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Kyoto University
Existence and non-existence of the nonlinear Schrödinger equations for one and high dimensional case

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0. Introduction

In this report, we will introduce the results of [S] and related results. We consider the following nonlinear Schrödinger equations:

\[- \Delta u + (1 + b(x))u = f(u) \quad \text{in} \ R^N,\]
\[u \in H^1(R^N).\] (*)

We mainly considered the one-dimensional case in [S] but, in this report, we consider not only one-dimensional case but also the high-dimensional case. Here, we assume that the potential \( b(x) \in C(R, R) \) satisfies the following assumptions:

(b.1) \( 1 + b(x) \geq 0 \) for all \( x \in R^N \).
(b.2) \( \lim_{|x| \to \infty} b(x) = 0 \).
(b.3) There exist \( \beta_0 > 2 \) and \( C_0 > 0 \) such that \( b(x) \leq C_0 e^{-\beta_0|x|} \) for all \( x \in R^N \).

We also assume that the nonlinearity \( f(u) \in C(R, R) \) satisfies the following

(f.0) \( f(u) = |u|^{p-1}u \) for \( p \in (1, \frac{N+2}{N-2}) \) when \( N \geq 3 \) and \( p \in (1, \infty) \) when \( N = 2 \).
(f.1) There exists \( \eta_0 > 0 \) such that \( \lim_{|u| \to 0} \frac{f(u)}{|u|^{1+\eta_0}} = 0 \).
(f.2) There exists \( u_0 > 0 \) such that

\[ F(u) < \frac{1}{2} u^2 \quad \text{for all} \ u \in (0, u_0), \]
\[ F(u_0) = \frac{1}{2} u_0^2, \quad f(u_0) > u_0. \]
(f.3) There exists \( \mu_0 > 2 \) such that \( 0 < \mu_0 F(u) \leq uf(u) \) for all \( u \neq 0 \).
To consider the $(*)$, the following equation plays an important roles:

$$-\Delta u + u = f(u) \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N). \tag{0.1}$$

From (b.2), the equation $-\Delta u + u = f(u)$ appears as a limit when $|x|$ goes to $\infty$ in $(*)$. To show the existence of positive solution of $(*)$ in our arguments, the uniqueness (up to translation) of positive solutions of (0.1) is also important. Under the condition (f.0), it is well-known that the uniqueness (up to translation) of the positive solutions of (0.1). When $N = 1$, it is known that the conditions (f.1) and (f.2) are sufficient conditions for (0.1) to have an unique (up to translation) positive solution:

**Remark 0.1.** In Section 5 of [BeL1], Berestycki-Lions showed that if $f(u)$ is of locally Lipschitz continuous and $f(u) = 0$, then (f.2) is a necessary and sufficient condition for the existence of a non-trivial solution of (1.0). Moreover, it also was shown that the uniqueness (up to translation) of positive solutions under the (f.2). In Section 2 of [JT1], Jeanjean-Tanaka showed that when $f(u)$ is of continuous, (f.1) and (f.2) are sufficient conditions for (0.1) to have a unique positive solution.

The condition (f.3) is so called Ambrosetti-Rabinowitz condition, which guarantees the boundedness of (PS)-sequences for the functional corresponding to the equation $(*)$ and (0.1). To state an our result for one-dimensional case, we also need the following assumption for $b(x)$.

(b.4) When $N = 1$, there exists $x_0 \in \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} b(x - x_0)e^{2|x|} dx \in [-\infty, 2).$$

Our first theorem is the following.

**Theorem 0.2.** When $N \geq 2$, we assume that (b.1)–(b.3) and (f.0) hold. Then $(*)$ has at least a positive solution. When $N = 1$, we assume that (b.1)–(b.4) and (f.1)–(f.3) hold. Then $(*)$ has at least a positive solution.

In [S], we give a proof of Theorem 0.2 for the one-dimensional case. To prove the Theorem 0.2, we developed the arguments of [BaL] and [Sp]. We remark that, for high-dimensional case, the proof of Theorem 0.2 almost are parallel to the proof of [BaL]. However, for the proof of the one-dimensional case, we essentially developed the arguments of [BaL] and [Sp]. Bahri-Li [BaL] showed that there exists a positive solution of

$$-\Delta u + u = (1 - b(x))|u|^{p-1}u \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \tag{0.2}$$
where \( N \geq 3, 1 < p < \frac{N+2}{N-2} \) and \( b(x) \in C(\mathbb{R}, \mathbb{R}) \) satisfies (b.2)-(b.3) and \( (b.1)' \ 1 - b(x) \geq 0 \) for all \( x \in \mathbb{R}^N \).

