\gamma_2,\gamma_3]]^{-1}[\gamma_2,[\gamma_1,\gamma_4]][\gamma_2,[\gamma_1,\gamma_3]]^{-2}[\gamma_1,[\gamma_3,\gamma_4]]^{4}\\
& [\gamma_1,[\gamma_2,\gamma_4]]^{-3}[\gamma_1,[\gamma_2,\gamma_3]]^{4}[\gamma_1,[\gamma_1,\gamma_4]][\gamma_1,[\gamma_1,\gamma_3]]^{-1}\\
& [\gamma_3,\gamma_4]^{-2}[\gamma_1,\gamma_4]^{2}[\gamma_1,\gamma_3]^{-3}[\gamma_1,\gamma_2]\gamma_4^{2}\gamma_3^{-3}\gamma_2^{2}\gamma_1^{-1},
\end{aligned}$$

$$\begin{aligned}
\gamma_2 \mapsto\ & [\gamma_4,[\gamma_2,\gamma_4]]^{-3}[\gamma_4,[\gamma_1,\gamma_4]]^{3}[\gamma_3,[\gamma_2,\gamma_4]]^{6}[\gamma_3,[\gamma_2,\gamma_3]]^{-3}[\gamma_3,[\gamma_1,\gamma_4]]^{-6}[\gamma_3,[\gamma_1,\gamma_3]]^{3}\\
& [\gamma_2,[\gamma_3,\gamma_4]]^{-2}[\gamma_2,[\gamma_2,\gamma_4]]^{-4}[\gamma_2,[\gamma_2,\gamma_3]]^{6}[\gamma_2,[\gamma_1,\gamma_4]]^{2}[\gamma_2,[\gamma_1,\gamma_3]]^{-3}[\gamma_1,[\gamma_3,\gamma_4]]^{2}\\
& [\gamma_1,[\gamma_2,\gamma_4]]^{4}[\gamma_1,[\gamma_2,\gamma_3]]^{-6}[\gamma_1,[\gamma_1,\gamma_4]]^{-2}[\gamma_1,[\gamma_1,\gamma_3]]^{3}\\
& [\gamma_2,\gamma_4]^{2}[\gamma_2,\gamma_3]^{-3}[\gamma_1,\gamma_4]^{-2}[\gamma_1,\gamma_3]^{3}\gamma_2\gamma_1^{-1},
\end{aligned}$$

$$\begin{aligned}
\gamma_3 \mapsto\ & [\gamma_4,[\gamma_3,\gamma_4]]^{-1}[\gamma_4,[\gamma_2,\gamma_4]]^{-3}[\gamma_4,[\gamma_1,\gamma_4]]^{3}[\gamma_3,[\gamma_3,\gamma_4]][\gamma_3,[\gamma_2,\gamma_4]]^{7}[\gamma_3,[\gamma_2,\gamma_3]]^{-4}\\
& [\gamma_3,[\gamma_1,\gamma_4]]^{-7}[\gamma_3,[\gamma_1,\gamma_3]]^{4}[\gamma_2,[\gamma_3,\gamma_4]]^{-4}[\gamma_2,[\gamma_2,\gamma_4]]^{-3}[\gamma_2,[\gamma_2,\gamma_3]]^{5}[\gamma_2,[\gamma_1,\gamma_4]]^{2}\\
& [\gamma_2,[\gamma_1,\gamma_3]]^{-3}[\gamma_1,[\gamma_3,\gamma_4]]^{4}[\gamma_1,[\gamma_2,\gamma_4]]^{3}[\gamma_1,[\gamma_2,\gamma_3]]^{-5}[\gamma_1,[\gamma_1,\gamma_4]]^{-2}[\gamma_1,[\gamma_1,\gamma_3]]^{3}\\
& [\gamma_3,\gamma_4][\gamma_2,\gamma_4][\gamma_2,\gamma_3]^{-2}[\gamma_1,\gamma_4]^{-1}[\gamma_1,\gamma_3]^{2}\gamma_4\gamma_3^{-1}\gamma_2\gamma_1^{-1},
\end{aligned}$$

$$\begin{aligned}
\gamma_4 \mapsto\ & [\gamma_4,[\gamma_3,\gamma_4]]^{-35}[\gamma_4,[\gamma_2,\gamma_4]]^{9}[\gamma_4,[\gamma_1,\gamma_4]]^{-9}[\gamma_3,[\gamma_3,\gamma_4]]^{26}[\gamma_3,[\gamma_2,\gamma_4]]^{-25}[\gamma_3,[\gamma_2,\gamma_3]]^{17}\\
& [\gamma_3,[\gamma_1,\gamma_4]]^{25}[\gamma_3,[\gamma_1,\gamma_3]]^{-17}[\gamma_2,[\gamma_3,\gamma_4]]^{13}[\gamma_2,[\gamma_2,\gamma_4]]^{2}[\gamma_2,[\gamma_2,\gamma_3]]^{-2}[\gamma_2,[\gamma_1,\gamma_4]]^{-19}\\
& [\gamma_2,[\gamma_1,\gamma_3]]^{23}[\gamma_2,[\gamma_1,\gamma_2]]^{-5}[\gamma_1,[\gamma_3,\gamma_4]]^{-13}[\gamma_1,[\gamma_2,\gamma_4]]^{13}[\gamma_1,[\gamma_2,\gamma_3]]^{-16}[\gamma_1,[\gamma_1,\gamma_4]]^{4}\\
& [\gamma_1,[\gamma_1,\gamma_3]]^{-5}[\gamma_1,[\gamma_1,\gamma_2]]^{4}\\
& [\gamma_3,\gamma_4]^{-10}[\gamma_2,\gamma_4]^{2}[\gamma_2,\gamma_3]^{-3}[\gamma_1,\gamma_4]^{-2}[\gamma_1,\gamma_3]^{3}[\gamma_1,\gamma_2]^{-3}\gamma_4^{5}\gamma_3^{-6}\gamma_2^{3}\gamma_1^{-3}.
\end{aligned}$$

$$\begin{aligned}
f_1 &= T_5^{-4}\cdot S_7^{2}\cdot T_1\cdot T_4^{-1}\cdot S_4\cdot T_5^{-1}\\
f_2 &= I_{134}^{-2}\cdot I_{234}^{2}\cdot I_{123}
\end{aligned}$$

$$\begin{aligned}\widetilde{\tau}_2(M_{0258}\cdot f_1\cdot f_2) = &-T(\gamma_1,\gamma_4,\gamma_2,\gamma_4) - 2T(\gamma_1,\gamma_3,\gamma_3,\gamma_4) + 2T(\gamma_1,\gamma_3,\gamma_2,\gamma_4)\\ &- 2T(\gamma_1,\gamma_3,\gamma_2,\gamma_3) - T(\gamma_1,\gamma_3,\gamma_1,\gamma_4) + T(\gamma_1,\gamma_2,\gamma_3,\gamma_4)\\ &+ T(\gamma_1,\gamma_2,\gamma_2,\gamma_3) - T(\gamma_1,\gamma_2,\gamma_1,\gamma_3) + 2T(\gamma_2,\gamma_3,\gamma_3,\gamma_4)\\ &- T(\gamma_2,\gamma_3,\gamma_2,\gamma_4) + Y(\gamma_1,\gamma_4) + 2Y(\gamma_1,\gamma_3) - Y(\gamma_1,\gamma_2)\\ &+ Y(\gamma_2,\gamma_4) + 2Y(\gamma_2,\gamma_3) + 5Y(\gamma_3,\gamma_4)\\ \mapsto\ & Y(\gamma_1,\gamma_3) + Y(\gamma_1,\gamma_4) + Y(\gamma_2,\gamma_3) \neq 0 \in \mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2.\end{aligned}$$

<u>12_n0279</u>

An admissible presentation of $\pi_1(M_{0279})$:

$$\begin{aligned}&z_1z_2z_4,\quad z_1z_3^{-1}z_2z_9^{-1},\quad z_5z_8^{-1}z_6^{-1},\quad z_6z_7z_8z_9,\quad z_2^{-1}z_3z_2z_5^{-1},\\ &i_-(\gamma_1)z_5z_8z_2z_9^{-1}z_5^{-1},\quad i_-(\gamma_2)z_5z_6^{-1}z_5^{-1},\quad i_-(\gamma_3)z_9^{-1}z_6^{-1}z_5^{-1},\quad i_-(\gamma_4)z_2^{-1}z_3z_1z_2^2,\\ &i_+(\gamma_1)z_5z_2z_9^{-1}z_5^{-1},\quad i_+(\gamma_2)z_5z_9z_6^{-1},\quad i_+(\gamma_3)z_2^{-1}z_6^{-1},\quad i_+(\gamma_4)z_2^{-1}z_1z_2^2.\end{aligned}$$

The action of M_{0279} on N_4:

$$\gamma_1 \mapsto [\gamma_4,[\gamma_1,\gamma_4]]^{-1}[\gamma_3,\gamma_4][\gamma_1,\gamma_3]\gamma_4\gamma_3\gamma_1^2,$$

$$\begin{aligned}\gamma_2 \mapsto\ &[\gamma_4,[\gamma_3,\gamma_4]]^{-2}[\gamma_4,[\gamma_1,\gamma_4]]^{-1}[\gamma_3,[\gamma_3,\gamma_4]]^{-1}[\gamma_3,[\gamma_2,\gamma_4]]^{-1}[\gamma_3,[\gamma_2,\gamma_3]]^{-1}[\gamma_3,[\gamma_1,\gamma_4]]^{-1}\\ &[\gamma_2,[\gamma_3,\gamma_4]]^2[\gamma_2,[\gamma_1,\gamma_4]][\gamma_1,[\gamma_3,\gamma_4]]^{-1}[\gamma_1,[\gamma_2,\gamma_4]]^{-1}[\gamma_1,[\gamma_2,\gamma_3]]^{-1}[\gamma_1,[\gamma_1,\gamma_4]]^{-3}\\ &[\gamma_1,[\gamma_1,\gamma_2]][\gamma_3,\gamma_4][\gamma_2,\gamma_3][\gamma_1,\gamma_4]^2[\gamma_1,\gamma_2]^{-1}\gamma_4^{-1}\gamma_2,\end{aligned}$$

$$\begin{aligned}\gamma_3 \mapsto\ &[\gamma_4,[\gamma_3,\gamma_4]]^{-2}[\gamma_4,[\gamma_2,\gamma_4]]^2[\gamma_4,[\gamma_1,\gamma_4]]^{-2}[\gamma_3,[\gamma_3,\gamma_4]]^{-1}[\gamma_3,[\gamma_2,\gamma_4]]^{-1}[\gamma_3,[\gamma_2,\gamma_3]]^{-1}\\ &[\gamma_3,[\gamma_1,\gamma_4]]^{-1}[\gamma_2,[\gamma_3,\gamma_4]]^2[\gamma_2,[\gamma_1,\gamma_4]]^2[\gamma_1,[\gamma_3,\gamma_4]]^{-1}[\gamma_1,[\gamma_2,\gamma_4]]^{-1}[\gamma_1,[\gamma_2,\gamma_3]]^{-1}\\ &[\gamma_1,[\gamma_1,\gamma_4]]^{-4}[\gamma_1,[\gamma_1,\gamma_2]][\gamma_3,\gamma_4][\gamma_2,\gamma_4]^{-1}[\gamma_2,\gamma_3][\gamma_1,\gamma_4]^3[\gamma_1,\gamma_2]^{-1}\gamma_4^{-2}\gamma_2,\end{aligned}$$

$$\begin{aligned}\gamma_4 \mapsto\ &[\gamma_4,[\gamma_3,\gamma_4]]^2[\gamma_4,[\gamma_2,\gamma_4]]^{-2}[\gamma_4,[\gamma_1,\gamma_4]]^{-5}[\gamma_3,[\gamma_2,\gamma_4]]^{-3}[\gamma_3,[\gamma_1,\gamma_4]]^{-1}[\gamma_2,[\gamma_3,\gamma_4]]^2\\ &[\gamma_2,[\gamma_2,\gamma_4]][\gamma_2,[\gamma_2,\gamma_3]][\gamma_2,[\gamma_1,\gamma_4]][\gamma_1,[\gamma_1,\gamma_4]]^2\\ &[\gamma_3,\gamma_4][\gamma_2,\gamma_4]^{-1}[\gamma_2,\gamma_3]^{-1}[\gamma_1,\gamma_4]^{-2}\gamma_4^3\gamma_3\gamma_2^{-1}\gamma_1.\end{aligned}$$

$$\begin{aligned}f_1 &= S_5^{-1}\cdot T_5^2\cdot T_1^{-1}\cdot S_1^2\cdot E_{12}\cdot T_1^{-2}\cdot T_5^{-2}\cdot T_1\cdot T_2\cdot T_1\cdot T_4\cdot T_5\cdot T_4\\ f_2 &= I_{134}^{-1}\cdot I_{234}^{-1}\cdot I_{124}^{-1}\end{aligned}$$

$$\begin{aligned}\widetilde{\tau}_2(M_{0279}\cdot f_1\cdot f_2) = &\ 2T(\gamma_1,\gamma_4,\gamma_3,\gamma_4) + T(\gamma_1,\gamma_2,\gamma_2,\gamma_4) + T(\gamma_1,\gamma_2,\gamma_1,\gamma_4)\\ &+ T(\gamma_2,\gamma_4,\gamma_3,\gamma_4) - Y(\gamma_1,\gamma_2) + 2Y(\gamma_3,\gamma_4)\\ \mapsto\ & Y(\gamma_1,\gamma_4) \neq 0 \in \mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2.\end{aligned}$$

