数理解析研究所講究録
第 1747 巻 2011 年 23-34

CENTRALIZERS IN 3-MANIFOLD GROUPS

STEFAN FRIEDL

1. Introduction

In this paper we will study centralizers in fundamental groups of 3-manifolds. By a 3–manifold we will always mean a compact, orientable, connected, irreducible 3-manifold with empty or toroidal boundary.

Let π be a group. The *centralizer* of an element $g \in \pi$ is defined to be the subgroup

$$C_\pi(g) := \{h \in \pi \mid gh = hg\}.$$

Determining centralizers is an important step towards understanding a group. The goal of this note is to give a new proof of the following theorem.

Theorem 1.1. *Let N be a 3-manifold. We write $\pi = \pi_1(N)$. Let $g \in \pi$. If $C_\pi(g)$ is non-cyclic, then one of the following holds:*

(1) *there exists a JSJ torus or a boundary torus T and $h \in \pi$ such that $g \in h\pi_1(T)h^{-1}$ and such that*
$$C_\pi(g) = h\pi_1(T)h^{-1},$$
(2) *there exists a Seifert fibered component M and $h \in \pi$ such that $g \in h\pi_1(M)h^{-1}$ and such that*
$$C_\pi(g) = hC_{\pi_1(M)}(h^{-1}gh)h^{-1}.$$

If N is Seifert fibered, then the theorem holds trivially, and if N is hyperbolic, then it follows from well-known properties of hyperbolic 3-manifold groups (we refer to Section 3.1 for details). If N is neither Seifert fibered nor hyperbolic, then by the Geometrization Theorem N has a non-trivial JSJ decomposition, in particular N is Haken, and in that case the theorem was proved by Jaco and Shalen [8, Theorem VI.1.6] and independently by Johannson [9, Proposition 32.9]. In this note we will give an alternative proof of Theorem 1.1 for 3-manifolds with non-trivial JSJ decomposition using the Geometrization Theorem proved by Perelman. Our proof involves basic facts about fundamental groups of Seifert fibered spaces and hyperbolic 3-manifolds and it consists of a careful study of the fundamental group of the graph of groups corresponding to the JSJ decomposition.

In order to determine centralizers of 3-manifolds it thus suffices to understand centralizers of Seifert fibered spaces. For the reader's convenience we recall the results of Jaco–Shalen and Johannson. Let N be a Seifert fibered 3-manifold with a given Seifert fiber structure. Then there exists a projection map $p\colon N \to B$ where B is the base orbifold. We denote by $B' \to B$ the orientation cover, note that this is either the identity or a 2-fold cover. Following [8] we refer to $p_*^{-1}(\pi_1(B'))$ as the *canonical subgroup* of $\pi_1(N)$. If f is a regular fiber of the Seifert fibration, then we refer to the subgroup of $\pi_1(N)$ generated by f as the *fiber subgroup*. Recall that if N is non-spherical, then the fiber subgroup is

Received February 1, 2011.

infinite cyclic and normal. (Note that the fact that the fiber subgroup is normal implies in particular that it is well-defined, and not just up to conjugacy.)

Remark 1.2. Note that the definition of the canonical subgroup and of the fiber subgroup depend on the Seifert fiber structure. By [13, Theorem 3.8] (see also [12] and [8, II.4.11]) a Seifert fibered 3-manifold N admits a unique Seifert fiber structure unless N is either covered by S^3, $S^2 \times \mathbb{R}$, or the 3-torus, or $N = S^1 \times D^2$ or if N is an I-bundle over the torus or the Klein bottle.

The following theorem, together with Theorem 1.1, now classifies centralizers of non-spherical 3-manifolds.

Theorem 1.3. *Let N be a non-spherical Seifert fibered 3-manifold with a given Seifert fiber structure. Let $g \in \pi = \pi_1(N)$ be a non-trivial element. Then the following hold:*

(1) *if g lies in the fiber group, then $C_\pi(g)$ equals the canonical subgroup,*
(2) *if g does not lie in the fiber group, then the intersection of $C_\pi(g)$ with the canonical subgroup is abelian, in particular $C_\pi(g)$ admits an abelian subgroup of index at most two,*
(3) *if g does not lie in the canonical subgroup, then $C_\pi(g)$ is infinite cyclic.*

The first statement is [8, Proposition II.4.5]. The second and the third statement follow from [8, Proposition II.4.7]. Using Theorems 1.1 and 1.3 one can now immediately obtain results on root structures and the divisibility of elements in 3-manifold groups. We refer to [1, Section 4] for details.

Note that given a group π and an element $g \in \pi$ the set of conjugacy classes of g is in a canonical bijection to the set $\pi/C_g(\pi)$. We thus obtain the following corollary to Theorem 1.1.

Theorem 1.4. *Let N be a 3-manifold. If N is not a Seifert fibered 3-manifold, then the number of conjugacy classes is infinite for any $g \in \pi_1(N)$.*

This result was first obtained by de la Harpe and Préaux [5] using different methods. They consider a slightly larger class of 3–manifolds, but extending our approach to the class of 3–manifolds considered in [5] poses no problems. We also refer to [5] for an application of this result to the von Neumann algebra $W^*_\lambda(\pi_1(N))$.

Acknowledgment. We would like to thank Matthias Aschenbrenner, Pierre de la Harpe, Saul Schleimer, Stephan Tillmann and Henry Wilton for helpful conversations.

2. Graphs of groups

In this section we summarize some basic definitions and facts concerning graphs of groups and their fundamental groups. We refer to [2, 3, 14] for missing details.