For one dimensional case, Spradlin [Sp] proved that there exists a positive solution of the equation
\[
-u'' + u = (1 - b(x))f(u) \quad \text{in} \quad \mathbb{R}, \quad u \in H^1(\mathbb{R}).
\] (0.3)

They also assumed that \( b(x) \in C(\mathbb{R}, \mathbb{R}) \) satisfies (b.l)' and (b.2)-(b.3) and \( f(u) \) satisfies (f.1)-(f.3) and
\( (f.4) \frac{f(u)}{u} \) is an increasing function for all \( u > 0 \).

When (f.0) or (f.4) holds, we can consider the Nehari manifold and they argued on Nehari manifold in [BaL] and [Sp]. In our situation, when \( N = 1 \), we cannot argue on Nehari manifold. This was one of the difficulties which had to overcome in [S].

From the above results and Theorem 0.2, it seems that, when \( N = 1 \), Theorem 0.2 holds without condition (b.4). However (b.4) is an essential assumption for (*) to have non-trivial solutions. In what follows, we will show a result about the non-existence of nontrivial solutions for (*).

In next our result, we will assume that \( N = 1 \) and \( b(x) \) satisfies the following condition:

(b.5) There exist \( \mu > 0 \) and \( m_2 \geq m_1 > 0 \) such that
\[
m_1 \mu e^{-\mu|x|} \leq b(x) \leq m_2 \mu e^{-\mu|x|} \quad \text{for all} \quad x \in \mathbb{R}.
\]

Here, we remark that, if (b.5) holds for \( \mu > 2 \), then \( b(x) \) satisfies (b.1)-(b.3) and
\[
\frac{2\mu}{\mu - 2} m_1 \leq \int_{-\infty}^{\infty} b(x) e^{2|x|} \, dx \leq \frac{2\mu}{\mu - 2} m_2.
\]

Thus, when \( m_2 < 1 \) and \( \mu \) is very large, the condition (b.4) also holds.

Our second result is the following:

**Theorem 0.3.** Assume \( N = 1 \), (b.5) holds and \( f(u) = |u|^{p-1}u \ (p > 1) \).

(i) If \( m_1 > 1 \), there exists \( \mu_1 > 0 \) such that (*) does not have non-trivial solution for all \( \mu \geq \mu_1 \).

(ii) If \( m_2 < 1 \), there exists \( \mu_2 > 0 \) such that (*) has at least a non-trivial solution for all \( \mu \geq \mu_2 \).

(iii) There exists \( \mu_3 > 0 \) such that (*) does not have sign-changing solutions for all \( \mu \geq \mu_3 \).

From Theorem 0.3, we see that Theorem 0.2 does not hold except for condition (b.4). This is a drastically different situation from the high-dimensional cases. This is one of the interesting points in our results.
We remark that the condition (b.4) implies \( \int_{-\infty}^{\infty} b(x) \, dx < 2 \) and the assumption of (ii) of Theorem 0.3 also means \( \int_{-\infty}^{\infty} b(x) \, dx < 2 \). Thus we expect that the difference from existence and non-existence of non-trivial solutions of (*) depends on the quantity of integrate of \( b(x) \).

We can obtain this expectation from another viewpoint, which is a perturbation problem. Setting \( b_{\mu}(x) = m \mu e^{-\mu|x|} \), \( b_{\mu}(x) \) satisfies (b.5) and, when \( \mu \to \infty \), \( b_{\mu}(x) \) converges to the delta function \( 2m \delta_{0} \) in distribution sense. Thus (*) approaches to the equation

\[
-u'' + (1 + 2m \delta_{0})u = |u|^{p-1}u \quad \text{in } \mathbb{R}, \quad u \in H^{1}(\mathbb{R}),
\]

in distribution sense. Here, if \( u \) is a solution of (0.4) in distribution sense, we can see that \( u \) is of \( C^{2} \)-function in \( \mathbb{R} \setminus \{0\} \) and continuous in \( \mathbb{R} \) and \( u \) satisfies

\[
u'(0) - u'(-0) = 2mu(0).
\]