<u>12_n0382</u>

An admissible presentation of $\pi_1(M_{0382})$:

$$\begin{aligned}&i_-(\gamma_1)z_2z_1^{-1}z_3z_2^{-1},\quad i_-(\gamma_2)z_2z_3^{-1}z_2z_1^{-2}z_4z_2^{-1},\quad i_-(\gamma_3)z_4^{-1}z_1^{-1}z_4z_2^{-1},\quad i_-(\gamma_4)z_2^2z_1^{-1}z_4,\\ &i_+(\gamma_1)z_3z_2^{-1},\quad i_+(\gamma_2)z_2z_1^{-2}z_4z_1^{-1},\quad i_+(\gamma_3)z_1^{-1},\quad i_+(\gamma_4)z_4z_2z_1^{-1}z_4.\end{aligned}$$

The action of M_{0382} on N_4:

$$\begin{aligned}\gamma_1 \mapsto\ &[\gamma_4,[\gamma_3,\gamma_4]]^{-2}[\gamma_4,[\gamma_1,\gamma_4]][\gamma_3,[\gamma_3,\gamma_4]]^{-2}[\gamma_3,[\gamma_2,\gamma_3]][\gamma_3,[\gamma_1,\gamma_4]]^4[\gamma_3,[\gamma_1,\gamma_3]]^{10}\\ &[\gamma_2,[\gamma_3,\gamma_4]]^{-1}[\gamma_2,[\gamma_2,\gamma_3]]^{-1}[\gamma_2,[\gamma_1,\gamma_4]]^{-2}[\gamma_2,[\gamma_1,\gamma_3]]^{-8}[\gamma_2,[\gamma_1,\gamma_2]][\gamma_1,[\gamma_3,\gamma_4]]^{-2}\\ &[\gamma_1,[\gamma_2,\gamma_4]]^2[\gamma_1,[\gamma_2,\gamma_3]]^8[\gamma_3,\gamma_4]^2[\gamma_1,\gamma_4]^{-1}[\gamma_1,\gamma_3]^{-4}[\gamma_1,\gamma_2]^2\gamma_4^{-1}\gamma_3^{-4}\gamma_2^2\gamma_1,\end{aligned}$$

$$\begin{aligned}\gamma_2 \mapsto\ & [\gamma_3,[\gamma_2,\gamma_4]]^{-8}[\gamma_3,[\gamma_2,\gamma_3]]^{-13}[\gamma_2,[\gamma_3,\gamma_4]]^{5}[\gamma_2,[\gamma_2,\gamma_4]]^{3}[\gamma_2,[\gamma_2,\gamma_3]]^{13}\\ & [\gamma_3,\gamma_4]^{2}[\gamma_2,\gamma_4]^{-2}[\gamma_2,\gamma_3]^{-8}\gamma_4\gamma_3^{4}\gamma_2^{-1}\gamma_1^{-1},\end{aligned}$$

$$\begin{aligned}\gamma_3 \mapsto\ & [\gamma_4,[\gamma_3,\gamma_4]]^{-1}[\gamma_3,[\gamma_3,\gamma_4]]^{-6}[\gamma_3,[\gamma_2,\gamma_4]]^{-13}[\gamma_3,[\gamma_2,\gamma_3]]^{-35}[\gamma_2,[\gamma_3,\gamma_4]]^{13}[\gamma_2,[\gamma_2,\gamma_4]]^{3}\\ & [\gamma_2,[\gamma_2,\gamma_3]]^{22}[\gamma_3,\gamma_4][\gamma_2,\gamma_4]^{-2}[\gamma_2,\gamma_3]^{-13}\gamma_4\gamma_3^{6}\gamma_2^{-2},\end{aligned}$$

$$\begin{aligned}\gamma_4 \mapsto\ & [\gamma_4,[\gamma_3,\gamma_4]]^{-5}[\gamma_4,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_3,\gamma_4]]^{-14}[\gamma_3,[\gamma_2,\gamma_4]]^{7}[\gamma_3,[\gamma_2,\gamma_3]]^{2}4[\gamma_2,[\gamma_3,\gamma_4]]^{-4}\\ & [\gamma_2,[\gamma_2,\gamma_3]]^{-2}[\gamma_3,\gamma_4]^{6}[\gamma_2,\gamma_4]^{-1}[\gamma_2,\gamma_3]^{-4}\gamma_4^{-1}\gamma_3^{-7}\gamma_2^{3}.\end{aligned}$$

$$\begin{aligned} f_1 &= T_2\cdot T_5^2\cdot T_3\cdot S_4\cdot T_1^{-1}\cdot T_5^{-6}\\ f_2 &= I_{234}\cdot I_{123}\end{aligned}$$

$$\begin{aligned}\widetilde{\tau}_2(M_{0382}\cdot f_1\cdot f_2) &= -T(\gamma_1,\gamma_2,\gamma_3,\gamma_4)+T(\gamma_2,\gamma_3,\gamma_3,\gamma_4)-Y(\gamma_1,\gamma_2)+2Y(\gamma_2,\gamma_3)\\ &\mapsto Y(\gamma_2,\gamma_3)\neq 0\in\mathfrak{h}_{g,1}(2)/\operatorname{Image}\tau_2.\end{aligned}$$

<u>12_n0394</u>

An admissible presentation of $\pi_1(M_{0394})$:

$$\begin{aligned}& i_-(\gamma_1)z_1^{-1}z_2^{-1}z_3, \quad i_-(\gamma_2)z_3^{-1}z_4z_2z_3z_2^{-1}z_1, \quad i_-(\gamma_3)z_4z_2z_3z_2^{-1}z_1, \quad i_-(\gamma_4)z_4,\\ & i_+(\gamma_1)z_2^{-1}z_3, \quad i_+(\gamma_2)z_3^{-1}z_1z_3^{-1}z_4z_2z_3z_2^{-1}, \quad i_+(\gamma_3)z_2z_3z_2^{-1}, \quad i_+(\gamma_4)z_2z_4.\end{aligned}$$

The action of M_{0394} on N_4:

$$\begin{aligned}\gamma_1 \mapsto\ & [\gamma_3,[\gamma_2,\gamma_4]]^{2}[\gamma_3,[\gamma_2,\gamma_3]]^{-2}[\gamma_3,[\gamma_1,\gamma_4]]^{-1}[\gamma_3,[\gamma_1,\gamma_3]]^{-2}[\gamma_2,[\gamma_3,\gamma_4]]^{-2}[\gamma_2,[\gamma_2,\gamma_4]]\\ & [\gamma_2,[\gamma_2,\gamma_3]]^{-1}[\gamma_2,[\gamma_1,\gamma_4]]^{-1}[\gamma_2,[\gamma_1,\gamma_3]][\gamma_2,[\gamma_1,\gamma_2]][\gamma_1,[\gamma_3,\gamma_4]][\gamma_1,[\gamma_2,\gamma_3]]^{-1}\\ & [\gamma_2,\gamma_4]^{-1}[\gamma_2,\gamma_3][\gamma_1,\gamma_4][\gamma_1,\gamma_2]^{-1}\gamma_4\gamma_3^{-2}\gamma_2^{-1}\gamma_1^{2},\end{aligned}$$

$$\begin{aligned}\gamma_2 \mapsto\ & [\gamma_4,[\gamma_3,\gamma_4]]^{-1}[\gamma_4,[\gamma_2,\gamma_4]]^{-1}[\gamma_3,[\gamma_3,\gamma_4]]^{2}[\gamma_3,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_1,\gamma_4]][\gamma_3,[\gamma_1,\gamma_3]]\\ & [\gamma_2,[\gamma_3,\gamma_4]]^{2}[\gamma_2,[\gamma_1,\gamma_3]]^{-1}[\gamma_1,[\gamma_3,\gamma_4]]^{-2}[\gamma_1,[\gamma_2,\gamma_4]]^{-1}[\gamma_1,[\gamma_2,\gamma_3]][\gamma_1,[\gamma_1,\gamma_3]]^{3}\\ & [\gamma_1,[\gamma_1,\gamma_2]][\gamma_3,\gamma_4][\gamma_2,\gamma_4][\gamma_1,\gamma_3]^{-2}[\gamma_1,\gamma_2]^{-1}\gamma_3\gamma_2,\end{aligned}$$

$$\begin{aligned}\gamma_3 \mapsto\ & [\gamma_4,[\gamma_3,\gamma_4]]^{-1}[\gamma_4,[\gamma_2,\gamma_4]]^{-1}[\gamma_3,[\gamma_3,\gamma_4]]^{2}[\gamma_3,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_2,\gamma_3]][\gamma_3,[\gamma_1,\gamma_4]]\\ & [\gamma_3,[\gamma_1,\gamma_3]][\gamma_2,[\gamma_3,\gamma_4]]^{2}[\gamma_2,[\gamma_1,\gamma_3]]^{-2}[\gamma_1,[\gamma_3,\gamma_4]]^{-2}[\gamma_1,[\gamma_2,\gamma_4]]^{-1}[\gamma_1,[\gamma_2,\gamma_3]]\\ & [\gamma_1,[\gamma_1,\gamma_3]]^{4}[\gamma_1,[\gamma_1,\gamma_2]][\gamma_3,\gamma_4][\gamma_2,\gamma_4][\gamma_2,\gamma_3][\gamma_1,\gamma_3]^{-3}[\gamma_1,\gamma_2]^{-1}\gamma_3^{2}\gamma_2,\end{aligned}$$

$$\gamma_4 \mapsto [\gamma_3,[\gamma_1,\gamma_4]][\gamma_3,[\gamma_1,\gamma_3]]^{-2}[\gamma_1,[\gamma_3,\gamma_4]]^{-1}[\gamma_1,[\gamma_1,\gamma_3]]^{-1}[\gamma_1,\gamma_3]\gamma_4\gamma_3^{-1}\gamma_1.$$

$$\begin{aligned} f_1 &= S_7\cdot T_1^{-2}\cdot T_4^{-1}\cdot S_5^{-1}\cdot T_2\cdot T_5^2\\ f_2 &= I_{234}^{-1}\cdot I_{123}^{-1}\end{aligned}$$

$$\begin{aligned}\widetilde{\tau}_2(M_{0394}\cdot f_1\cdot f_2) &= -T(\gamma_1,\gamma_3,\gamma_3,\gamma_4)+T(\gamma_1,\gamma_2,\gamma_3,\gamma_4)+T(\gamma_1,\gamma_2,\gamma_2,\gamma_3)\\ &\quad +T(\gamma_2,\gamma_3,\gamma_3,\gamma_4)+Y(\gamma_1,\gamma_2)+2Y(\gamma_3,\gamma_4)\\ &\mapsto Y(\gamma_1,\gamma_3)\neq 0\in\mathfrak{h}_{g,1}(2)/\operatorname{Image}\tau_2.\end{aligned}$$

<u>12_n0464</u>

An admissible presentation of $\pi_1(M_{0464})$:

$$\begin{gathered} z_1z_2z_6z_7, \quad z_2z_9z_7, \quad z_3z_4z_5z_{10}^{-1}, \quad z_4z_5z_8, \quad z_1z_2z_3^{-1}z_2^{-1}, \quad z_8z_6z_8^{-1}z_9^{-1}, \\ i_-(\gamma_1)z_2z_{10}z_5^{-1}z_9^{-1}z_2^{-1}, \quad i_-(\gamma_2)z_2z_{10}z_5^{-1}z_3^{-1}z_2^{-1}, \quad i_-(\gamma_3)z_2z_8^{-1}z_2^{-1}, \quad i_-(\gamma_4)z_2z_1, \\ i_+(\gamma_1)z_2z_9^{-1}z_2^{-1}, \quad i_+(\gamma_2)z_2z_5^{-1}z_2^{-1}, \quad i_+(\gamma_3)z_1^{-1}z_8^{-1}z_9^{-1}z_2^{-1}z_1, \quad i_+(\gamma_4)z_1^{-1}z_7^{-1}z_1. \end{gathered}$$