2.1. **Graphs.** A *graph* $\mathcal{Y}$ consists of a set $V = V(\mathcal{Y})$ of *vertices* and a set $E = E(\mathcal{Y})$ of *edges*, and two maps $E \to V \times V\colon e \mapsto (o(e), t(e))$ and $E \to E\colon e \mapsto \overline{e}$, subject to the following condition: for each $e \in E$ we have $\overline{\overline{e}} = e$, $\overline{e} \neq e$, and $o(e) = t(\overline{e})$. We sometimes also denote $\overline{e}$ by e^{-1}. Throughout this paper, all graphs are understood to be connected and finite (i.e., their vertex sets and edge sets are finite).

2.2. **The fundamental group of a graph of groups.** Let $\mathcal{Y}$ be a graph. A *graph $\mathcal{G}$ of groups based on $\mathcal{Y}$* consists of families $\{G_v\}_{v\in V(\mathcal{Y})}$ and $\{G_e\}_{e\in E(\mathcal{Y})}$ of groups satisfying $G_e = G_{\overline{e}}$ for every $e \in E(\mathcal{Y})$, together with a family $\{\varphi_e\}_{e\in E(\mathcal{Y})}$ of monomorphisms $\varphi_e\colon G_e \to G_{t(e)}$ $(e \in E(\mathcal{Y}))$. We will refer to $\mathcal{Y}$ as the *underlying graph* of $\mathcal{G}$.

Let $\mathcal{G}$ be a graph of groups based on a graph $\mathcal{Y}$. We recall the construction of the fundamental group $G = \pi_1(\mathcal{G})$ of $\mathcal{G}$ from [14, I.5.1]. First, consider the *path group* $\pi(\mathcal{G})$ which is generated by the groups G_v $(v \in V(\mathcal{Y}))$ and the elements $e \in E(\mathcal{Y})$ subject to the relations

$$e\varphi_e(g)\overline{e} = \varphi_{\overline{e}}(g) \qquad (e \in E(\mathcal{Y}), g \in G_e).$$

By a *path in $\mathcal{Y}$* from a vertex v to a vertex w we mean a sequence $(e_1, e_2, \dots, e_n)$ where $o(e_1) = v, t(e_i) = o(e_{i+1}), i = 1, \dots, n-1$ and $t(e_n) = w$.

By a *path in $\mathcal{G}$* from a vertex v to a vertex w we mean a sequence

$$(g_0, e_1, g_1, e_2, \dots, e_n, g_n),$$

of elements in E where $(e_1, \dots, e_n)$ is a path of length n in $\mathcal{Y}$ from v to w and where $g_0 \in G_v$ and where $g_i \in G_{t(e_i)}$ for $i = 1, \dots, n$. We write $l(\gamma) = n$ and call it the length of γ. We say that the path γ *represents* the element

$$g = g_0e_1g_1e_2\cdots e_ng_n$$

of $\pi(\mathcal{G})$.

Let now w be a fixed vertex of $\mathcal{Y}$. We will refer to a path from w to w as a *loop based at w*. The fundamental group $\pi_1(\mathcal{G}, w)$ of $\mathcal{G}$ (with base point w) is defined to be the subgroup of $\pi(\mathcal{G})$ consisting of elements represented by loops based at w. If $w' \in V(\mathcal{Y})$ is another base point, and g is an element of $\pi(\mathcal{G})$ represented by a path from w' to w, then $\pi_1(\mathcal{G}, w') \to \pi_1(\mathcal{G}, w)\colon t \mapsto g^{-1}tg$ is an isomorphism. By abuse of notation we write $\pi_1(\mathcal{G})$ to denote $\pi_1(\mathcal{G}, w)$ if the particular choice of base point is irrelevant.

Now let $v \in V$. Pick a path g from v to w. Then the map $G_v \to \pi_1(\mathcal{G}, w)$ given by $t \mapsto g^{-1}tg$ defines a group morphism which is injective (see again [14, I.5.2, Corollary 1]). In particular the vertex groups define subgroups of $\pi_1(\mathcal{G}, w)$ which are well-defined up to conjugation. Given a graph of groups $\mathcal{G}$ and a base vertex w it is always understood that for each vertex v we picked once and for all a path from v to w.

We will later on make use of the following operations on paths. Given a path p in $\mathcal{G}$ from v_1 to v_2 we write $o(p) = v_1$ and $t(p) = v_2$. Given two paths

$$\begin{aligned} p &:= (g_0, e_1, g_1, e_2, \dots, e_n, g_n), \text{ and} \\ q &:= (h_0, f_1, h_1, f_2, \dots, f_m, h_m), \end{aligned}$$

with $t(p) = o(q)$ we define

$$p * q := (g_0, e_1, g_1, e_2, \dots, e_n, g_n \cdot h_0, f_1, h_1, f_2, \dots, f_m, h_m)$$

which is a path from $o(p)$ to $t(q)$. Furthermore, given a path

$$p := (g_0, e_1, g_1, e_2, \dots, e_n, g_n)$$

we define the inverse path to be

$$p^{-1} := (g_n^{-1}, \overline{e_n}, \dots, g_1^{-1}, \overline{e_1}, g_0^{-1}).$$

Note that p^{-1} is a path from $t(p)$ to $o(p)$.

2.3. **Reduced paths.** A path $(g_0, e_1, g_1, e_2, \dots, e_n, g_n)$ in $\mathcal{G}$ is *reduced* if it satisfies one of the following conditions:

(1) $n = 0$, or
(2) $n > 0$ and $g_i \notin \varphi_{e_i}(G_{e_i})$ for each index i such that $e_{i+1} = \overline{e_i}$.

Given $g \in \pi(\mathcal{G})$ we define its length $l(g)$ to be the length of a reduced path representing it. Note that this is well-defined (see [14, p. 4]), i.e. any g is represented by a reduced path and the definition is independent of the choice of the reduced path. Also note that

$$l(g) = \min\{l(p) \mid p \text{ a path which represents } g\}.$$

Note that $l(g) = 0$ if and only if g lies in G_v for some $v \in V$.