Moreover, since \( u \) is a homoclinic orbit of \( -u'' + u = f(u) \) in \((\infty, 0) \) or \((0, \infty) \), respectively, \( u \) satisfies

\[
-\frac{1}{2} u'(x)^2 + \frac{1}{2} u(x)^2 - \frac{1}{p+1} |u(x)|^{p+1} = 0 \quad \text{for } x \neq 0.
\]

When \( x \to \pm 0 \) in (0.6), from (f.1), we find

\[
u'(-0) = -u'(0), \quad |u'(\pm 0)| < |u(0)|.
\]

Thus, from (0.5) and (0.7), it easily see that (0.4) has an unique positive solution when \( |m| < 1 \) and (0.4) has no non-trivial solutions when \( |m| \geq 1 \). Therefore we can regard Theorem 0.3 as results of a perturbation problem of (0.4).

To prove Theorem 0.3, we develop the shooting arguments which used in [BE]. Bianchi and Egnell [BE] argued about the existence and non-existence of radial solutions for

\[
-\Delta u = K(|x|) |u|^\frac{N+2}{N-2}, \quad u > 0 \quad \text{in } \mathbb{R}^{N}, \quad u(x) = O(|x|^{2-N}) \quad \text{as } |x| \to \infty.
\]

Here \( N \geq 3 \) and \( K(|x|) \) is a radial continuous function. Roughly speaking their approach, by setting \( u(r) = u(|x|) \), they reduce (0.8) to an ordinary differential equation and considered solutions of two initial value problems of that ordinary differential equation which have initial conditions \( u(0) = \lambda \) and \( \lim_{r \to \infty} r^{N-2}u(r) = \lambda \). And, examining whether those solutions have suitable matchings at \( r = 1 \), they argued about the existence and non-existence of radial solutions.
In [S], to prove Theorem 0.3, we also consider two initial value problems from $\pm\infty$, that is, for $\lambda_1, \lambda_2 > 0$, we consider the following two problems:

\[-u'' + (1 + b(x))u = f(u),
\lim_{x \to -\infty} e^{-x}u(x) = \lim_{x \to -\infty} e^{-x}u'(x) = \lambda_1,\tag{0.9}\]

and

\[-u'' + (1 + b(x))u = f(u),
\lim_{x \to \infty} e^{x}u(x) = -\lim_{x \to \infty} e^{x}u(x) = \lambda_2.\tag{0.10}\]

Then (0.9) and (0.10) have an unique solution respectively and write those solutions as $u_1(x; \lambda_1)$ and $u_2(x; \lambda_2)$ respectively. We set

\[
\Gamma_1 = \{(u_1(0; \lambda_1), u_1'(0; \lambda_1)) \in \mathbb{R}^2 | \lambda_1 > 0\},
\Gamma_2 = \{(u_2(0; \lambda_2), u_1'(0; \lambda_2)) \in \mathbb{R}^2 | \lambda_2 > 0\}.
\]

Then, $\Gamma_1 \cap \Gamma_2 = \emptyset$ is equivalent to the non-existence of solutions for $(\ast)$. Thus it is important to study shapes of $\Gamma_1$ and $\Gamma_2$. In respect to the details of proofs of Theorem 0.3, see [S].

In next sections, we state about the outline of the proof of Theorem 0.2. We will consider the one-dimensional case in Section 1 and treat the high-dimensional case in Section 2.

1. The outline of the proof of Theorem 0.2 for $N = 1$

In this section, we consider the case $N = 1$. We will developed a variational approach which was used in [BaL] and [Sp].

In what follows, since we seek positive solutions of $(\ast)$, without loss of generalities, we assume $f(u) = 0$ for $u < 0$. To prove Theorem 0.2, we seek non-trivial critical points of the functional

\[I(u) = \frac{1}{2}||u||_{H^1(\mathbb{R})}^2 + \frac{1}{2} \int_{-\infty}^{\infty} b(x)u^2 \, dx - \int_{-\infty}^{\infty} F(u) \, dx \in C^1(H^1(\mathbb{R}), \mathbb{R}),\]

whose critical points are positive solutions of $(\ast)$. Here we use the following notations:

\[||u||_{H^1(\mathbb{R})}^2 = ||u'||_{L^2(\mathbb{R})}^2 + ||u||_{L^2(\mathbb{R})}^2,\]

\[||u||_{L^p(\mathbb{R})}^p = \int_{\mathbb{R}} |u|^p \, dx \quad \text{for} \quad p > 1.\]