The action of M_{0464} on N_4:

$$\begin{aligned} \gamma_1 \mapsto\ & [\gamma_4,[\gamma_1,\gamma_4]][\gamma_3,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_1,\gamma_4]]^3[\gamma_3,[\gamma_1,\gamma_3]][\gamma_2,[\gamma_1,\gamma_4]][\gamma_2,[\gamma_1,\gamma_3]] \\ & [\gamma_1,[\gamma_3,\gamma_4]]^{-1}[\gamma_3,\gamma_4][\gamma_2,\gamma_4][\gamma_2,\gamma_3][\gamma_1,\gamma_4][\gamma_1,\gamma_3][\gamma_1,\gamma_2]\gamma_4\gamma_3\gamma_2\gamma_1, \end{aligned}$$

$$\begin{aligned} \gamma_2 \mapsto\ & [\gamma_4,[\gamma_1,\gamma_4]]^2[\gamma_3,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_1,\gamma_4]]^3[\gamma_3,[\gamma_1,\gamma_3]]^2[\gamma_1,[\gamma_3,\gamma_4]]^{-2}[\gamma_1,[\gamma_1,\gamma_4]] \\ & [\gamma_1,[\gamma_1,\gamma_3]][\gamma_3,\gamma_4][\gamma_2,\gamma_4][\gamma_2,\gamma_3][\gamma_1,\gamma_4]^{-1}[\gamma_1,\gamma_3]^{-1}\gamma_4\gamma_3\gamma_2, \end{aligned}$$

$$\begin{aligned} \gamma_3 \mapsto\ & [\gamma_4,[\gamma_1,\gamma_4]][\gamma_3,[\gamma_1,\gamma_4]][\gamma_3,[\gamma_1,\gamma_3]][\gamma_1,[\gamma_3,\gamma_4]]^{-1}[\gamma_1,[\gamma_1,\gamma_4]][\gamma_1,[\gamma_1,\gamma_3]] \\ & [\gamma_3,\gamma_4][\gamma_1,\gamma_4]^{-1}[\gamma_1,\gamma_3]^{-1}\gamma_4\gamma_3, \end{aligned}$$

$$\gamma_4 \mapsto [\gamma_3,[\gamma_1,\gamma_3]]^{-1}[\gamma_1,[\gamma_1,\gamma_4]]^{-1}[\gamma_1,[\gamma_1,\gamma_3]]^{-1}[\gamma_1,\gamma_4][\gamma_1,\gamma_3]\gamma_4\gamma_1.$$

$$\begin{aligned} f_1 &= S_5^{-1}\cdot T_2\cdot T_5\cdot S_4\cdot T_4 \\ f_2 &= 1 \end{aligned}$$

$$\begin{aligned} \widetilde{\tau}_2(M_{0464}\cdot f_1\cdot f_2) &= -T(\gamma_1,\gamma_3,\gamma_1,\gamma_4) - Y(\gamma_1,\gamma_3) - Y(\gamma_1,\gamma_4) \\ &\mapsto Y(\gamma_1,\gamma_3) + Y(\gamma_1,\gamma_4) \neq 0 \in \mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2. \end{aligned}$$

$\underline{12_n0483}$

An admissible presentation of $\pi_1(M_{0483})$:

$$\begin{gathered} z_8^{-1}z_1z_4z_9z_4^{-1}, \quad z_5z_6z_7^{-1}z_6^{-1}z_8, \quad z_2z_3z_2^{-1}z_1, \quad z_3^{-1}z_2z_3z_5^{-1}, \quad z_4z_9^{-1}z_4^{-1}z_3, \\ i_-(\gamma_1)z_1z_2^{-1}z_1^{-1}, \quad i_-(\gamma_2)z_1z_4^{-1}z_8^{-1}, \quad i_-(\gamma_3)z_6^{-1}, \quad i_-(\gamma_4)z_6^{-1}z_3, \\ i_+(\gamma_1)z_4^{-1}z_2^{-1}, \quad i_+(\gamma_2)z_4^{-1}, \quad i_+(\gamma_3)z_5z_6^{-1}z_8, \quad i_+(\gamma_4)z_8^{-1}z_3. \end{gathered}$$

The action of M_{0483} on N_4:

$$\gamma_1 \mapsto [\gamma_2,[\gamma_2,\gamma_4]]^{-1}[\gamma_2,[\gamma_1,\gamma_2]][\gamma_1,[\gamma_2,\gamma_4]][\gamma_2,\gamma_4][\gamma_1,\gamma_4]^{-1}[\gamma_1,\gamma_2]^{-1}\gamma_2^{-1}\gamma_1,$$

$$\gamma_2 \mapsto [\gamma_4,[\gamma_1,\gamma_4]][\gamma_2,[\gamma_1,\gamma_4]][\gamma_1,[\gamma_1,\gamma_4]]^{-1}[\gamma_2,\gamma_4]^{-2}[\gamma_1,\gamma_4]\gamma_4^{-1}\gamma_2,$$

$$\begin{aligned} \gamma_3 \mapsto\ & [\gamma_3,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_1,\gamma_4]]^{-1}[\gamma_2,[\gamma_1,\gamma_3]]^{-1}[\gamma_2,[\gamma_1,\gamma_2]][\gamma_1,[\gamma_2,\gamma_3]] \\ & [\gamma_1,\gamma_2]^{-1}\gamma_3\gamma_2^{-1}\gamma_1, \end{aligned}$$

$$\begin{aligned} \gamma_4 \mapsto\ & [\gamma_4,[\gamma_2,\gamma_4]][\gamma_4,[\gamma_1,\gamma_4]]^{-1}[\gamma_3,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_1,\gamma_4]]^{-1}[\gamma_2,[\gamma_1,\gamma_4]]^{-2}[\gamma_2,[\gamma_1,\gamma_3]]^{-1} \\ & [\gamma_2,[\gamma_1,\gamma_2]][\gamma_1,[\gamma_2,\gamma_4]][\gamma_1,[\gamma_2,\gamma_3]][\gamma_1,[\gamma_1,\gamma_4]][\gamma_2,\gamma_4][\gamma_1,\gamma_4]^{-1}[\gamma_1,\gamma_2]^{-1}\gamma_4\gamma_3\gamma_2^{-1}\gamma_1. \end{aligned}$$

$$\begin{aligned} f_1 &= S_3\cdot T_4^{-1}\cdot S_5^{-1}\cdot T_2^{-1}\cdot T_1^{-1} \\ f_2 &= I_{134}\cdot I_{234}^{-1} \end{aligned}$$

$$\begin{aligned} \widetilde{\tau}_2(M_{0483}\cdot f_1\cdot f_2) &= T(\gamma_1,\gamma_4,\gamma_3,\gamma_4) - T(\gamma_2,\gamma_4,\gamma_3,\gamma_4) + Y(\gamma_3,\gamma_4) \\ &\mapsto Y(\gamma_1,\gamma_4) + Y(\gamma_2,\gamma_4) \neq 0 \in \mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2. \end{aligned}$$

$\underline{12_n0535}$

An admissible presentation of $\pi_1(M_{0535})$:

$$z_1z_2z_3,\quad z_2z_6z_7^{-1},\quad z_{10}^{-1}z_4z_5z_8^{-1}z_7,\quad z_1z_{10}z_9,\quad z_2z_3z_2^{-1}z_4^{-1},\quad z_2z_6^{-1}z_2^{-1}z_5^{-1},$$
$$i_-(\gamma_1)z_{10}^{-1},\quad i_-(\gamma_2)z_{10}^{-1}z_1^{-1}z_3^{-1}z_1^{-1}z_{10},\quad i_-(\gamma_3)z_7^{-1}z_1^{-1}z_{10},\quad i_-(\gamma_4)z_6z_3^{-1}z_7,$$
$$i_+(\gamma_1)z_7^{-1}z_9,\quad i_+(\gamma_2)z_7^{-1}z_1^{-1}z_3^{-1}z_{10}z_7,\quad i_+(\gamma_3)z_7^{-1}z_3z_{10}z_7,\quad i_+(\gamma_4)z_7^{-1}z_6z_3^{-1}z_7.$$

The action of M_{0535} on N_4:

$$\begin{aligned}\gamma_1 \mapsto\ & [\gamma_4,[\gamma_3,\gamma_4]]^2[\gamma_4,[\gamma_1,\gamma_4]]^{-1}[\gamma_3,[\gamma_3,\gamma_4]][\gamma_3,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_1,\gamma_3]]^{-1}[\gamma_2,[\gamma_3,\gamma_4]]\\ & [\gamma_2,[\gamma_2,\gamma_4]]^3[\gamma_2,[\gamma_1,\gamma_4]]^2[\gamma_2,[\gamma_1,\gamma_3]]^{-1}[\gamma_2,[\gamma_1,\gamma_2]]^{-1}[\gamma_1,[\gamma_3,\gamma_4]]^2[\gamma_1,[\gamma_2,\gamma_4]]^{-1}\\ & [\gamma_1,[\gamma_1,\gamma_3]]^{-1}[\gamma_3,\gamma_4]^{-1}[\gamma_2,\gamma_4]^{-2}[\gamma_1,\gamma_3][\gamma_1,\gamma_2]\gamma_4\gamma_1,\end{aligned}$$

$$\begin{aligned}\gamma_2 \mapsto\ & [\gamma_4,[\gamma_2,\gamma_4]]^{-4}[\gamma_4,[\gamma_1,\gamma_4]]^2[\gamma_3,[\gamma_3,\gamma_4]]^{-1}[\gamma_3,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_1,\gamma_4]]^6[\gamma_3,[\gamma_1,\gamma_3]]\\ & [\gamma_2,[\gamma_3,\gamma_4]]^{-2}[\gamma_2,[\gamma_2,\gamma_4]]^{-3}[\gamma_2,[\gamma_1,\gamma_4]]^{13}[\gamma_2,[\gamma_1,\gamma_3]]^6[\gamma_2,[\gamma_1,\gamma_2]]^4[\gamma_1,[\gamma_3,\gamma_4]]^{-6}\\ & [\gamma_1,[\gamma_2,\gamma_4]]^{-14}[\gamma_1,[\gamma_2,\gamma_3]]^{-3}[\gamma_1,[\gamma_1,\gamma_4]][\gamma_1,[\gamma_1,\gamma_3]]^3[\gamma_1,[\gamma_1,\gamma_2]]^4\\ & [\gamma_3,\gamma_4]^3[\gamma_2,\gamma_4]^3[\gamma_2,\gamma_3][\gamma_1,\gamma_4]^3[\gamma_1,\gamma_3][\gamma_1,\gamma_2]^2\gamma_4^3\gamma_3\gamma_2^2\gamma_1^3,\end{aligned}$$

$$\begin{aligned}\gamma_3 \mapsto\ & [\gamma_4,[\gamma_3,\gamma_4]]^{-3}[\gamma_4,[\gamma_2,\gamma_4]]^{-3}[\gamma_4,[\gamma_1,\gamma_4]]^{-3}[\gamma_3,[\gamma_3,\gamma_4]]^{-1}[\gamma_3,[\gamma_2,\gamma_4]]^{-1}[\gamma_3,[\gamma_1,\gamma_4]]\\ & [\gamma_3,[\gamma_1,\gamma_3]][\gamma_2,[\gamma_3,\gamma_4]]^{-1}[\gamma_2,[\gamma_2,\gamma_4]]^{-4}[\gamma_2,[\gamma_1,\gamma_4]]^{-2}[\gamma_2,[\gamma_1,\gamma_3]][\gamma_2,[\gamma_1,\gamma_2]]\\ & [\gamma_1,[\gamma_3,\gamma_4]]^{-4}[\gamma_1,[\gamma_2,\gamma_4]]^{-3}[\gamma_1,[\gamma_1,\gamma_4]]^{-4}[\gamma_1,[\gamma_1,\gamma_3]]^2[\gamma_1,[\gamma_1,\gamma_2]]\\ & [\gamma_3,\gamma_4][\gamma_2,\gamma_4]^3[\gamma_1,\gamma_4]^3[\gamma_1,\gamma_3]^{-1}[\gamma_1,\gamma_2]^{-1}\gamma_4^{-2}\gamma_1^{-1},\end{aligned}$$