We say that $s = (g_0, e_1, g_1, e_2, \dots, e_n, g_n)$ is *cyclically reduced* if s is reduced and if one of the following holds:

(1) $n = 0$, or
(2) $e_1 \neq \overline{e_n}$, or
(3) $e_1 = \overline{e_n}$ but $g_n g_0$ is not conjugate to an element in $\mathrm{Im}(\varphi_{e_n})$.

Note that a reduced loop $s = (g_0, e_1, g_1, e_2, \dots, e_n, g_n)$ is cyclically reduced if and only if the element it represents has minimal length in its conjugacy class in the path group $\pi(\mathcal{G})$.

We say that $g \in \pi_1(\mathcal{G}, w)$ is *cyclically reduced* if there exists a cyclically reduced loop which represents it. It is straightforward to see that g is cyclically reduced if and only if any reduced loop representing it is cyclically reduced. Also note that if g is cyclically reduced, then $l(g^n) = n \cdot l(g)$.

Any element g of the path groups $\pi(\mathcal{G})$ is conjugate in $\pi(\mathcal{G})$ to a cyclically reduced element s, we can thus define $cl(g) = l(s)$. Note that this is independent of the choice of s. Note that if g is cyclically reduced, then a straightforward argument shows that $l(g^n) = n \cdot l(g)$. In particular given any g we have $cl(g^n) = n \cdot cl(g)$.

3. Fundamental groups of 3-manifolds

In the next two sections we cover properties of fundamental groups of hyperbolic 3-manifold groups and of Seifert fibered spaces, before we return to the study of 3-manifold groups in general.

3.1. **Fundamental groups of hyperbolic 3-manifolds.** Let N be a 3-manifold. We say that N is hyperbolic if the interior admits a complete metric of finite volume and constant sectional curvature equal to -1.

Throughout this section we write

$$U := \left\{ \begin{pmatrix} \varepsilon & a \\ 0 & \varepsilon \end{pmatrix} \text{ with } \varepsilon \in \{-1, 1\} \text{ and } a \in \mathbb{C} \right\} \subset \mathrm{SL}(2, \mathbb{C}).$$

Note that U is an abelian subgroup of $\mathrm{SL}(2,\mathbb{C})$. Recall that $A \in \mathrm{SL}(2,\mathbb{C})$ is called *parabolic* if it is conjugate to an element in U. We say that A is *loxodromic* if A is diagonalizable with eigenvalues λ, λ^{-1} such that $|\lambda| > 1$. We recall the following well known proposition.

Proposition 3.1. *Let N be a hyperbolic 3-manifold. Then the following hold:*

(1) *There exists a faithful discrete representation $\rho\colon \pi_1(N) \to \mathrm{SL}(2,\mathbb{C})$.*

(2) *Let* $g \in \pi_1(N)$*, then* $\rho(g)$ *is either parabolic or loxodromic.*
(3) *An element* $g \in \pi_1(N)$ *is conjugate to an element in a boundary component if and only if* $\rho(g)$ *is parabolic.*
(4) *Let* T *be a boundary torus, then there exists a matrix* $P \in \mathrm{SL}(2,\mathbb{C})$ *such that* $P\rho(\pi_1(T))P^{-1} \subset U$*.*
(5) *Let* $g \in \pi_1(N)$*. Then* $C_g(\pi_1(N))$ *is either infinite cyclic or a free abelian group of rank two. The latter case occurs precisely when* g *is conjugate to an element in a boundary component* T *and in that case* $C_g(\pi_1(N))$ *is a conjugate of* $\pi_1(T)$*.*

We include the proof of the proposition for completeness' sake.

Proof. (1) A hyperbolic 3-manifold N admits a faithful discrete representation $\pi_1(N) \to \mathrm{Isom}(\mathbb{H}^3) = \mathrm{PSL}(2,\mathbb{C})$. Thurston (see [15, Section 1.6]) showed that this representation lifts to a faithful discrete representation $\pi_1(N) \to \mathrm{SL}(2,\mathbb{C})$.
(2) This follows immediately from considering the Jordan transform of $\rho(g)$ and from the fact that the infinite cyclic group generated by $\rho(g)$ is discrete in $\mathrm{SL}(2,\mathbb{C})$.
(3) This is well-known, see e.g. [10, p. 115].
(4) This statement follows easily from the fact that $\pi_1(T) \subset \mathrm{SL}(2,\mathbb{C})$ is a discrete subgroup isomorphic to $\mathbb{Z}^2$.
(5) By (1) we can view $\pi = \pi_1(N)$ as a discrete, torsion-free subgroup of $\mathrm{SL}(2,\mathbb{C})$. Note that the centralizer of any non-trivial matrix in $\mathrm{SL}(2,\mathbb{C})$ is abelian (this can be seen easily using the Jordan normal form of such a matrix). Now let $g \in \pi \subset \mathrm{SL}(2,\mathbb{C})$ be non-trivial. Since π is torsion-free and discrete in $\mathrm{SL}(2,\mathbb{C})$ it follows easily that $C_\pi(g)$ is in fact either infinite cyclic or a free abelian group of rank two. It now follows from [16, Proposition 5.4.4] (see also [13, Corollary 4.6] for the closed case) that there exists a boundary component S and $h \in \pi_1(N)$ such that

$$C_\pi(g) = h\pi_1(S)h^{-1}.$$

□

Given a group π we say that an element g is *divisible by an integer* n if there exists an $h \in \pi$ with $g = h^n$. We say g is *infinitely divisible* if g is divisible by infinitely many integers. The following lemma is an immediate consequences of Proposition 3.1 (5).