From (f.1)–(f.2), we can see that $I(u)$ satisfies a mountain pass geometry (See Section 3 in [JT2]), that is, $I(u)$ satisfies...
(i) $I(0) = 0$.
(ii) There exist $\delta > 0$ and $\rho > 0$ such that $I(u) \geq \delta$ for all $||u||_{H^1(R)} = \rho$.
(iii) There exists $u_0 \in H^1(R)$ such that $I(u_0) < 0$ and $||u_0||_{H^1(R)} > \rho$.

From the mountain pass geometry (i)–(iii), we can define a standard minimax value $c > 0$ by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

(1.1)

$$\Gamma = \{ \gamma(t) \in C([0,1], H^1(R)) | \gamma(0) = 0, I(\gamma(1)) < 0 \}.$$

And, by a standard way, we can construct $(PS)_c$-sequence $(u_n)_{n=1}^{\infty}$, that is, $(u_n)_{n=1}^{\infty}$ satisfies

$$I(u_n) \to c \quad (n \to \infty),$$

$$I'(u_n) \to 0 \quad \text{in} \ H^{-1}(R) \quad (n \to \infty).$$

Moreover, since $(u_n)_{n=1}^{\infty}$ is bounded in $H^1(R)$ from (f.3), $(u_n)_{n=1}^{\infty}$ has a subsequence $(u_{n_j})_{j=1}^{\infty}$ which weakly converges to some $u_0$ in $H^1(R)$. If $(u_{n_j})_{j=1}^{\infty}$ strongly converges to $u_0$ in $H^1(R)$, $c$ is a non-trivial critical value of $I(u)$ and our proof is completed. However, since the embedding $L^p(R) \subset H^1(R)$ ($p > 1$) is not compact, there may not exist a subsequence $(u_{n_j})_{j=1}^{\infty}$ which strongly converges in $H^1(R)$. Therefore, in our situation, we don’t know $c$ is a critical value.

In our situation, from the lack of the compactness mentioned the above, we must use the concentration-compactness approach as [BaL] and [Sp]. In the concentration-compactness approach, we examine in detail what happens in bounded (PS)-sequence. When we state the concentration-compactness argument for the (PS)-sequences of $I(u)$, the limit problem (0.1) plays an important role. Setting

$$I_0(u) = \frac{1}{2}||u||_{H^1(R)}^2 - \int_{-\infty}^{\infty} F(u) \, dx \in C^1(H^1(R), R),$$

the critical points of $I_0(u)$ correspond to the solutions of limit problem (0.1). The equation (0.1) has an unique positive solution, identifying ones which obtain by translations. Thus let $\omega(x)$ be an unique positive solution of (0.1) with $\max_{x \in R} \omega(x) = \omega(0)$ and we set $c_0 = I_0(\omega)$. Since $I_0$ also satisfies the mountain pass geometry (i)–(iii), we see $c_0 > 0$ and $c_0$ is an unique non-trivial critical value.

For the bounded (PS)-sequences of $I(u)$, we have the following:

**Proposition 1.1.** Suppose (b.1)–(b.2) and (f.1)–(f.2) hold. If $(u_n)_{n=1}^{\infty}$ is a bounded (PS)-sequence of $I(u)$, then there exist a subsequence $n_j \to \infty$, $k \in N \cup \{0\}$, $k$-sequences
$(x_1^j, \ldots, x_k^j)_{j=1}^{\infty} \subset \mathbb{R}$, and a critical point $u_0$ of $I(u)$ such that

\[
I(u_{n_j}) \to I(u_0) + kc_0 \quad (j \to \infty),
\]

\[
\left\|u_{n_j}(x) - u_0(x) - \sum_{\ell=1}^{k} \omega(x-x_j^\ell)\right\|_{H^1(\mathbb{R})} \to 0 \quad (j \to \infty),
\]

$|x_j^\ell - x_j^{\ell'}| \to \infty \quad (j \to \infty) \quad (\ell \neq \ell')$,

$|x_j^\ell| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \ldots, k)$.