$$\begin{aligned}\gamma_4 \mapsto\ & [\gamma_4,[\gamma_3,\gamma_4]]^5[\gamma_4,[\gamma_2,\gamma_4]]^{-6}[\gamma_4,[\gamma_1,\gamma_4]]^{-1}[\gamma_3,[\gamma_2,\gamma_4]]^{-1}[\gamma_3,[\gamma_1,\gamma_4]]^3[\gamma_2,[\gamma_2,\gamma_4]]^2\\ & [\gamma_2,[\gamma_1,\gamma_4]]^{10}[\gamma_2,[\gamma_1,\gamma_3]]^2[\gamma_1,[\gamma_3,\gamma_4]]^{-2}[\gamma_1,[\gamma_2,\gamma_4]]^{-9}[\gamma_1,[\gamma_2,\gamma_3]]^{-2}[\gamma_1,[\gamma_1,\gamma_4]]\\ & [\gamma_1,[\gamma_1,\gamma_2]][\gamma_3,\gamma_4]^2[\gamma_2,\gamma_4]^{-2}[\gamma_1,\gamma_3][\gamma_1,\gamma_2]^2\gamma_4^4\gamma_3\gamma_2\gamma_1^2.\end{aligned}$$

$$f_1 = S_3\cdot T_2^2\cdot T_4\cdot S_2\cdot T_1^{-1}\cdot T_2^{-2}\cdot T_5^2\cdot T_4^{-1}\cdot T_5^{-1}$$
$$f_2 = I_{124}^{-2}\cdot I_{134}^{-1}$$

$$\begin{aligned}\widetilde{\tau}_2(M_{0535}\cdot f_1\cdot f_2) &= -3T(\gamma_1,\gamma_4,\gamma_3,\gamma_4)-3T(\gamma_1,\gamma_4,\gamma_2,\gamma_4)-T(\gamma_1,\gamma_2,\gamma_3,\gamma_4)\\ &\quad -T(\gamma_1,\gamma_2,\gamma_1,\gamma_4)-T(\gamma_2,\gamma_4,\gamma_3,\gamma_4)\\ &\quad -5Y(\gamma_1,\gamma_4)-2Y(\gamma_1,\gamma_2)-2Y(\gamma_2,\gamma_4)\\ &\mapsto Y(\gamma_1,\gamma_4)+Y(\gamma_2,\gamma_4)\neq 0\in \mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2.\end{aligned}$$

$\underline{12_n0650}$

An admissible presentation of $\pi_1(M_{0650})$:

$$z_2z_3z_1^{-1}z_4z_1,\quad z_2z_6z_8^{-1}z_6^{-1}z_{11}^{-1},\quad z_1z_5^{-1}z_1^{-1}z_4,\quad z_3z_6z_9^{-1}z_6^{-1},\quad z_9z_8^{-1}z_7z_8,\quad z_8z_7z_{10}^{-1}z_7^{-1},$$
$$z_{10}z_6^{-1}z_{11}z_6,\quad i_-(\gamma_1)z_2z_6^{-1}z_2^{-1},\quad i_-(\gamma_2)z_2z_7z_6^{-1}z_2^{-1},\quad i_-(\gamma_3)z_6z_8z_6^{-1}z_2^{-1},\quad i_-(\gamma_4)z_2z_3z_1^{-1},$$
$$i_+(\gamma_1)z_{11}z_6^{-1}z_2^{-1},\quad i_+(\gamma_2)z_2z_3^{-1}z_2^{-1},\quad i_+(\gamma_3)z_1z_6z_8z_6^{-1},\quad i_+(\gamma_4)z_1^{-1}.$$

The action of M_{0650} on N_4:

$$\gamma_1 \mapsto [\gamma_4,[\gamma_2,\gamma_4]]^{-1}[\gamma_3,[\gamma_2,\gamma_4]]^{-2}[\gamma_3,[\gamma_2,\gamma_3]]^{-1}[\gamma_3,[\gamma_1,\gamma_4]][\gamma_2,[\gamma_3,\gamma_4]]$$
$$[\gamma_3,\gamma_4][\gamma_1,\gamma_4][\gamma_1,\gamma_3]\gamma_4\gamma_3\gamma_1,$$

$$\gamma_2 \mapsto [\gamma_4,[\gamma_2,\gamma_4]]^{-2}[\gamma_3,[\gamma_2,\gamma_4]]^{-2}[\gamma_3,[\gamma_2,\gamma_3]]^{-2}[\gamma_3,[\gamma_1,\gamma_4]][\gamma_2,[\gamma_3,\gamma_4]]^2[\gamma_2,[\gamma_1,\gamma_4]]$$
$$[\gamma_2,[\gamma_1,\gamma_3]][\gamma_3,\gamma_4][\gamma_2,\gamma_4]^2[\gamma_2,\gamma_3]^2[\gamma_1,\gamma_4][\gamma_1,\gamma_3]\gamma_4\gamma_3\gamma_2\gamma_1,$$

$$\gamma_3 \mapsto [\gamma_4,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_2,\gamma_3]][\gamma_2,[\gamma_3,\gamma_4]]^{-1}[\gamma_3,\gamma_4][\gamma_2,\gamma_4]^{-1}[\gamma_2,\gamma_3]^{-1}\gamma_4\gamma_3,$$

$$\gamma_4 \mapsto [\gamma_3,[\gamma_2,\gamma_3]][\gamma_2,\gamma_4]^{-1}[\gamma_2,\gamma_3]^{-1}\gamma_4\gamma_2^{-1}.$$

$$f_1 = T_2^{-1}\cdot S_7\cdot S_4\cdot T_1^{-1}\cdot T_5^{-1}$$
$$f_2 = I_{123}$$

$$\begin{aligned}\widetilde{\tau}_2(M_{0650}\cdot f_1\cdot f_2) &= -T(\gamma_1,\gamma_2,\gamma_3,\gamma_4) - T(\gamma_1,\gamma_2,\gamma_3,\gamma_4) - T(\gamma_1,\gamma_2,\gamma_2,\gamma_3)\\ &\quad - Y(\gamma_1,\gamma_2) - Y(\gamma_2,\gamma_4)\\ &\mapsto Y(\gamma_2,\gamma_3) + Y(\gamma_2,\gamma_4) \neq 0 \in \mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2.\end{aligned}$$

$\underline{12_n0801}$

An admissible presentation of $\pi_1(M_{0801})$:

$$z_1^{-1}z_6z_7z_8^{-1}z_9^{-1},\quad z_3z_4z_9z_6^{-1},\quad z_2z_4z_5,\quad z_2z_6z_7^{-1}z_6^{-1},\quad z_2z_3^{-1}z_2^{-1}z_1,$$
$$i_-(\gamma_1)z_6z_7z_8^{-1}z_6,\quad i_-(\gamma_2)z_1z_2z_8z_7^{-1}z_6^{-1},\quad i_-(\gamma_3)z_9z_6^{-1}z_2^{-1},\quad i_-(\gamma_4)z_5^{-1}z_9^{-1}z_5^{-1},$$
$$i_+(\gamma_1)z_6z_9,\quad i_+(\gamma_2)z_6z_2z_6z_9^{-1}z_6^{-1},\quad i_+(\gamma_3)z_5z_9z_6^{-1},\quad i_+(\gamma_4)z_4z_9^{-1}z_5^{-1}.$$

The action of M_{0801} on N_4:

$$\gamma_1 \mapsto [\gamma_4,[\gamma_3,\gamma_4]]^{-7}[\gamma_4,[\gamma_2,\gamma_4]]^{-9}[\gamma_3,[\gamma_3,\gamma_4]]^{-7}[\gamma_3,[\gamma_2,\gamma_4]]^{-20}[\gamma_3,[\gamma_2,\gamma_3]]^{-20}[\gamma_2,[\gamma_3,\gamma_4]]^7$$
$$[\gamma_2,[\gamma_2,\gamma_4]]^{-13}[\gamma_2,[\gamma_2,\gamma_3]]^{-17}[\gamma_2,[\gamma_1,\gamma_4]]^{-1}[\gamma_2,[\gamma_1,\gamma_2]]^4[\gamma_1,[\gamma_2,\gamma_4]]$$
$$[\gamma_3,\gamma_4]^4[\gamma_2,\gamma_4]^6[\gamma_2,\gamma_3]^8[\gamma_1,\gamma_2]^{-2}\gamma_4^{-2}\gamma_3^{-3}\gamma_2^{-1},$$

$$\gamma_2 \mapsto [\gamma_4,[\gamma_2,\gamma_4]]^{-1}[\gamma_3,[\gamma_2,\gamma_4]]^{-4}[\gamma_3,[\gamma_2,\gamma_3]][\gamma_2,[\gamma_3,\gamma_4]]^3[\gamma_2,[\gamma_2,\gamma_4]]^6[\gamma_2,[\gamma_2,\gamma_3]]^8$$
$$[\gamma_2,[\gamma_1,\gamma_4]]^4[\gamma_2,[\gamma_1,\gamma_3]]^5[\gamma_2,[\gamma_1,\gamma_2]]^{-1}[\gamma_1,[\gamma_2,\gamma_4]]^{-4}[\gamma_1,[\gamma_2,\gamma_3]]^{-5}$$
$$[\gamma_2,\gamma_4]^{-3}[\gamma_2,\gamma_3]^{-3}[\gamma_1,\gamma_2]^2\gamma_4\gamma_3^2\gamma_2^2,$$

$$\gamma_3 \mapsto [\gamma_4,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_2,\gamma_4]]^2[\gamma_3,[\gamma_2,\gamma_3]]^3[\gamma_2,[\gamma_3,\gamma_4]]^{-1}[\gamma_2,[\gamma_2,\gamma_4]]^3[\gamma_2,[\gamma_2,\gamma_3]]^5$$
$$[\gamma_2,[\gamma_1,\gamma_2]]^{-1}[\gamma_2,\gamma_4]^{-1}[\gamma_2,\gamma_3]^{-2}\gamma_2^{-1},$$

$$\gamma_4 \mapsto [\gamma_4,[\gamma_3,\gamma_4]]^7[\gamma_4,[\gamma_2,\gamma_4]]^{-1}[\gamma_4,[\gamma_1,\gamma_4]]^2[\gamma_3,[\gamma_3,\gamma_4]]^3[\gamma_3,[\gamma_2,\gamma_4]]^3[\gamma_3,[\gamma_2,\gamma_3]]^4$$
$$[\gamma_3,[\gamma_1,\gamma_4]]^8[\gamma_3,[\gamma_1,\gamma_3]]^5[\gamma_2,[\gamma_3,\gamma_4]]^{-3}[\gamma_2,[\gamma_2,\gamma_4]][\gamma_2,[\gamma_2,\gamma_3]]^2[\gamma_2,[\gamma_1,\gamma_4]]^{14}$$
$$[\gamma_2,[\gamma_1,\gamma_3]]^{18}[\gamma_2,[\gamma_1,\gamma_2]]^5[\gamma_1,[\gamma_3,\gamma_4]]^{-5}[\gamma_1,[\gamma_2,\gamma_4]]^{-13}[\gamma_1,[\gamma_2,\gamma_3]]^{-15}[\gamma_1,[\gamma_1,\gamma_2]]$$
$$[\gamma_3,\gamma_4]^4[\gamma_2,\gamma_3]^2[\gamma_1,\gamma_4][\gamma_1,\gamma_3]^2[\gamma_1,\gamma_2]^4\gamma_4^3\gamma_3^4\gamma_2^3\gamma_1.$$

$$\begin{aligned} f_1 &= S_5^2 \cdot T_2^{-2} \cdot T_3^{-1} \cdot T_4^2 \cdot S_4 \cdot T_1^{-1} \cdot T_5^{-1} \cdot E_{12} \cdot T_2 \cdot T_1^2 \cdot T_2 \cdot T_1^2 \cdot T_2 \cdot T_5^{-2} \\ f_2 &= I_{134}^{-1} \cdot I_{124}^{-1} \cdot I_{123}^{-1} \end{aligned}$$

$$\begin{aligned} \widetilde{\tau}_2(M_{0801} \cdot f_1 \cdot f_2) = &-T(\gamma_1,\gamma_4,\gamma_3,\gamma_4) - T(\gamma_1,\gamma_3,\gamma_3,\gamma_4) - T(\gamma_1,\gamma_2,\gamma_2,\gamma_4) \\ &- T(\gamma_1,\gamma_2,\gamma_2,\gamma_3) - T(\gamma_1,\gamma_2,\gamma_1,\gamma_4) - T(\gamma_1,\gamma_2,\gamma_1,\gamma_3) \\ &+ Y(\gamma_3,\gamma_4) \\ &\mapsto Y(\gamma_2,\gamma_3) + Y(\gamma_2,\gamma_4) \neq 0 \in \mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2. \end{aligned}$$