Lemma 3.2. *Let* $\pi \subset \mathrm{SL}(2,\mathbb{C})$ *be a discrete torsion-free group. Then* π *does not contain any non-trivial elements which are infinitely divisible.*

Let π be a group. We say that a subgroup $H \subset \pi$ is *division closed* if for any $g \in \pi$ and $n > 0$ with $g^n \in H$ the element g already lies in H. The following lemma is an immediate consequence of Proposition 3.1 (2) and (5) and from the observation that $A \subset \mathrm{SL}(2,\mathbb{C})$ is parabolic (respectively loxodromic) if and only if a non-trivial power of A is parabolic (respectively loxodromic).

Lemma 3.3. *Let* N *be a 3-manifold such that the interior of* N *is a hyperbolic 3-manifold of finite volume. Let* T *be a boundary component of* N*. Then* $\pi_1(T) \subset \pi_1(N)$ *is division closed.*

Let π be a group. We say that a subgroup H is *malnormal* if $gHg^{-1} \cap H$ is trivial for any $g \notin H$. The following lemma is well-known.

Lemma 3.4. *Let N be a hyperbolic 3-manifold.*

(1) *Let T be a boundary torus. Then $\pi_1(T) \subset \pi_1(N)$ is malnormal.*
(2) *Let T_1 and T_2 be distinct boundary tori. Then for any $g \in \pi_1(N)$ we have $\pi_1(T_1) \cap g\pi_1(T_2)g^{-1} = \{e\}$.*

3.2. **Fundamental groups of Seifert fibered manifolds.** Let N be a Seifert fibered space with regular fiber c. First note that if T is a boundary torus, then the Seifert fibration restricted to T induces a product structure. It follows that $c \in \pi_1(T)$ and that c is indivisible in $\pi_1(T) \cong \mathbb{Z}^2$.

The following results summarize the key properties of fundamental groups of Seifert fibered spaces which are relevant to our discussion.

Theorem 3.5. *Let N be a Seifert fibered 3-manifold with regular fiber c. Then there exists an $s \in \mathbb{N}$ with the following property: If T is a boundary component, and if $g \notin \pi_1(T)$ but some power of g lies in $\pi_1(T)$, then there exists $d \leq s$ such that $g^d = c$ or $g^d = c^{-1}$.*

Proof. Let N be a Seifert fibered 3-manifold with boundary. Let s be the maximum order of a singular fiber of the fibration. Let T be a boundary component, and let $g \notin \pi_1(T)$ such that some power of g lies in $\pi_1(T)$. We denote by $p : N \to B$ the projection to the base orbifold. We denote by b the boundary curve of B corresponding to T. Note that $p(g) \notin \langle b \rangle$ but a power of $p(g)$ lies in $\langle b \rangle$. It follows easily from [8, Remark II.3.1] that $p(g)$ is of finite order. In particular g corresponds to a singular fiber, and then it follows from the definition of s that there exists a $d \leq s$ such that $g^d = c$ or $g^d = c^{-1}$. □

Lemma 3.6. *Let N be a Seifert fibered 3-manifold with regular fiber c and let T be a boundary component. Let $g \in \pi_1(T)$ which is not a power of c, then $C_g(\pi_1(N)) = \pi_1(T)$.*

Proof. We denote by $p : N \to B$ the projection to the base orbifold. Note that $p(g) \in \pi_1(B)$ is non-trivial. It follows easily from [8, Remark II.3.1] that $C_{p(g)}(\pi_1(B))$ is the group generated by the boundary curve of N corresponding to T. It follows easily that $C_g(\pi_1(N)) = \pi_1(T)$. □

The following lemma is also well-known. It can be proved in a similar fashion as Lemma 3.6 by considering the equivalent problem in the fundamental group of the base manifold.

Lemma 3.7. *Let N be a Seifert fibered 3-manifold. Denote by $c \in \pi_1(N)$ the element represented by a regular fiber.*

(1) *Let T be a boundary torus and $g \in \pi_1(N) \setminus \pi_1(T)$, then $\pi_1(T) \cap g\pi_1(T)g^{-1} = \langle c \rangle$.*
(2) *Let T_1 and T_2 be distinct boundary tori. Then for any $g \in \pi_1(N)$ we have $\pi_1(T_1) \cap g\pi_1(T_2)g^{-1} = \langle c \rangle$.*

We conclude with the following lemma.

Lemma 3.8. *Let N be a non-spherical Seifert fibered manifold. Then $\pi_1(N)$ does not contain non-trivial elements which are infinitely divisible.*

Proof. Let N be a Seifert fibered manifold. Then there exists a finite cover N' which is an S^1-bundle over a surface S (see e.g. [7, p. 391] for details). We write $\Gamma = \pi_1(S)$, $\pi = \pi_1(N)$ and $\pi' = \pi_1(N')$. If N is non-spherical then the long exact sequence in homotopy implies that there exists a short exact sequence

$$1 \to \mathbb{Z} \to \pi' \to \Gamma \to 1.$$

Since $\mathbb{Z}$ and Γ are well-known not to admit any non-trivial infinitely divisible elements, it follows easily that π' does not admit a non-trivial infinitely divisible element. We write $n = [\pi : \pi']$. Since N is non-spherical we know that π is torsion-free. Note that if $g \in \pi$ is non-trivial, then g^n lies in π' and it is also non-trivial. It is now easy to see that π can not admit a non-trivial infinitely divisible element either. □

3.3. **3-manifolds and graphs of groups.** In this section we recall the well-known interpretation of 3-manifold groups as the fundamental group of a graph of groups. Let N be an irreducible, closed, oriented 3-manifold. Recall that the JSJ tori are a minimal collection $\{T_1, \dots, T_k\}$ of tori such that the complements of the tori are either atoroidal or Seifert fibered.