**Proof.** We can easily get Proposition 1.1 from Theorem 5.1 of [JT1]. Theorem 5.1 of [JT1] required the assumption $\lim_{u \to \infty} f(u)u^{-p} = 0$ ($p > 1$). However we take off that assumption for one dimensional case by improving Step 2 of Theorem 5.1 of [JT1]. In fact we have only to change $\sup_{z \in \mathbb{R}^N} \int_{B_{1}(z)}|v_n^1|^2 dx \to 0$ in Step2 to $||v_n^1||_{L^{\infty}(\mathbb{R})} \to 0$.

If the minimax value $c$ satisfies $c \in (0, c_0)$, from Proposition 1.1, we see that $I(u)$ has at least a non-trivial critical point. In fact, let $(u_n)_{n=1}^{\infty}$ be a bounded $(PS)_c$-sequence of $I(u)$, from Proposition 1.1, there exists a subsequence $n_j \to \infty$, $k \in \mathbb{N} \cup \{0\}$ and a critical point $u_0$ of $I(u)$ such that

\[
I(u_{n_j}) \to I(u_0) + kc_0 \quad (j \to \infty).
\]

Here, if $u_0 = 0$, we get $I(u_{n_j}) \to kc_0$ as $j \to \infty$. However this contradicts to the fact that $I(u_n) \to c \in (0, c_0)$ as $n \to \infty$. Thus $u_0 \neq 0$ and $u_0$ is a non-trivial critical point of $I(u)$. From the above argument, we have the following corollary.

**Corollary 1.2.** Suppose $I(u)$ has no non-trivial critical points and let $(u_n)_{n=1}^{\infty}$ be a $(PS)$-sequence of $I(u)$. Then, only $kc_0$’s ($k \in \mathbb{N} \cup \{0\}$) can be limit points of $\{I(u_n) | n \in \mathbb{N}\}$.

**Remark 1.3.** Corollary 1.2 essentially depends on the uniqueness of the positive solution of (0.1).

As mentioned the above, when $c \in (0, c_0)$, $I(u)$ has at least a non-trivial critical point. However, unfortunately, under the condition (b.1)–(b.4), it may be $c = c_0$. Thus we need consider another minimax value. To define another minimax value, we use a path $\gamma_0(t) \in C(\mathbb{R}, H^1(\mathbb{R}))$ which is defined as follows: for small $\epsilon_0 > 0$, we set

\[
h(x) = \begin{cases} 
\omega(x) & x \in [0, \epsilon_0], \\
x^4 + u_0 & x \in (-\epsilon_0, 0), \\
\epsilon_0^4 + u_0 & x \in (-\infty, -\epsilon_0),
\end{cases}
\]

\[
\gamma_0(t)(x) = \begin{cases} 
h(x-t) & x \geq 0, \\
h(-x-t) & x < 0.
\end{cases}
\]
Here, we remark that $u_0$ was given in (f.2). This path $\gamma_0(t)$ was introduced in [JT2]. Choosing a proper $\epsilon_0 > 0$ sufficiently small, $\gamma_0(t)$ achieves the mountain pass value of $I_0(u)$ and satisfies the followings:

**Lemma 1.4.** Suppose (f.1)–(f.2) hold. Then $\gamma_0(t)$ satisfies

(i) $\gamma_0(0)(x) = \omega(x)$.
(ii) $I_0(\gamma_0(t)) < I_0(\omega) = c_0$ for all $t \neq 0$.
(iii) $\lim_{t \to -\infty} ||\gamma_0(t)||_{H^1(\mathbb{R})} = 0$, $\lim_{t \to \infty} ||\gamma_0(t)||_{H^1(\mathbb{R})} = \infty$.

**Proof.** See Section 3 in [JT2].

**Remark 1.5.** When $f(u)/u$ is a increasing function, we can use a simpler path than $\gamma_0(t)$.

In fact, setting $\tilde{\gamma}_0(t) = t\omega : [0, \infty) \to H^1(\mathbb{R})$, we also have

(i) $\tilde{\gamma}_0(1)(x) = \omega(x)$.
(ii) $I_0(\tilde{\gamma}_0(t)) < I_0(\omega) = c_0$ for all $t \neq 1$.
(iii) $\tilde{\gamma}_0(0) = 0$, $\lim_{t \to \infty} ||\tilde{\gamma}_0(t)||_{H^1(\mathbb{R})} = \infty$.