<u>12_n0815</u>

An admissible presentation of $\pi_1(M_{0815})$:

$z_1z_9z_6,\quad z_1z_2^{-1}z_4^{-1},\quad z_4z_{11}^{-1}z_5,\quad z_{10}^{-1}z_5^{-1}z_6z_7z_8,\quad z_8^{-1}z_6^{-1}z_9z_6,\quad z_7^{-1}z_6^{-1}z_3z_6,\quad z_4z_3^{-1}z_4^{-1}z_{10},$
$i_-(\gamma_1)z_4z_3^{-1}z_4^{-1},\quad i_-(\gamma_2)z_4z_{11},\quad i_-(\gamma_3)z_9,\quad i_-(\gamma_4)z_2^{-1}z_9^{-1},$
$i_+(\gamma_1)z_2^{-1}z_3^{-1}z_4^{-1},\quad i_+(\gamma_2)z_{11}z_1,\quad i_+(\gamma_3)z_9z_3^{-1}z_1,\quad i_+(\gamma_4)z_9z_2^{-1}z_9^{-1}.$

The action of M_{0815} on N_4:

$$\begin{aligned} \gamma_1 \mapsto\ &[\gamma_4,[\gamma_3,\gamma_4]][\gamma_4,[\gamma_1,\gamma_4]][\gamma_2,[\gamma_1,\gamma_4]]^{-1}[\gamma_1,[\gamma_3,\gamma_4]]^{-1}[\gamma_1,[\gamma_2,\gamma_4]] \\ &[\gamma_3,\gamma_4]^{-1}[\gamma_1,\gamma_4]^{-1}[\gamma_1,\gamma_2]\gamma_4^{-1}\gamma_2\gamma_1, \end{aligned}$$

$$\begin{aligned} \gamma_2 \mapsto\ &[\gamma_4,[\gamma_2,\gamma_4]]^{-4}[\gamma_4,[\gamma_1,\gamma_4]]^{-2}[\gamma_2,[\gamma_3,\gamma_4]]^{-3}[\gamma_2,[\gamma_2,\gamma_4]]^4[\gamma_1,[\gamma_3,\gamma_4]]^{-1}[\gamma_1,[\gamma_2,\gamma_4]]^2 \\ &[\gamma_1,[\gamma_1,\gamma_4]][\gamma_3,\gamma_4][\gamma_2,\gamma_4]^{-3}[\gamma_1,\gamma_4]^{-1}\gamma_4\gamma_2, \end{aligned}$$

$$\begin{aligned} \gamma_3 \mapsto\ &[\gamma_4,[\gamma_3,\gamma_4]][\gamma_3,[\gamma_2,\gamma_4]]^{-4}[\gamma_3,[\gamma_1,\gamma_4]]^{-2}[\gamma_2,[\gamma_3,\gamma_4]]^2[\gamma_2,[\gamma_2,\gamma_4]][\gamma_2,[\gamma_2,\gamma_3]]^3 \\ &[\gamma_2,[\gamma_1,\gamma_4]]^3[\gamma_2,[\gamma_1,\gamma_3]]^2[\gamma_2,[\gamma_1,\gamma_2]]^{-3}[\gamma_1,[\gamma_3,\gamma_4]][\gamma_1,[\gamma_2,\gamma_4]]^{-4}[\gamma_1,[\gamma_1,\gamma_3]] \\ &[\gamma_1,[\gamma_1,\gamma_2]]^{-2}[\gamma_3,\gamma_4]^2[\gamma_2,\gamma_4]^{-2}[\gamma_2,\gamma_3]^{-2}[\gamma_1,\gamma_4]^{-1}[\gamma_1,\gamma_3]^{-1}[\gamma_1,\gamma_2]^2\gamma_4^2\gamma_3\gamma_2^{-2}\gamma_1^{-1}, \end{aligned}$$

$$\begin{aligned} \gamma_4 \mapsto\ &[\gamma_3,[\gamma_2,\gamma_4]]^2[\gamma_3,[\gamma_1,\gamma_4]][\gamma_2,[\gamma_3,\gamma_4]]^{-2}[\gamma_2,[\gamma_2,\gamma_4]][\gamma_2,[\gamma_1,\gamma_4]]^{-1}[\gamma_1,[\gamma_3,\gamma_4]]^{-1} \\ &[\gamma_1,[\gamma_2,\gamma_4]]^2[\gamma_1,[\gamma_1,\gamma_4]][\gamma_2,\gamma_4]^{-2}[\gamma_1,\gamma_4]^{-1}\gamma_4^{-1}\gamma_3^{-1}\gamma_2^2\gamma_1. \end{aligned}$$

$$\begin{aligned} f_1 &= T_5 \cdot T_3^2 \cdot T_1^{-1} \cdot T_4 \cdot S_1 \cdot T_2^{-1} \\ f_2 &= I_{124}^{-1} \end{aligned}$$

$$\begin{aligned} \widetilde{\tau}_2(M_{0815} \cdot f_1 \cdot f_2) &= T(\gamma_2,\gamma_4,\gamma_1,\gamma_4) - T(\gamma_1,\gamma_2,\gamma_2,\gamma_4) - Y(\gamma_1,\gamma_2) + Y(\gamma_1,\gamma_4) \\ &\mapsto Y(\gamma_1,\gamma_4) + Y(\gamma_2,\gamma_4) \neq 0 \in \mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2. \end{aligned}$$

<u>12_n0210</u>

An admissible presentation of $\pi_1(M_{0210})$:

$i_-(\gamma_1)z_3^{-1}z_4,\quad i_-(\gamma_2)z_3^{-2}z_2,\quad i_-(\gamma_3)z_5^{-1}z_3^{-1}z_2,\quad i_-(\gamma_4)z_2^{-1}z_1z_6^{-1}z_5z_6^{-1}z_5,$
$i_-(\gamma_5)z_5^{-1}z_6z_5^{-1}z_1z_6^{-1}z_5z_6^{-1}z_5,\quad i_-(\gamma_6)z_5^{-1}z_6z_5^{-1}z_1z_3^{-1}z_5z_6^{-1}z_5,\quad i_+(\gamma_1)z_4,\quad i_+(\gamma_2)z_4z_3^{-1}z_2z_3^{-1},$
$i_+(\gamma_3)z_6^{-1}z_2z_3^{-1},\quad i_+(\gamma_4)z_5z_2^{-1}z_1z_6^{-1}z_5,\quad i_+(\gamma_5)z_5^{-1}z_6z_2^{-1}z_1z_6^{-1}z_5,\quad i_+(\gamma_6)z_5^{-1}z_6z_3^{-1}z_5z_6^{-1}z_5$

The action of M_{0210} on N_4:

$$
\begin{aligned}
\gamma_1 \mapsto\ & [\gamma_6,[\gamma_5,\gamma_6]][\gamma_6,[\gamma_4,\gamma_6]]^{-1}[\gamma_6,[\gamma_3,\gamma_6]]^{-2}[\gamma_6,[\gamma_2,\gamma_6]]^{2}[\gamma_6,[\gamma_1,\gamma_6]]^{-1}[\gamma_5,[\gamma_4,\gamma_6]]^{-2}\\
& [\gamma_5,[\gamma_3,\gamma_6]]^{-1}[\gamma_5,[\gamma_2,\gamma_6]][\gamma_5,[\gamma_1,\gamma_6]][\gamma_4,[\gamma_4,\gamma_6]]^{2}[\gamma_4,[\gamma_4,\gamma_5]][\gamma_4,[\gamma_3,\gamma_6]]\\
& [\gamma_4,[\gamma_2,\gamma_6]]^{-1}[\gamma_4,[\gamma_1,\gamma_6]]^{-1}[\gamma_4,[\gamma_1,\gamma_5]]^{-1}[\gamma_4,[\gamma_1,\gamma_4]][\gamma_3,[\gamma_3,\gamma_6]][\gamma_3,[\gamma_2,\gamma_6]]^{-3}\\
& [\gamma_3,[\gamma_2,\gamma_5]]^{-1}[\gamma_3,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_1,\gamma_6]]^{5}[\gamma_3,[\gamma_1,\gamma_5]]^{2}[\gamma_3,[\gamma_1,\gamma_4]]^{-2}[\gamma_2,[\gamma_3,\gamma_6]]^{2}\\
& [\gamma_2,[\gamma_3,\gamma_5]][\gamma_2,[\gamma_3,\gamma_4]]^{-1}[\gamma_2,[\gamma_2,\gamma_3]][\gamma_2,[\gamma_1,\gamma_6]]^{-5}[\gamma_2,[\gamma_1,\gamma_5]]^{-2}[\gamma_2,[\gamma_1,\gamma_4]]^{2}\\
& [\gamma_2,[\gamma_1,\gamma_3]]^{-2}[\gamma_2,[\gamma_1,\gamma_2]]^{2}[\gamma_1,[\gamma_3,\gamma_6]]^{-5}[\gamma_1,[\gamma_3,\gamma_5]]^{-2}[\gamma_1,[\gamma_3,\gamma_4]]^{2}[\gamma_1,[\gamma_2,\gamma_6]]^{5}\\
& [\gamma_1,[\gamma_2,\gamma_5]]^{2}[\gamma_1,[\gamma_2,\gamma_4]]^{-2}[\gamma_1,[\gamma_1,\gamma_3]][\gamma_1,[\gamma_1,\gamma_2]]^{-1}\\
& [\gamma_5,\gamma_6]^{2}[\gamma_4,\gamma_6]^{-2}[\gamma_4,\gamma_5]^{-1}[\gamma_3,\gamma_6]^{-1}[\gamma_2,\gamma_6][\gamma_2,\gamma_3]^{-1}[\gamma_1,\gamma_6][\gamma_1,\gamma_5][\gamma_1,\gamma_4]^{-1}\\
& [\gamma_1,\gamma_3]^{2}[\gamma_1,\gamma_2]^{-2}\gamma_6^{2}\gamma_5\gamma_4^{-1}\gamma_3\gamma_2^{-1}\gamma_1^{2},
\end{aligned}
$$