We denote by $\mathcal{G}(N)$ the corresponding JSJ graph, i.e. the vertex set $V = V(\mathcal{G})$ of $\mathcal{G}$ consists of the set of components of N cut along $T_1, \dots, T_k$ pieces and the set $E = E(\mathcal{G})$ of (unoriented) edges consists of the set of JSJ tori $T_1, \dots, T_k$. We sometimes denote the JSJ tori by $T_e, e \in E$ and we denote the components of N cut along $\cup_{e \in E} T_e$ by $N_v, v \in V$. We equip each T_e with an orientation, we thus obtain two canonical embeddings $i_\pm$ of T_e into N cut along T_e. We then denote by $o(e) \in V$ the unique vertex with $i_-(T_e) \in N_{i(e)}$ and we denote by $t(e) \in V$ the unique vertex with $i_+(T_e) \in N_{f(e)}$.

Suppose that N has a non-trivial JSJ decomposition. Then given a Seifert fibered component N_v of the JSJ decomposition of N we denote by $c_v \in \pi_1(N_v)$ the group element defined by a corresponding regular fiber. Note that c_v is well-defined up to inversion (see [17, Lemma 1] or [4]).

We conclude this section with the following theorem.

Theorem 3.9. *Let N be a closed, oriented 3-manifold. Denote by $\mathcal{G} = \mathcal{G}(N)$ the corresponding JSJ graph. If e is an edge such that $o(e)$ and $t(e)$ correspond to Seifert fibered spaces, then $\varphi_e^{-1}(c_{t(e)}) \neq c_{o(e)}^{\pm 1}$.*

Proof. If $\varphi_e^{-1}(c_{t(e)})$ was equal to $c_{o(e)}^{\pm 1}$, then $N_{o(e)}$ and $N_{t(e)}$ would have Seifert fiber structures which (after an isotopy) match along the edge torus. But this contradicts the minimality of the JSJ decomposition. □

4. Proof of the main results

4.1. **Divisibility in 3-manifold groups.** We will first prove the following theorem.

Theorem 4.1. *Let N be a 3-manifold. If N is not spherical, then $\pi_1(N)$ does not contain any non-trivial elements which are infinitely divisible.*

Proof. Let N be a non-spherical 3-manifold and let $x \in \pi_1(N)$ be a non-trivial element. Since the statement of theorem is independent of the choice of base point and conjugation we can without loss of generality assume that $l(x) = cl(x)$. We write $l = l(x)$.

First suppose that $l > 0$. Suppose we have $y \in \pi_1(N)$ and n such that $y^n = x$. Note that $0 < cl(x) = cl(y^n) = n \cdot cl(y)$. It now follows immediately that $n \leq l = cl(x)$.

Now suppose that $l = 0$. Note that this means that x lies in a vertex group $\pi_1(N_w)$. We now define

$$d := \max\{n \in \mathbb{N} \mid x = y^n \text{ for some } y \in \pi_1(N_w)\}.$$

Note that $d < \infty$ by Lemmas 3.2 and 3.8. Furthermore, given a Seifert fibered component N_v we define

$$s_v := \text{maximum of the orders of the singular fibers of } N_v.$$

Finally we define s to be the maximum over all s_v. If there are no Seifert fibered components, then we set $s = 1$. The following claim now implies the theorem.

Claim 4.2. *If there exists $y \in \pi_1(N)$ and $n \in \mathbb{N}$ with $y^n = x$, then $n \le ds$.*

Suppose we have $y \in \pi_1(N)$ and n such that $y^n = x$. Note that $0 = l(x) = cl(x) = cl(y^n) = n \cdot cl(y)$. It now follows that $cl(y) = 0$. If $l(y) = 0$, then $y \in \pi_1(N_w)$, hence the conclusion holds trivially by the definition of d. Now suppose that $l(y) > 0$. Then there exists a reduced path $p = (g_0, e_1, g_1, \dots, e_l, g_l)$ from w to a vertex v and $z \in \pi_1(N_v)$ such that y is represented by $p * z * p^{-1}$. Among all such pairs (p, z) we pick a pair which minimizes the length of p.

Since p is minimal and $l(p) > 0$ we see that $g_l z g_l^{-1} \notin \mathrm{Im}(\varphi_{e_l})$. On the other hand $p * z^n * p^{-1}$ represents $y^n = x$, hence this path is reduced, which implies that $g_l z^n g_l^{-1} \in \mathrm{Im}(\varphi_{e_l})$. It follows that $\mathrm{Im}(\varphi_{e_l})$ is not division closed, using Lemma 3.3 we conclude that N_v is Seifert fibered.

We denote by c_v the regular fiber of N_v. Recall that by Theorem 3.5 there exists $r | s_v$ with $g_l z^r g_l^{-1} = c_v$. It also follows from Theorem 3.5 that $g_l z^n g_l^{-1} = c_v^m \in \mathrm{Im}(\varphi_{e_l})$ for some m. Note that $n = mr$.