Moreover, if $f(u)/u$ is a increasing function, in what follows, we can also construct a simpler proofs by aruging on Nehari manifold $N = \{u \in H^1(\mathbb{R}) \setminus \{0\} \mid I'(u)u = 0\}$. (See [Sp].)

Now, for $R > 0$, we consider a path $\gamma_R \in C(\mathbb{R}^2, H^1(\mathbb{R}))$ which is defined by

$$\gamma_R(s, t)(x) = \max\{\gamma_0(s)(x + R), \gamma_0(t)(x - R)\}.$$ 

In our proof of Theorem 0.2 in [S], the following proposition is a key proposition.

**Proposition 1.6.** Suppose (b.1)–(b.3) and (f.1)–(f.2) hold. Then, for any $L > 0$, we have

$$\lim_{R \to \infty} e^{2R} \max_{(s, t) \in [-L, L]^2} I(\gamma_R(s, t)) - 2c_0 \leq \frac{\lambda_0^2}{2} \left( \int_{-\infty}^{\infty} b(x)e^{2|x|}dx - 2 \right).$$

(1.2)

Here $\lambda_0 = \lim_{x \to \pm\infty} \omega(x)e^{\vert x\vert}$.

**Proof.** See [S].

By using a translation, without loss of generalities, we assume $x_0 = 0$ in (b.4). If (b.4) with $x_0 = 0$ holds, from Proposition 1.6, for any $L > 0$, there exists $R_0 > 0$ such that

$$\max_{(s, t) \in [-L, L]^2} I(\gamma_{R_0}(s, t)) < 2c_0.$$
To prove the Theorem 0.2, we also need a map $m : H^1(\mathbb{R}) \setminus \{0\} \to \mathbb{R}$ which is defined by the following: for any $u \in H^1(\mathbb{R}) \setminus \{0\}$, a function

$$T_u(s) = \int_{-\infty}^{\infty} \tan^{-1}(x-s)|u(x)|^2 \, dx : \mathbb{R} \to \mathbb{R}$$

is strictly decreasing and $\lim_{s \to \infty} T_u(s) = -||u||_{L^2(\mathbb{R})}^2 < 0$ and $\lim_{s \to -\infty} T_u(s) = ||u||_{L^2(\mathbb{R})}^2 > 0$. Thus, from the theorem of the intermediate value, $T_u(s)$ has an unique $s = m(u)$ such that $T_u(m(u)) = 0$. We also find that $m(u)$ is of continuous by the implicit function theorem to $(u, s) \mapsto T_u(s)$. The map $m(u)$ was introduced in [Sp]. We remark that $m(u)$ is regarded as a kind of center of mass of $|u(x)|^2$ and we can check the followings.

**Lemma 1.7.** We have

(i) $m(\gamma_0(t)) = 0$ for all $t \in \mathbb{R}$.

(ii) $m(\gamma_R(s, t)) > 0$ for all $-R < s < t < R$.

(iii) $m(\gamma_R(s, t)) < 0$ for all $-R < t < s < R$.

**Proof.** Since $\gamma_0(t)(x)$ is an even function, we have (i). We Note that

$$\gamma_R(s, t)(x) = \begin{cases} \gamma_0(s)(x + R) & \text{for } x \in (-\infty, \frac{s-t}{2}], \\ \gamma_0(t)(x - R) & \text{for } x \in (\frac{s-t}{2}, \infty). \end{cases}$$

Since $\gamma_R(s, s)(x)$ is also an even function, we have

$$m(\gamma_R(s, s)) = 0 \quad \text{for all } s \in \mathbb{R},$$

and we get (ii)–(iii).

In what follows, we will complete the proof of Theorem 0.2 for $N = 1$.

**Proof of Theorem 0.2 for $N = 1$.** First of all, we defined a minimax value $c_1 > 0$ by

$$c_1 = \inf_{\gamma \in \Gamma_1} \max_{t \in [0,1]} I(\gamma(t)),$$

$$\Gamma_1 = \{ \gamma(t) \in C([0,1], H^1(\mathbb{R})) \mid \gamma(0) = 0, \ I(\gamma(1)) < 0, \ |m(\gamma(t))| < 1 \}.$$  

Noting $\Gamma_1 \subset \Gamma$, we have

$$0 < c \leq c_1.$$ 

Since $\Gamma_1$ is not invariant by standard deformation flows of $I(u)$, $c_1$ may not be a critical point of $I(u)$. We will use $c_1$ to divide the case. We divide the case into the following three cases:

(i) $c_1 < c_0$. 