$$
\begin{aligned}
\gamma_2 \mapsto\ & [\gamma_6,[\gamma_2,\gamma_6]]^{-3}[\gamma_6,[\gamma_1,\gamma_6]]^{3}[\gamma_5,[\gamma_2,\gamma_6]]^{-2}[\gamma_5,[\gamma_2,\gamma_5]]^{-1}[\gamma_5,[\gamma_1,\gamma_6]]^{2}[\gamma_5,[\gamma_1,\gamma_5]]\\
& [\gamma_4,[\gamma_2,\gamma_6]]^{2}[\gamma_4,[\gamma_2,\gamma_5]][\gamma_4,[\gamma_1,\gamma_6]]^{-2}[\gamma_4,[\gamma_1,\gamma_5]]^{-1}[\gamma_3,[\gamma_2,\gamma_6]]^{-2}[\gamma_3,[\gamma_2,\gamma_5]]^{-1}\\
& [\gamma_3,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_2,\gamma_3]]^{-1}[\gamma_3,[\gamma_1,\gamma_6]]^{2}[\gamma_3,[\gamma_1,\gamma_5]][\gamma_3,[\gamma_1,\gamma_4]]^{-1}[\gamma_3,[\gamma_1,\gamma_3]]\\
& [\gamma_2,[\gamma_5,\gamma_6]]^{2}[\gamma_2,[\gamma_4,\gamma_6]]^{-2}[\gamma_2,[\gamma_4,\gamma_5]]^{-1}[\gamma_2,[\gamma_3,\gamma_6]]^{-1}[\gamma_2,[\gamma_2,\gamma_6]]^{3}[\gamma_2,[\gamma_2,\gamma_5]]\\
& [\gamma_2,[\gamma_2,\gamma_4]]^{-1}[\gamma_2,[\gamma_1,\gamma_6]]^{-5}[\gamma_2,[\gamma_1,\gamma_5]]^{-2}[\gamma_2,[\gamma_1,\gamma_4]]^{2}[\gamma_2,[\gamma_1,\gamma_3]]^{-1}[\gamma_1,[\gamma_5,\gamma_6]]^{-2}\\
& [\gamma_1,[\gamma_4,\gamma_6]]^{2}[\gamma_1,[\gamma_4,\gamma_5]][\gamma_1,[\gamma_3,\gamma_6]][\gamma_1,[\gamma_2,\gamma_6]]^{-3}[\gamma_1,[\gamma_2,\gamma_5]]^{-1}[\gamma_1,[\gamma_2,\gamma_4]]\\
& [\gamma_1,[\gamma_1,\gamma_6]]^{5}[\gamma_1,[\gamma_1,\gamma_5]]^{2}[\gamma_1,[\gamma_1,\gamma_4]]^{-2}[\gamma_1,[\gamma_1,\gamma_3]]\\
& [\gamma_2,\gamma_6]^{2}[\gamma_2,\gamma_5][\gamma_2,\gamma_4]^{-1}[\gamma_2,\gamma_3][\gamma_1,\gamma_6]^{-2}[\gamma_1,\gamma_5]^{-1}[\gamma_1,\gamma_4][\gamma_1,\gamma_3]^{-1}\gamma_2\gamma_1^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\gamma_3 \mapsto\ & [\gamma_6,[\gamma_2,\gamma_6]]^{-2}[\gamma_6,[\gamma_1,\gamma_6]]^{2}[\gamma_5,[\gamma_2,\gamma_6]]^{-2}[\gamma_5,[\gamma_2,\gamma_5]]^{-1}[\gamma_5,[\gamma_1,\gamma_6]]^{2}[\gamma_5,[\gamma_1,\gamma_5]]\\
& [\gamma_4,[\gamma_2,\gamma_6]]^{2}[\gamma_4,[\gamma_2,\gamma_5]][\gamma_4,[\gamma_1,\gamma_6]]^{-2}[\gamma_4,[\gamma_1,\gamma_5]]^{-1}[\gamma_3,[\gamma_3,\gamma_6]]^{-1}[\gamma_3,[\gamma_2,\gamma_6]]^{-1}\\
& [\gamma_3,[\gamma_2,\gamma_5]]^{-1}[\gamma_3,[\gamma_2,\gamma_4]][\gamma_3,[\gamma_2,\gamma_3]]^{-1}[\gamma_3,[\gamma_1,\gamma_6]][\gamma_3,[\gamma_1,\gamma_5]][\gamma_3,[\gamma_1,\gamma_4]]^{-1}\\
& [\gamma_3,[\gamma_1,\gamma_3]][\gamma_2,[\gamma_5,\gamma_6]]^{2}[\gamma_2,[\gamma_4,\gamma_6]]^{-2}[\gamma_2,[\gamma_4,\gamma_5]]^{-1}[\gamma_2,[\gamma_2,\gamma_6]]^{2}[\gamma_2,[\gamma_2,\gamma_5]]\\
& [\gamma_2,[\gamma_2,\gamma_4]]^{-1}[\gamma_2,[\gamma_1,\gamma_6]]^{-2}[\gamma_2,[\gamma_1,\gamma_5]]^{-2}[\gamma_2,[\gamma_1,\gamma_4]]^{2}[\gamma_2,[\gamma_1,\gamma_3]]^{-1}[\gamma_1,[\gamma_5,\gamma_6]]^{-2}\\
& [\gamma_1,[\gamma_4,\gamma_6]]^{2}[\gamma_1,[\gamma_4,\gamma_5]][\gamma_1,[\gamma_2,\gamma_6]]^{-2}[\gamma_1,[\gamma_2,\gamma_5]]^{-1}[\gamma_1,[\gamma_2,\gamma_4]][\gamma_1,[\gamma_1,\gamma_6]]^{2}\\
& [\gamma_1,[\gamma_1,\gamma_5]]^{2}[\gamma_1,[\gamma_1,\gamma_4]]^{-2}[\gamma_1,[\gamma_1,\gamma_3]]\\
& [\gamma_3,\gamma_6][\gamma_2,\gamma_5][\gamma_2,\gamma_4]^{-1}[\gamma_2,\gamma_3][\gamma_1,\gamma_5]^{-1}[\gamma_1,\gamma_4][\gamma_1,\gamma_3]^{-1}\gamma_6^{-1}\gamma_2\gamma_1^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\gamma_4 \mapsto\ & [\gamma_6,[\gamma_5,\gamma_6]][\gamma_6,[\gamma_3,\gamma_6]]^{-2}[\gamma_6,[\gamma_2,\gamma_6]]^{2}[\gamma_6,[\gamma_1,\gamma_6]]^{-2}[\gamma_5,[\gamma_3,\gamma_6]][\gamma_5,[\gamma_2,\gamma_6]]^{-1}\\
& [\gamma_5,[\gamma_1,\gamma_6]][\gamma_3,[\gamma_5,\gamma_6]]^{-1}[\gamma_3,[\gamma_3,\gamma_6]][\gamma_3,[\gamma_2,\gamma_6]]^{-7}[\gamma_3,[\gamma_2,\gamma_5]]^{-3}[\gamma_3,[\gamma_2,\gamma_3]]^{-2}\\
& [\gamma_3,[\gamma_1,\gamma_6]]^{7}[\gamma_3,[\gamma_1,\gamma_5]]^{3}[\gamma_3,[\gamma_1,\gamma_3]]^{2}[\gamma_2,[\gamma_5,\gamma_6]][\gamma_2,[\gamma_3,\gamma_6]]^{6}[\gamma_2,[\gamma_3,\gamma_5]]^{2}\\
& [\gamma_2,[\gamma_2,\gamma_5]][\gamma_2,[\gamma_2,\gamma_3]]^{4}[\gamma_2,[\gamma_1,\gamma_6]]^{-7}[\gamma_2,[\gamma_1,\gamma_5]]^{-3}[\gamma_2,[\gamma_1,\gamma_3]]^{-7}[\gamma_2,[\gamma_1,\gamma_2]]^{5}\\
& [\gamma_1,[\gamma_5,\gamma_6]]^{-1}[\gamma_1,[\gamma_3,\gamma_6]]^{-6}[\gamma_1,[\gamma_3,\gamma_5]]^{-2}[\gamma_1,[\gamma_2,\gamma_6]]^{6}[\gamma_1,[\gamma_2,\gamma_5]]^{2}[\gamma_1,[\gamma_2,\gamma_3]]^{2}\\
& [\gamma_1,[\gamma_1,\gamma_6]][\gamma_1,[\gamma_1,\gamma_3]][\gamma_1,[\gamma_1,\gamma_2]]^{-1}\\
& [\gamma_5,\gamma_6][\gamma_3,\gamma_6]^{-1}[\gamma_3,\gamma_5][\gamma_2,\gamma_6][\gamma_2,\gamma_5]^{-1}[\gamma_2,\gamma_3]^{-3}[\gamma_1,\gamma_6]^{-1}[\gamma_1,\gamma_5][\gamma_1,\gamma_3]^{3}[\gamma_1,\gamma_2]^{-3}\\
& \gamma_6^{2}\gamma_5\gamma_3^{2}\gamma_2^{-2}\gamma_1^{2},
\end{aligned}
$$

$$
\begin{aligned}
\gamma_5 \mapsto\ & [\gamma_6,[\gamma_4,\gamma_6]]^{3}[\gamma_6,[\gamma_3,\gamma_6]]^{-1}[\gamma_6,[\gamma_2,\gamma_6]]^{4}[\gamma_6,[\gamma_1,\gamma_6]]^{-4}[\gamma_4,[\gamma_3,\gamma_6]][\gamma_4,[\gamma_2,\gamma_6]]^{-3}\\
& [\gamma_4,[\gamma_1,\gamma_6]]^{3}[\gamma_3,[\gamma_5,\gamma_6]]^{-1}[\gamma_3,[\gamma_2,\gamma_4]]^{-1}[\gamma_3,[\gamma_1,\gamma_4]][\gamma_2,[\gamma_5,\gamma_6]][\gamma_2,[\gamma_3,\gamma_6]]^{2}\\
& [\gamma_2,[\gamma_2,\gamma_6]]^{-2}[\gamma_2,[\gamma_2,\gamma_4]][\gamma_2,[\gamma_2,\gamma_3]][\gamma_2,[\gamma_1,\gamma_6]]^{3}[\gamma_1,[\gamma_5,\gamma_6]]^{-1}[\gamma_1,[\gamma_3,\gamma_6]]^{-2}\\
& [\gamma_1,[\gamma_2,\gamma_6]]^{2}[\gamma_1,[\gamma_2,\gamma_4]]^{-1}[\gamma_1,[\gamma_2,\gamma_3]]^{-1}[\gamma_1,[\gamma_1,\gamma_6]]^{-3}\\
& [\gamma_5,\gamma_6]^{-1}[\gamma_4,\gamma_6]^{-2}[\gamma_3,\gamma_6]^{-1}[\gamma_3,\gamma_4][\gamma_2,\gamma_6]^{-1}[\gamma_2,\gamma_4]^{-1}[\gamma_2,\gamma_3]^{-1}[\gamma_1,\gamma_6][\gamma_1,\gamma_4][\gamma_1,\gamma_3]\\
& \gamma_6^{-2}\gamma_4\gamma_2\gamma_1^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\gamma_6 \mapsto\ & [\gamma_6,[\gamma_4,\gamma_6]]^{3}[\gamma_6,[\gamma_3,\gamma_6]]^{-3}[\gamma_6,[\gamma_2,\gamma_6]]^{6}[\gamma_6,[\gamma_1,\gamma_6]]^{-6}[\gamma_4,[\gamma_3,\gamma_6]]^{3}[\gamma_4,[\gamma_2,\gamma_6]]^{-5}\\
& [\gamma_4,[\gamma_1,\gamma_6]]^{5}[\gamma_3,[\gamma_3,\gamma_6]]^{-4}[\gamma_3,[\gamma_2,\gamma_6]]^{6}[\gamma_3,[\gamma_2,\gamma_4]]^{-1}[\gamma_3,[\gamma_2,\gamma_3]][\gamma_3,[\gamma_1,\gamma_6]]^{-6}\\
& [\gamma_3,[\gamma_1,\gamma_4]][\gamma_3,[\gamma_1,\gamma_3]]^{-1}[\gamma_2,[\gamma_3,\gamma_6]]^{2}[\gamma_2,[\gamma_3,\gamma_4]][\gamma_2,[\gamma_2,\gamma_6]]^{-4}[\gamma_2,[\gamma_1,\gamma_6]]^{11}\\
& [\gamma_2,[\gamma_1,\gamma_4]]^{-1}[\gamma_2,[\gamma_1,\gamma_3]]^{2}[\gamma_2,[\gamma_1,\gamma_2]]^{-1}[\gamma_1,[\gamma_3,\gamma_6]]^{-2}[\gamma_1,[\gamma_3,\gamma_4]]^{-1}[\gamma_1,[\gamma_2,\gamma_6]]^{2}\\
& [\gamma_1,[\gamma_2,\gamma_4]][\gamma_1,[\gamma_2,\gamma_3]]^{-1}[\gamma_1,[\gamma_1,\gamma_6]]^{-9}[\gamma_1,[\gamma_1,\gamma_3]]^{-1}[\gamma_1,[\gamma_1,\gamma_2]]
\end{aligned}
$$

$$[\gamma_5,\gamma_6]^{-1}[\gamma_4,\gamma_6]^{-2}[\gamma_3,\gamma_6]^{3}[\gamma_2,\gamma_6]^{-5}[\gamma_2,\gamma_3]^{-1}[\gamma_1,\gamma_6]^{5}[\gamma_1,\gamma_3][\gamma_1,\gamma_2]^{-1}$$
$$\gamma_6^{-2}\gamma_4\gamma_3^{-1}\gamma_2^{2}\gamma_1^{-2}.$$

$$f_1 = E_{23}\cdot S_8\cdot S_9^{-2}\cdot T_4^2\cdot S_{11}^{-2}\cdot T_5^2\cdot S_{10}\cdot S_7^{-1}\cdot T_1^{-1}\cdot T_4^2\cdot T_7^{-2}\cdot T_2\cdot T_5\cdot T_8^{-1}$$
$$f_2 = I_{234}^{-1}\cdot I_{235}^{2}\cdot I_{236}^{3}\cdot I_{246}\cdot I_{134}\cdot I_{135}^{-2}\cdot I_{136}^{-3}\cdot I_{146}^{-1}\cdot I_{356}^{-1}$$