We can now apply Lemmas 3.4 and 3.7, Theorem 3.9 and the fact that p is reduced to conclude that

$$(g_0, e_1, g_1, \dots, e_{l-1}, g_{l-1}\varphi_{e_l}^{-1}(c_v^m)g_{l-1}^{-1}, e_{l-1}^{-1}, \dots, g_1^{-1}, e_1^{-1}, g_0^{-1})$$

is reduced. It follows that $l = 1$. Note that

$$x = g_0 \varphi_{e_1}^{-1}(c_v^m) g_0^{-1} = (g_0 \varphi_{e_1}^{-1}(c_v) g_0^{-1})^m.$$

It follows that $m \le d$. We also have $r \le s_v \le s$. We now conclude that $n = mr \le ds$. □

4.2. Commuting elements in 3-manifold groups.

Theorem 4.3. *Let N be a 3-manifold. Let $x, y \in \pi_1(N)$ with $x = yxy^{-1}$. Then one of the following holds:*

1. *x and y generate a cyclic group in $\pi_1(N)$, or*
2. *there exists a JSJ torus T such that x and y lie in a conjugate of $\pi_1(T) \subset \pi_1(N)$, or*
3. *there exists a Seifert fibered component M of the JSJ decomposition such that x and y lie in a conjugate of $\pi_1(M) \subset \pi_1(N)$.*

Proof. Let N be a 3-manifold. Denote by $\mathcal{G} = \mathcal{G}(N)$ the corresponding JSJ graph with vertex set V and edge set E. We denote by $w \in V$ the vertex which contains the base point of N. We denote the vertex groups by $G_v = \pi_1(N_v), v \in V$.

The theorem holds trivially for Seifert fibered spaces, we can therefore assume that N is not a Seifert fibered space, in particular that N is not spherical. Suppose we have $x, y \in \pi_1(N)$ with $x = yxy^{-1}$. By the symmetry of x and y we can without loss of generality assume that $cl(x) \le cl(y)$. Note that the statement of the theorem does not change under conjugation and change of base point, we can therefore without loss of generality assume that $cl(x) = l(x)$.

We represent y by a reduced loop $p = (h_0, f_1, h_1, \dots, f_{l-1}, h_{l-1}, f_l, h_l)$ based at w. If $l = 0$, then $l(x) = 0$ as well since $l(x) = cl(x) \le cl(y) \le l(y) = 0$. In that case we are done by Proposition 3.1 (5). We thus henceforth only consider the case that $l \ge 1$.

After conjugating x and y with h_l we can without loss of generality assume that $h_l = 1$. Recall that p being reduced implies that for $i = 2, \dots, l$ the following holds:

$$f_i \neq \overline{f_{i-1}} \text{ or } f_i = \overline{f_{i-1}} \text{ and } h_{i-1} \notin \operatorname{Im}(\varphi_{f_{i-1}}). \tag{1}$$

We first study the case that $l(x) = 0$, i.e. $x \in G_w$. Clearly we can assume that x is non-trivial.

Now consider

$$p * x * p^{-1} = (h_0, f_1, h_1, \dots, f_l, x, f_l^{-1}, \dots, h_1^{-1}, f_1^{-1}, h_0^{-1}).$$

This path is not reduced since yxy^{-1} can be represented by a path of length zero. It follows that $x \in \operatorname{Im}(\varphi_{f_l})$. We can now represent $x = yxy^{-1}$ by the following path:

$$(h_0, f_1, h_1, \dots, f_{l-1}, h_{l-1}\varphi_{f_l}^{-1}(x)h_{l-1}^{-1}, f_{l-1}^{-1}, \dots, h_1^{-1}, f_1^{-1}, h_0^{-1}). \tag{2}$$

Case 1: $l = 1$, i.e. $y = (h_0, f_1, 1)$. In that case $yxy^{-1} = x$ is represented by $h_0\varphi_{f_1}^{-1}(x)h_0^{-1}$. It follows that $x \in \operatorname{Im}(\varphi_{f_1})$ and $x \in h_0 \operatorname{Im}(\varphi_{\overline{f_1}})h_0^{-1}$. But if $t(f_1) = o(f_1)$ is hyperbolic this is not possible by Lemma 3.4 since the two boundary tori of $N_{t(f_1)} = N_{o(f_1)}$ corresponding to the edge f_1 are obviously different. If $t(f_1) = o(f_1)$ is Seifert fibered, then we can similarly exclude this case by appealing to Lemma 3.7 and Theorem 3.9.

Case 2: The vertex $o(f_l)$ is hyperbolic. It follows easily from (1) and Lemma 3.4 that the path (2) is reduced. Since the path represents x this implies in particular that $l = 1$. We thus reduced Case 2 to Case 1.

Case 3: The vertex $o(f_l)$ is Seifert fibered and $\varphi_{f_l}^{-1}(x) \notin \langle c_{o(f_l)} \rangle$. Note that Lemma 3.7 together with Theorem 3.9 and (1) implies that the path (2) is reduced, i.e. $l = 1$. We thus also reduced Case 3 to Case 1.

Case 4: The vertex $o(f_l)$ is Seifert fibered, $\varphi_{f_l}^{-1}(x) \in \langle c_{o(f_l)} \rangle$ and $l > 1$. Note that by Theorem 3.5 (2) this implies that $h_{l-1}\varphi_{f_l}^{-1}(x)h_{l-1}^{-1} \in \operatorname{Im}(\varphi_{f_{l-1}})$. We can thus represent x by

$$(h_0, f_1, \dots, f_{l-2}, h_{l-2} \cdot \varphi_{f_{l-1}}^{-1}(h_{l-1}\varphi_{f_l}^{-1}(x)h_{l-1}^{-1}) \cdot h_{l-2}^{-1}, f_{l-2}^{-1}, \dots, f_1^{-1}, h_0^{-1}).$$

If $o(f_{l-1})$ is hyperbolic, then the argument of Case 2 immediately shows that $l = 2$. If $o(f_{l-1})$ is Seifert fibered, then it follows from Theorems 3.5 and 3.9 and from Lemma 3.7 (2) that $h_{l-2} \cdot \varphi_{f_{l-1}}^{-1}(h_{l-1}\varphi_{f_l}^{-1}(x)h_{l-1}^{-1}) \cdot h_{l-2}^{-1} \notin \langle c_{o(f_{l-1})} \rangle$. The argument of Case 3 immediately shows that again $l = 2$.