(ii) \( c_1 = c_0 \).

(iii) \( c_1 > c_0 \).

**Proof of Theorem 0.2 for the case (i).** Since the inequality \( c_1 < c_0 \) implies \( 0 < c < c_0 \), from Corollary 1.2, we can see \( I(u) \) has at least a non-trivial critical point.

**Proof of Theorem 0.2 for the case (ii).** In this case, if \( c < c_1 = c_0 \), then \( I(u) \) has at least a non-trivial critical point from Corollary 1.2. Thus we may consider the case \( c = c_1 = c_0 \). In this case, for any \( \epsilon > 0 \), there exists \( \gamma_\epsilon(t) \in \Gamma_1 \) such that

\[
\gamma_\epsilon(t) = \frac{\max I(\gamma_\epsilon(t)) < c + \epsilon}{c}.
\]

Since \( \gamma_\epsilon \in \Gamma_1 \subset \Gamma \) and \( \Gamma \) is an invariant set by standard deformation flows of \( I(u) \), by a standard Ekland principle, there exists \( u_\epsilon \in H^1(\mathbb{R}) \) such that

\[
c \leq I(u_\epsilon) \leq \max_{t \in [0,1]} I(\gamma_\epsilon(t)) < c + \epsilon,
\]

\[
||I'(u_\epsilon)|| < 2\sqrt{\epsilon},
\]

\[
\inf_{t \in [0,1]} ||u_\epsilon - \gamma_\epsilon(t)||_{H^1(\mathbb{R})} < \epsilon.
\]

Then, from Proposition 1.1, there exist a subsequence \( \epsilon_j \rightarrow 0 \), \( k \in \mathbb{N} \cup \{0\} \), \( k \)-sequences \( (x_j^1)_{j=1}^\infty, \ldots, (x_j^k)_{j=1}^\infty \subset \mathbb{R} \), and a critical point \( u_0 \) of \( I(u) \) such that

\[
I(u_{\epsilon_j}) \rightarrow I(u_0) + kc_0 \quad (j \rightarrow \infty),
\]

\[
\left| u_{\epsilon_j}(x) - u_0(x) - \sum_{\ell=1}^k \omega(x - x_j^\ell) \right|_{H^1(\mathbb{R})} \rightarrow 0 \quad (j \rightarrow \infty),
\]

\[
x_j^\ell - x_j^{\ell'} \rightarrow \infty \quad (j \rightarrow \infty) \quad (\ell \neq \ell'),
\]

\[
x_j^\ell \rightarrow \infty \quad (j \rightarrow \infty) \quad (\ell = 1, 2, \ldots, k).
\]

Now, if \( u_0 \neq 0 \), our proof is completed. So we suppose \( u_0 = 0 \). Then, from (1.4), it must be \( k = 1 \). Thus, we have

\[
||u_{\epsilon_j}(x) - \omega(x - x_j^1)||_{H^1(\mathbb{R})} \rightarrow 0 \quad (j \rightarrow \infty),
\]

\[
x_j^1 \rightarrow \infty \quad (j \rightarrow \infty).
\]

On the other hand, we remark that, since \( m(\omega) = 0 \) and \( m \) is of continuous, there exists \( \delta > 0 \) such that

\[
|m(u)| < 1 \quad \text{for all} \quad u \in B_\delta(\omega) = \{v \in H^1(\mathbb{R}) | ||v - \omega||_{H^1(\mathbb{R})} < \delta\}.\]
Thus, from (1.3) and (1.5), for some $\epsilon_0 \in (0, \frac{\delta}{2})$ and $t_0 \in [0, 1]$, we have

$$|m(\gamma_{\epsilon_0}(t_0)) - x_j^1| < 1.$$ 

This contradicts to $\gamma_{\epsilon_0} \in \Gamma_1$. Therefore $u_0 \neq 0$ and $I(u)$ has at least a non-trivial critical point.