$$\begin{aligned}
&\widetilde{\tau}_2(M_{0210}\cdot f_1\cdot f_2)\\
&= T(\gamma_1,\gamma_6,\gamma_4,\gamma_6) + 7T(\gamma_1,\gamma_6,\gamma_3,\gamma_6) - 7T(\gamma_1,\gamma_6,\gamma_2,\gamma_6) + T(\gamma_1,\gamma_5,\gamma_4,\gamma_6) + 3T(\gamma_1,\gamma_5,\gamma_3,\gamma_6)\\
&\quad - 6T(\gamma_1,\gamma_5,\gamma_2,\gamma_6) - 2T(\gamma_1,\gamma_5,\gamma_2,\gamma_5) + 2T(\gamma_1,\gamma_5,\gamma_1,\gamma_6) - T(\gamma_1,\gamma_4,\gamma_5,\gamma_6)\\
&\quad - 9T(\gamma_1,\gamma_4,\gamma_3,\gamma_6) - 3T(\gamma_1,\gamma_4,\gamma_3,\gamma_5) + T(\gamma_1,\gamma_4,\gamma_3,\gamma_4) + 6T(\gamma_1,\gamma_4,\gamma_2,\gamma_6)\\
&\quad + 2T(\gamma_1,\gamma_4,\gamma_2,\gamma_5) - 2T(\gamma_1,\gamma_4,\gamma_1,\gamma_6) - T(\gamma_1,\gamma_4,\gamma_1,\gamma_5) - 3T(\gamma_1,\gamma_3,\gamma_5,\gamma_6)\\
&\quad + 4T(\gamma_1,\gamma_3,\gamma_4,\gamma_6) + T(\gamma_1,\gamma_3,\gamma_4,\gamma_5) - 2T(\gamma_1,\gamma_3,\gamma_3,\gamma_6) - T(\gamma_1,\gamma_3,\gamma_3,\gamma_5) + T(\gamma_1,\gamma_3,\gamma_3,\gamma_4)\\
&\quad + 2T(\gamma_1,\gamma_3,\gamma_2,\gamma_6) - 2T(\gamma_1,\gamma_3,\gamma_2,\gamma_5) - 4T(\gamma_1,\gamma_3,\gamma_2,\gamma_4) + 2T(\gamma_1,\gamma_3,\gamma_1,\gamma_6)\\
&\quad + 3T(\gamma_1,\gamma_3,\gamma_1,\gamma_5) + T(\gamma_1,\gamma_3,\gamma_1,\gamma_4) + 2T(\gamma_1,\gamma_2,\gamma_5,\gamma_6) - 2(\gamma_1,\gamma_2,\gamma_4,\gamma_6) - T(\gamma_1,\gamma_2,\gamma_4,\gamma_5)\\
&\quad + 2T(\gamma_1,\gamma_2,\gamma_3,\gamma_6) + 3T(\gamma_1,\gamma_2,\gamma_3,\gamma_5) - T(\gamma_2,\gamma_6,\gamma_4,\gamma_6) - 7T(\gamma_2,\gamma_6,\gamma_3,\gamma_6)\\
&\quad - T(\gamma_2,\gamma_5,\gamma_4,\gamma_6) - 3T(\gamma_2,\gamma_5,\gamma_3,\gamma_6) + 2T(\gamma_2,\gamma_5,\gamma_2,\gamma_6) + T(\gamma_2,\gamma_4,\gamma_5,\gamma_6)\\
&\quad + 9T(\gamma_2,\gamma_4,\gamma_3,\gamma_6) + 3T(\gamma_2,\gamma_4,\gamma_3,\gamma_5) - T(\gamma_2,\gamma_4,\gamma_3,\gamma_4) - 3T(\gamma_2,\gamma_4,\gamma_2,\gamma_6)\\
&\quad - T(\gamma_2,\gamma_4,\gamma_2,\gamma_5) + 3T(\gamma_2,\gamma_3,\gamma_5,\gamma_6) - 4T(\gamma_2,\gamma_3,\gamma_4,\gamma_6) - T(\gamma_2,\gamma_3,\gamma_4,\gamma_5)\\
&\quad + 2T(\gamma_2,\gamma_3,\gamma_3,\gamma_6) + T(\gamma_2,\gamma_3,\gamma_3,\gamma_5) - T(\gamma_2,\gamma_3,\gamma_3,\gamma_4) - T(\gamma_2,\gamma_3,\gamma_2,\gamma_6) + T(\gamma_2,\gamma_3,\gamma_2,\gamma_5)\\
&\quad + T(\gamma_2,\gamma_3,\gamma_2,\gamma_4) - 3T(\gamma_3,\gamma_6,\gamma_4,\gamma_6) - 2T(\gamma_3,\gamma_5,\gamma_4,\gamma_6) - 6T(\gamma_3,\gamma_5,\gamma_3,\gamma_6)\\
&\quad - T(\gamma_3,\gamma_4,\gamma_5,\gamma_6) + T(\gamma_3,\gamma_4,\gamma_4,\gamma_6) + 2T(\gamma_3,\gamma_4,\gamma_3,\gamma_6) + 2T(\gamma_3,\gamma_4,\gamma_3,\gamma_5)\\
&\quad + 4Y(\gamma_1,\gamma_6) + 2Y(\gamma_1,\gamma_5) - 6Y(\gamma_1,\gamma_3) + 4Y(\gamma_2,\gamma_6) + 2Y(\gamma_2,\gamma_5)\\
&\quad - 6Y(\gamma_2,\gamma_3) - 9Y(\gamma_3,\gamma_6) - 4Y(\gamma_3,\gamma_5) - Y(\gamma_3,\gamma_4)\\
&\mapsto Y(\gamma_1,\gamma_6) + Y(\gamma_2,\gamma_6) + Y(\gamma_3,\gamma_6) \neq 0 \in \mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2.
\end{aligned}$$

12_n0214

An admissible presentation of $\pi_1(M_{0214})$:

$$\begin{gathered}
i_-(\gamma_1)z_2z_3^{-1}z_2^{-1},\quad i_-(\gamma_2)z_2z_1^{-1}z_2,\quad i_-(\gamma_3)z_5^{-1}z_1^{-1}z_2,\quad i_-(\gamma_4)z_6^{-1}z_1z_3^{-1}z_5,\\
i_-(\gamma_5)z_5^{-1}z_4z_3^{-1}z_1z_3^{-1}z_5,\quad i_-(\gamma_6)z_5^{-1}z_4,\quad i_+(\gamma_1)z_2^2z_3^{-1}z_2^{-1},\quad i_+(\gamma_2)z_2^2z_6^{-1},\\
i_+(\gamma_3)z_1^{-1}z_2z_6^{-1},\quad i_+(\gamma_4)z_5z_3^{-1}z_1,\quad i_+(\gamma_5)z_3^{-1}z_5z_3^{-1}z_1,\quad i_+(\gamma_6)z_3^{-1}z_4.
\end{gathered}$$

The action of M_{0214} on N_4:

$$\begin{aligned}
\gamma_1 \mapsto\ & [\gamma_5,[\gamma_1,\gamma_5]][\gamma_4,[\gamma_4,\gamma_5]][\gamma_4,[\gamma_1,\gamma_5]]^{-1}[\gamma_1,[\gamma_4,\gamma_5]][\gamma_1,[\gamma_1,\gamma_5]][\gamma_1,[\gamma_1,\gamma_4]]^{-1}\\
& [\gamma_4,\gamma_5]^{-1}[\gamma_1,\gamma_5]^{-1}[\gamma_1,\gamma_4]\gamma_5\gamma_4^{-1},
\end{aligned}$$

$$\begin{aligned}
\gamma_2 \mapsto\ & [\gamma_5,[\gamma_4,\gamma_5]]^{8}[\gamma_5,[\gamma_3,\gamma_5]]^{3}[\gamma_5,[\gamma_2,\gamma_5]]^{-3}[\gamma_5,[\gamma_1,\gamma_5]]^{-1}[\gamma_4,[\gamma_4,\gamma_5]]^{-1}[\gamma_4,[\gamma_3,\gamma_5]]^{-5}\\
& [\gamma_4,[\gamma_3,\gamma_4]]^{2}[\gamma_4,[\gamma_2,\gamma_5]]^{5}[\gamma_4,[\gamma_2,\gamma_4]]^{-2}[\gamma_4,[\gamma_1,\gamma_5]]^{2}[\gamma_4,[\gamma_1,\gamma_4]]^{-1}[\gamma_3,[\gamma_4,\gamma_5]]^{3}\\
& [\gamma_3,[\gamma_2,\gamma_5]]^{3}[\gamma_3,[\gamma_2,\gamma_4]]^{-3}[\gamma_3,[\gamma_1,\gamma_5]]^{-5}[\gamma_3,[\gamma_1,\gamma_4]]^{5}[\gamma_2,[\gamma_4,\gamma_5]]^{-3}[\gamma_2,[\gamma_3,\gamma_5]]^{-2}\\
& [\gamma_2,[\gamma_3,\gamma_4]]^{2}[\gamma_2,[\gamma_2,\gamma_5]]^{-1}[\gamma_2,[\gamma_2,\gamma_4]][\gamma_2,[\gamma_2,\gamma_3]][\gamma_2,[\gamma_1,\gamma_5]]^{5}[\gamma_2,[\gamma_1,\gamma_4]]^{-5}\\
& [\gamma_2,[\gamma_1,\gamma_3]]^{-2}[\gamma_2,[\gamma_1,\gamma_2]]^{2}[\gamma_1,[\gamma_4,\gamma_5]]^{-1}[\gamma_1,[\gamma_3,\gamma_5]]^{4}[\gamma_1,[\gamma_3,\gamma_4]]^{-4}[\gamma_1,[\gamma_2,\gamma_5]]^{-4}\\
& [\gamma_1,[\gamma_2,\gamma_4]]^{4}[\gamma_1,[\gamma_1,\gamma_5]]^{-1}[\gamma_1,[\gamma_1,\gamma_4]][\gamma_1,[\gamma_1,\gamma_3]][\gamma_1,[\gamma_1,\gamma_2]]^{-1}\\
& [\gamma_4,\gamma_5]^{-3}[\gamma_3,\gamma_5]^{-1}[\gamma_3,\gamma_4][\gamma_2,\gamma_5][\gamma_2,\gamma_4]^{-1}[\gamma_2,\gamma_3]^{-1}[\gamma_1,\gamma_3]^{2}[\gamma_1,\gamma_2]^{-2}\gamma_5^{-3}\gamma_4^{3}\gamma_3\gamma_2^{-1}\gamma_1^{3},
\end{aligned}$$

$$\begin{aligned}
\gamma_3 \mapsto\ & [\gamma_5,[\gamma_4,\gamma_5]]^3[\gamma_5,[\gamma_1,\gamma_5]]^{-3}[\gamma_4,[\gamma_4,\gamma_5]][\gamma_4,[\gamma_3,\gamma_5]]^2[\gamma_4,[\gamma_3,\gamma_4]]^{-1}[\gamma_4,[\gamma_2,\gamma_5]]^{-2}\\
& [\gamma_4,[\gamma_2,\gamma_4]][\gamma_4,[\gamma_1,\gamma_5]]^7[\gamma_4,[\gamma_1,\gamma_4]]^{-2}[\gamma_3,[\gamma_4,\gamma_5]]^{-2}[\gamma_2,[\gamma_4,\gamma_5]]^2[\gamma_2,[\gamma_3,\gamma_4]]\\
& [\gamma_2,[\gamma_2,\gamma_4]]^{-1}[\gamma_1,[\gamma_4,\gamma_5]]^{-7}[\gamma_1,[\gamma_3,\gamma_4]]^{-2}[\gamma_1,[\gamma_2,\gamma_4]]^2[\gamma_1,[\gamma_1,\gamma_5]]^{-1}\\
& [\gamma_4,\gamma_5]^{-2}[\gamma_3,\gamma_4]^{-1}[\gamma_2,\gamma_4][\gamma_1,\gamma_5]^2[\gamma_1,\gamma_4]^{-4}\gamma_4^{-1}\gamma_1,
\end{aligned}$$

$$\begin{aligned}
\gamma_4 \mapsto\ & [\gamma_5,[\gamma_4,\gamma_5]]^{-4}[\gamma_5,[\gamma_3,\gamma_5]]^{-2}[\gamma_5,[\gamma_2,\gamma_5]]^3[\gamma_5,[\gamma_1,\gamma_5]]^{-4}[\gamma_4,[\gamma_4,\gamma_5]]^2[\gamma_4,[\gamma_3,\gamma_5]]^2\\
& [\gamma_4,[\gamma_2,\gamma_5]]^{-4}[\gamma_4,[\gamma_2,\gamma_4]][\gamma_4,[\gamma_1,\gamma_5]]^5[\gamma_4,[\gamma_1,\gamma_4]]^{-1}[\gamma_2,[\gamma_4,\gamma_5]][\gamma_2,[\gamma_3,\gamma_5]]^{-1}\\
& [\gamma_2,[\gamma_2,\gamma_5]][\gamma_2,[\gamma_1,\gamma_5]]^{-5}[\gamma_2,[\gamma_1,\gamma_4]]^5[\gamma_1,[\gamma_4,\gamma_5]]^{-2}[\gamma_1,[\gamma_3,\gamma_5]][\gamma_1,[\gamma_2,\gamma_5]]^{-1}\\
& [\gamma_1,[\gamma_1,\gamma_5]]^9[\gamma_1,[\gamma_1,\gamma_4]]^{-9}\\
& [\gamma_4,\gamma_5][\gamma_3,\gamma_5][\gamma_2,\gamma_5][\gamma_2,\gamma_4]^{-2}[\gamma_1,\gamma_5]^{-4}[\gamma_1,\gamma_4]^5\gamma_5^2\gamma_4^{-1}\gamma_2\gamma_1^{-2},
\end{aligned}$$