We now showed that $l = 2$, we thus see that x equals

$$h_0 \cdot \varphi_{f_{l-1}}^{-1}(h_{l-1}\varphi_{f_l}^{-1}(x)h_{l-1}^{-1}) \cdot h_0^{-1}.$$

If $o(f_1) = t(f_2)$ is hyperbolic, then $x \in \operatorname{Im}(\varphi_{f_2})$ and $x \in h_0 \operatorname{Im}(\varphi_{\overline{f_1}})h_0^{-1}$. It follows from Lemma 3.4 that $f_1 = \overline{f_2}$ and $h_0 \in \operatorname{Im}(\varphi_{\overline{f_1}})$. If we change the base point to $o(f_2) = t(f_1)$ we see that x is represented by $\varphi_{f_2}^{-1}(x) \in G_{o(f_2)}$ and y is represented by $\varphi_{f_1}(h_0)h_1 \in G_{o(f_2)}$. If on the other hand $o(f_1) = t(f_2)$ is Seifert fibered, then it follows from Theorem 3.9 that $x \notin \langle c_{t(f_2)} \rangle$. It now follows easily from Lemma 3.7 that $f_1 = \overline{f_2}$ and $h_0 \in \operatorname{Im}(\varphi_{\overline{f_1}})$. We conclude the argument as above.

We now turn to the case that $l(x) > 0$. We claim that Conclusion (1) holds. By Theorem 4.1 we can find $z \in \pi_1(N)$ which is indivisible and $n > 0$ with $x = z^n$. Without loss of generality assume that z is cyclically reduced. We claim that y is a power of z as well. We represent z by a reduced loop $q = (g_0, e_1, g_1, \dots, e_k, g_k)$. We now consider the path $p * q^n * p^{-1}$ which is given by

$$(h_0, f_1, h_1, \dots, f_l, h_l \cdot g_0, e_1, g_1, \dots, e_k, g_k \cdot h_l^{-1}, f_l^{-1}, \dots, h_1^{-1}, f_1^{-1}, h_0^{-1}).$$

This loop has to be reduced since $l > 0$ and therefore the loop is longer than the loop q^n which represents the same element. We conclude that one of the following conditions hold:

(1) $f_l = \overline{e_1}$ and $h_l g_0 \in \mathrm{Im}(\varphi_{f_l})$, or
(2) $e_k = f_l$ and $g_k h_l^{-1} \in \mathrm{Im}(\varphi_{e_k})$.

Note though that not both conclusions can hold, otherwise x would not be cyclically reduced. Now suppose that (1) holds and (2) does not hold. A straightforward induction argument now shows that $p = p' * q^{-1}$ for some reduced path p'. On the other hand, if (2) holds and (1) does not hold, then a straightforward induction argument shows that $p = q^{-1} * p'$ for some reduced path p'.

Claim 4.4. *If $l(p') = 0$, then p' represents the trivial element.*

If $l(p') = 0$, then we denote by y' the element represented by p'. Suppose that y' is non-trivial. In that case we have $y'x^n(y')^{-1} = x^n$ for any n, in particular $x^n y' x^{-n} = y'$. It follows from the discussion of Cases 1, 2, 3 and 4 above that $l(x^n) \le 2$ for any n. Since x is cyclically reduced and $l(x) > 0$ this case can not occur. This concludes the proof of the claim.

If p' represents the trivial element we are clearly done. If not, then $l(p') > 0$ and we can do an induction argument on the length of p' to show that y is in fact a power of z. □

4.3. **Malnormality of peripheral subgroups.** Using the methods of the proof of Theorem 4.3 we can now also prove the following theorem which was first proved by de la Harpe and Weber [6].

Theorem 4.5. *Let N be a compact, orientable, irreducible 3-manifold with toroidal boundary and S a boundary component. If the JSJ component which contains S is hyperbolic, then $\pi_1(S) \subset \pi_1(N)$ is malnormal.*

Proof. Let N be a compact, orientable, irreducible 3-manifold with toroidal boundary and S a boundary component. We denote by $\mathcal{G} = \mathcal{G}(N)$ the corresponding JSJ graph with vertex set V and edge set E. Suppose that the JSJ component N_w which contains S is hyperbolic. Now let $x \in \pi_1(S)$ and $g \in \pi_1(N) \setminus \pi_1(S)$.

We pick a base point on S. We represent g by a reduced loop $p = (h_0, f_1, h_1, \dots, f_{l-1}, h_{l-1}, f_l, h_l)$ based at w. If $l = 0$, then $g \in \pi_1(N_w)$, but since $\pi_1(S) \subset \pi_1(N_w)$ is malnormal by Lemma 3.4 (1) it follows that $gxg^{-1} \notin \pi_1(S)$. Now suppose that $l > 0$. We consider the path

$$p * x * p^{-1} = (h_0, f_1, h_1, \dots, f_l, h_l x h_l^{-1}, f_l^{-1}, \dots, h_1^{-1}, f_1^{-1}, h_0^{-1}).$$

This path is reduced if and only if $x \in \mathrm{Im}(\varphi_{f_l})$. But $\mathrm{Im}(\varphi_{f_l})$ is the image of a boundary torus in N_w distinct from S. It now follows from Lemma 3.4 (2) that $h_l x h_l^{-1} \notin \mathrm{Im}(\varphi_{f_l})$. We conclude that the path $p * x * p^{-1}$ is reduced, i.e. gxg^{-1} does not lie in $\pi_1(N_w)$, let alone in $\pi_1(S)$. □

4.4. **Proof of Theorem 1.1.** For the reader's convenience we recall the statement of Theorem 1.1.