$$\begin{aligned}
\gamma_5 \mapsto\ & [\gamma_5,[\gamma_4,\gamma_6]]^{-3}[\gamma_5,[\gamma_4,\gamma_5]]^{-5}[\gamma_5,[\gamma_3,\gamma_6]]^{-2}[\gamma_5,[\gamma_3,\gamma_5]]^{-3}[\gamma_5,[\gamma_2,\gamma_6]]^2[\gamma_5,[\gamma_2,\gamma_5]]^3\\
& [\gamma_5,[\gamma_1,\gamma_6]]^{-3}[\gamma_5,[\gamma_1,\gamma_5]]^{-3}[\gamma_4,[\gamma_5,\gamma_6]]^3[\gamma_4,[\gamma_4,\gamma_5]]^5[\gamma_4,[\gamma_3,\gamma_6]]^2[\gamma_4,[\gamma_3,\gamma_5]]^6\\
& [\gamma_4,[\gamma_3,\gamma_4]]^{-3}[\gamma_4,[\gamma_2,\gamma_6]]^{-2}[\gamma_4,[\gamma_2,\gamma_5]]^{-6}[\gamma_4,[\gamma_2,\gamma_4]]^3[\gamma_4,[\gamma_1,\gamma_6]]^3[\gamma_4,[\gamma_1,\gamma_5]]^6\\
& [\gamma_4,[\gamma_1,\gamma_4]]^{-3}[\gamma_3,[\gamma_5,\gamma_6]]^2[\gamma_3,[\gamma_4,\gamma_6]]^{-2}[\gamma_3,[\gamma_4,\gamma_5]]^{-1}[\gamma_3,[\gamma_3,\gamma_5]]^2[\gamma_3,[\gamma_3,\gamma_4]]^{-2}\\
& [\gamma_3,[\gamma_2,\gamma_5]]^{-2}[\gamma_3,[\gamma_2,\gamma_4]]^2[\gamma_3,[\gamma_1,\gamma_5]]^3[\gamma_3,[\gamma_1,\gamma_4]]^{-3}[\gamma_2,[\gamma_5,\gamma_6]]^{-2}[\gamma_2,[\gamma_4,\gamma_6]]^2\\
& [\gamma_2,[\gamma_4,\gamma_5]][\gamma_2,[\gamma_3,\gamma_5]]^{-1}[\gamma_2,[\gamma_2,\gamma_5]][\gamma_2,[\gamma_1,\gamma_5]]^{-3}[\gamma_2,[\gamma_1,\gamma_4]]^3[\gamma_1,[\gamma_5,\gamma_6]]^3\\
& [\gamma_1,[\gamma_4,\gamma_6]]^{-3}[\gamma_1,[\gamma_4,\gamma_5]]^{-1}[\gamma_1,[\gamma_3,\gamma_5]][\gamma_1,[\gamma_2,\gamma_5]]^{-1}[\gamma_1,[\gamma_1,\gamma_5]]^4[\gamma_1,[\gamma_1,\gamma_4]]^{-4}\\
& [\gamma_5,\gamma_6][\gamma_4,\gamma_5]^{-1}[\gamma_3,\gamma_5]^{-1}[\gamma_3,\gamma_4]^2[\gamma_2,\gamma_5][\gamma_2,\gamma_4]^{-2}[\gamma_1,\gamma_5]^{-2}[\gamma_1,\gamma_4]^3\gamma_6\gamma_5^2\gamma_4^{-2}\gamma_3^{-1}\gamma_2\gamma_1^{-1},
\end{aligned}$$

$$\begin{aligned}
\gamma_6 \mapsto\ & [\gamma_5,[\gamma_4,\gamma_6]]^{-3}[\gamma_5,[\gamma_4,\gamma_5]][\gamma_5,[\gamma_3,\gamma_6]]^{-2}[\gamma_5,[\gamma_3,\gamma_5]][\gamma_5,[\gamma_2,\gamma_6]]^2[\gamma_5,[\gamma_2,\gamma_5]]^{-1}\\
& [\gamma_5,[\gamma_1,\gamma_6]]^{-3}[\gamma_5,[\gamma_1,\gamma_5]]^3[\gamma_4,[\gamma_5,\gamma_6]]^3[\gamma_4,[\gamma_4,\gamma_5]]^4[\gamma_4,[\gamma_3,\gamma_6]]^2[\gamma_4,[\gamma_3,\gamma_5]]^2\\
& [\gamma_4,[\gamma_3,\gamma_4]]^{-3}[\gamma_4,[\gamma_2,\gamma_6]]^{-2}[\gamma_4,[\gamma_2,\gamma_5]]^{-2}[\gamma_4,[\gamma_2,\gamma_4]]^3[\gamma_4,[\gamma_1,\gamma_6]]^3[\gamma_4,[\gamma_1,\gamma_4]]^{-3}\\
& [\gamma_3,[\gamma_5,\gamma_6]]^2[\gamma_3,[\gamma_4,\gamma_6]]^{-2}[\gamma_3,[\gamma_4,\gamma_5]][\gamma_3,[\gamma_3,\gamma_5]]^2[\gamma_3,[\gamma_3,\gamma_4]]^{-2}[\gamma_3,[\gamma_2,\gamma_5]]^{-2}\\
& [\gamma_3,[\gamma_2,\gamma_4]]^2[\gamma_3,[\gamma_1,\gamma_5]]^3[\gamma_3,[\gamma_1,\gamma_4]]^{-3}[\gamma_2,[\gamma_5,\gamma_6]]^{-2}[\gamma_2,[\gamma_4,\gamma_6]]^2[\gamma_2,[\gamma_4,\gamma_5]]^{-1}\\
& [\gamma_2,[\gamma_1,\gamma_5]]^{-3}[\gamma_2,[\gamma_1,\gamma_4]]^3[\gamma_1,[\gamma_5,\gamma_6]]^3[\gamma_1,[\gamma_4,\gamma_6]]^{-3}[\gamma_1,[\gamma_4,\gamma_5]]^3[\gamma_1,[\gamma_1,\gamma_5]]^4\\
& [\gamma_1,[\gamma_1,\gamma_4]]^{-4}\\
& [\gamma_4,\gamma_5]^{-3}[\gamma_3,\gamma_5]^{-2}[\gamma_3,\gamma_4]^2[\gamma_2,\gamma_5]^2[\gamma_2,\gamma_4]^{-2}[\gamma_1,\gamma_5]^{-3}[\gamma_1,\gamma_4]^3\gamma_6\gamma_5\gamma_4^{-2}\gamma_3^{-1}\gamma_2\gamma_1^{-1}.
\end{aligned}$$

$$\begin{aligned}
f_1 &= S_6^{-2}\cdot S_{12}\cdot T_2^2\cdot T_5^4\cdot T_8^{-2}\cdot S_{13}\cdot S_{14}\cdot T_1^{-1}\cdot T_4^{-1}\cdot T_7\cdot S_{15}\cdot S_{16}\cdot E_{23}\\
&\quad \cdot T_2^2\cdot T_4\cdot T_8^{-1}\cdot T_5\cdot T_4^2\cdot T_5\cdot T_4^2\cdot U\\
f_2 &= I_{124}^{-5}\cdot I_{125}^2\cdot I_{234}\cdot I_{245}^4\cdot I_{246}\cdot I_{134}^{-1}\cdot I_{135}^{-1}\cdot I_{145}^{-5}\cdot I_{345}^{-1}\cdot I_{456}^2
\end{aligned}$$

$$\begin{aligned}
&\widetilde{\tau}_2(M_{0214}\cdot f_1\cdot f_2)\\
&= -T(\gamma_1,\gamma_5,\gamma_5,\gamma_6)+8T(\gamma_1,\gamma_5,\gamma_4,\gamma_5)-T(\gamma_1,\gamma_5,\gamma_2,\gamma_5)+4T(\gamma_1,\gamma_4,\gamma_5,\gamma_6)\\
&\quad +T(\gamma_1,\gamma_4,\gamma_4,\gamma_6)-2T(\gamma_1,\gamma_4,\gamma_4,\gamma_5)+7T(\gamma_1,\gamma_4,\gamma_3,\gamma_5)-7T(\gamma_1,\gamma_4,\gamma_3,\gamma_4)\\
&\quad -2T(\gamma_1,\gamma_4,\gamma_2,\gamma_6)-7T(\gamma_1,\gamma_4,\gamma_2,\gamma_5)+9T(\gamma_1,\gamma_4,\gamma_1,\gamma_5)+T(\gamma_1,\gamma_3,\gamma_5,\gamma_6)\\
&\quad -4T(\gamma_1,\gamma_3,\gamma_4,\gamma_5)+T(\gamma_1,\gamma_3,\gamma_1,\gamma_5)-T(\gamma_1,\gamma_2,\gamma_4,\gamma_6)-3T(\gamma_1,\gamma_2,\gamma_4,\gamma_5)\\
&\quad +T(\gamma_1,\gamma_2,\gamma_3,\gamma_5)-5T(\gamma_1,\gamma_2,\gamma_3,\gamma_4)+T(\gamma_1,\gamma_2,\gamma_2,\gamma_5)-T(\gamma_1,\gamma_2,\gamma_2,\gamma_4)-3T(\gamma_1,\gamma_2,\gamma_1,\gamma_5)\\
&\quad -6T(\gamma_1,\gamma_2,\gamma_1,\gamma_4)-T(\gamma_2,\gamma_6,\gamma_4,\gamma_6)-T(\gamma_2,\gamma_5,\gamma_5,\gamma_6)-4T(\gamma_2,\gamma_5,\gamma_4,\gamma_6)\\
&\quad -8T(\gamma_2,\gamma_5,\gamma_4,\gamma_5)-T(\gamma_2,\gamma_5,\gamma_3,\gamma_5)+T(\gamma_2,\gamma_4,\gamma_4,\gamma_6)+2T(\gamma_2,\gamma_4,\gamma_4,\gamma_5)-T(\gamma_2,\gamma_4,\gamma_3,\gamma_6)\\
&\quad -5T(\gamma_2,\gamma_4,\gamma_3,\gamma_5)+2T(\gamma_2,\gamma_4,\gamma_3,\gamma_4)-T(\gamma_2,\gamma_4,\gamma_2,\gamma_6)-3T(\gamma_2,\gamma_4,\gamma_2,\gamma_5)\\
&\quad +T(\gamma_2,\gamma_3,\gamma_4,\gamma_6)+4T(\gamma_2,\gamma_3,\gamma_4,\gamma_5)+T(\gamma_3,\gamma_5,\gamma_5,\gamma_6)+2T(\gamma_3,\gamma_5,\gamma_4,\gamma_5)\\
&\quad -T(\gamma_3,\gamma_4,\gamma_5,\gamma_6)-T(\gamma_3,\gamma_4,\gamma_4,\gamma_6)-8T(\gamma_3,\gamma_4,\gamma_4,\gamma_5)+3T(\gamma_4,\gamma_5,\gamma_5,\gamma_6)\\
&\quad -10Y(\gamma_1,\gamma_5)-4Y(\gamma_1,\gamma_4)-3Y(\gamma_1,\gamma_2)+Y(\gamma_2,\gamma_5)-7Y(\gamma_2,\gamma_4)
\end{aligned}$$

$$+Y(\gamma_3,\gamma_5)-5Y(\gamma_3,\gamma_4)+Y(\gamma_4,\gamma_6)-14Y(\gamma_4,\gamma_5)-Y(\gamma_5,\gamma_6)$$
$$\mapsto Y(\gamma_1,\gamma_4)+Y(\gamma_1,\gamma_5)+Y(\gamma_2,\gamma_5)\neq 0\in \mathfrak{h}_{g,1}(2)/\mathrm{Image}\,\tau_2.$$

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DEPARTMENT OF MATHEMATICS, TOKYO UNIVERSITY OF AGRICULTURE AND TECHNOLOGY, 2-24-16 NAKA-CHO, KOGANEI, TOKYO 184-8588, JAPAN
E-mail address: goda@cc.tuat.ac.jp

DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, 2-12-1 OH-OKAYAMA, MEGURO-KU, TOKYO 152-8552, JAPAN
E-mail address: sakasai@math.titech.ac.jp