Theorem 4.6. *Let N be a 3-manifold. We write $\pi = \pi_1(N)$. Let $g \in \pi$. If $C_\pi(g)$ is non-cyclic, then one of the following holds:*

(1) *there exists a JSJ torus or a boundary torus T and $h \in \pi$ such that $g \in h\pi_1(T)h^{-1}$ and such that*
$$C_\pi(g) = h\pi_1(T)h^{-1},$$
(2) *there exists a Seifert fibered component M and $h \in \pi$ such that $g \in h\pi_1(M)h^{-1}$ and such that*
$$C_\pi(g) = hC_{\pi_1(M)}(h^{-1}gh)h^{-1}.$$

Proof. Let N be a 3-manifold and let $g \in \pi = \pi_1(N)$. If for any $h \in C_\pi(g)$ the group generated by g and h is cyclic, then either $C_\pi(g)$ is cyclic, or g is infinitely divisible. Since the former case is excluded by Theorem 4.1 the latter case has to hold.

Now suppose that $C_\pi(g)$ is not cyclic and suppose that there exist an $h \in C_\pi(g)$ such that the group generated by g and h is not cyclic. It follows from Theorem 4.3 that one of the following three cases occurs:

(1) there exists a JSJ torus T such that g lies in a conjugate of $\pi_1(T) \subset \pi_1(N)$,
(2) there exists a Seifert fibered component M of the JSJ decomposition such that g lies in a conjugate of $\pi_1(M) \subset \pi_1(N)$,

First suppose there exists a JSJ torus T such that g lies in a conjugate of $\pi_1(T) \subset \pi_1(N)$. Without loss of generality we can assume that $g \in \pi_1(T)$. We first consider the case that the two JSJ components abutting T are different. We denote these two components by M_1 and M_2. By Proposition 3.1 (5) the following claim implies the theorem in this case.

Claim 4.7. *There exists an $i \in \{1,2\}$ such that*
$$C_\pi(g) = C_{\pi_1(M_i)}(g).$$

Let $h \in C_\pi(g)$. It follows easily from the proof of Theorem 4.3 that either $h \in \pi_1(M_1)$ or $h \in \pi_1(M_2)$. If M_1 is hyperbolic, then it follows from Lemma 3.2 and from Proposition 3.1 (5) that $h \in \pi_1(T)$. It follows that $C_\pi(g) = C_{\pi_1(M_2)}(g)$. Similarly we deal with the case that M_2 is hyperbolic. Finally assume that M_1 and M_2 are Seifert fibered. We denote by c_1 and c_2 the regular fibers of M_1 and M_2. If g is not a power of c_1, then it follows from Lemma 3.6 that $C_\pi(g) = C_{\pi_1(M_2)}(g)$, similarly if g is not a power of c_2. Recall that c_1 and c_2 are indivisible in $\pi_1(T)$ and that by Theorem 3.9 we have $c_1 \neq c_2^{\pm 1}$. It follows that g is either not a power of c_1 or not a power of c_2.

The case that the torus is non-separating can be dealt with similarly. We leave this to the reader. Also, if there exists a Seifert fibered component M of the JSJ decomposition such that g lies in a conjugate of $\pi_1(M) \subset \pi_1(N)$ and such that g does not lie in the image of a boundary torus, then it follows easily from the proof of Theorem 4.3 that
$$C_\pi(g) = C_{\pi_1(M)}(g).$$

□

REFERENCES

[1] M. Aschenbrenner, S. Friedl and H. Wilton, *3-manifold groups*, Preprint (2011).

[2] H. Bass, *Covering theory for graphs of groups*, J. Pure Appl. Algebra **89** (1993), 3-47.

[3] D. E. Cohen, *Combinatorial Group Theory: A Topological Approach,* London Mathematical Society Student Texts, **14**, Cambridge University Press, Cambridge, 1989.

[4] C. McA. Gordon and W. Heil, *Cyclic normal subgroups of fundamental groups of 3-manifolds*, Topology **14** (1975), 305-309.

[5] P. de la Harpe and J.-P. Préaux, *Groupes fondamentaux des variétés de dimension 3 et algébres d'opérateurs*, Annales Fac. Sciences Toulouse, Math., Sér. 6, **16** (2007) 561-589.

[6] P. de la Harpe and C. Weber, *On malnormal peripheral subgroups in fundamental groups of 3-manifolds*, in preparation (2011).

[7] J. Hempel, *Residual finiteness for 3-manifolds*, Combinatorial group theory and topology (Alta, Utah, 1984), 379–396, Ann. of Math. Stud., **111**, Princeton Univ. Press, Princeton, NJ, 1987.

[8] W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc. **21** (1979), no. 220.

[9] K. Johannson, *Homotopy equivalences of 3-manifolds with boundaries*, Lecture Notes in Mathematics, **761**, Springer, Berlin, 1979.

[10] A. Marden, *Outer circles. An introduction to hyperbolic 3-manifolds*, Cambridge University Press, Cambridge, 2007.

[11] P. Orlik, *Seifert manifolds*, Lecture Notes in Mathematics, **291**, Springer-Verlag (1972).

[12] P. Orlik, E. Vogt and H. Zieschang, *Zur Topologie gefaserter dreidimensionaler Mannigfaltigkeiten*, Topology, **6** (1967), 49-64.

[13] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.

[14] J.-P. Serre, *Trees*, Springer-Verlag, Berlin-New York, 1980.

[15] P. Shalen, *Representations of 3–manifold groups*, Handbook of Geometric Topology, 955–1044 (2002).

[16] W. Thurston, *The geometry and topology of 3-manifolds*, Princeton Lecture Notes (1979), available at
`http://www.msri.org/publications/books/gt3m/`

[17] J. Tollefson, *Involutions of Seifert fiber spaces*, Pacific J. Math. **74** (1978), 519–529.

[18] F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. Math. (2) **87**, 56-88 (1968).

MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, GERMANY
E-mail address: `sfriedl@gmail.com`