**Proof of the Theorem 0.2 for the case (iii).** First of all, we set $\delta = \frac{c_1 - c_0}{2} > 0$ and choose $L_0 > 0$ such that

$$\max_{(s, t) \in D_{2L_0} \setminus D_{L_0}} I(\gamma_R(s, t)) < c_0 + \delta < c_1 \quad \text{for all} \quad R > 3L_0. \quad (1.6)$$ 

Here we set $D_L = [L, L] \times [L, L] \subset \mathbb{R}^2$. Next, from Proposition 1.6, we can choose $R_0 > 3L_0$ such that

$$\max_{(s, t) \in D_{L_0}} I(\gamma_{R_0}(s, t)) < 2c_0. \quad (1.7)$$ 

Here we fix $\gamma_{R_0}(s, t)$ and define the following minimax value:

$$c_2 = \inf_{\gamma \in \Gamma_2} \max_{(s, t) \in D_{2L_0}} I(\gamma(s, t)),$$

$$\Gamma_2 = \{ \gamma(s, t) \in C(D_{2L_0}, H^1(\mathbb{R})) | \gamma(s, t) = \gamma_{R_0}(s, t) \text{ for all } (s, t) \in D_{2L_0} \setminus D_{L_0} \}.$$ 

Then we have the following lemma.

**Lemma 1.8.** We have

$$0 < c_0 < c_1 \leq c_2 < 2c_0.$$ 

We postpone the proof of Lemma 1.8 to end of this section. If Lemma 1.8 is true, then $\Gamma_2$ is an invariant set by the deformation flows of $I(u)$. Thus $I(u)$ has a (PS)-sequence $(u_n)_{n=1}^{\infty}$ such that

$$I(u_n) \to c_2 \in (c_0, 2c_0) \quad (n \to \infty).$$ 

From Corollary 1.2, we can see that $I(u)$ must have at least a non-trivial critical point. Combining the proofs of the cases (i)–(iii), we complete a proof of Theorem 0.2. 

**Proof of Lemma 1.8.** The inequality $c_0 < c_1$ is an assumption of the case (iii). From $\gamma_{R_0} \in \Gamma_2$ and (1.6)–(1.7), $c_2 < 2c_0$ is obvious. Thus we show $c_1 \leq c_2$. For any $\gamma(s, t) \in \Gamma_2$, we have

$$m(\gamma(s, t)) > 0 \quad \text{for all} \quad (s, t) \in D_1, \quad (1.8)$$

$$m(\gamma(s, t)) < 0 \quad \text{for all} \quad (s, t) \in D_2. \quad (1.9)$$
Here we set \(D_1 = \{(s, t) \in D_{2L_0} \setminus D_{L_0} | s < t\}\) and \(D_2 = \{(s, t) \in D_{2L_0} \setminus D_{L_0} | s > t\}\). From (1.8)–(1.9), a set \(\{(s, t) \in D_{2L_0} \setminus D_{L_0} | |m(\gamma(s, t))| < 1\}\) have a connected component which contains a path joining two points \(\gamma_{R_0}(-2L_0, -2L_0)\) and \(\gamma_{R_0}(2L_0, 2L_0)\). Thus we construct a path \(\gamma_1(t) \in \Gamma_1\) such that

\[
\{ \gamma_1(t) | t \in [1/3, 2/3] \} \subset \{ \gamma(s, t) | (s, t) \in D_{2L_0} \},
\]

\[
\max_{t \in [0, 1/3] \cup [2/3, 1]} I(\gamma_1(t)) \leq c_0.
\]

Thus we see

\[
c_1 \leq \max_{t \in [0, 1]} I(\gamma_1(t)) \leq \max_{(s,t) \in D_{2L_0}} I(\gamma(s, t)).
\]

Since \(\gamma(s, t) \in \Gamma_2\) is arbitrary, from (1.10), we have

\[
c_1 \leq c_2.
\]

Thus we get Lemma 1.8.

**Remark 1.9.** In our proofs of Theorem 0.2, the path \(\gamma_R(s, t)\) played an important role. In particular, the estimate (1.2) was an important. However, we don’t know that \(\gamma_R(s, t)\) is the best path to show the existence of positive solutions of (\(*\)). Using other path, we might be able to get better estimate than (1.2). Instead of \(\gamma_R(s, t)\), we can consider another path \(\overline{\gamma}_R \in C(R^2, H^1(R))\) which is defined by

\[
\overline{\gamma}_R(s, t)(x) = \gamma_0(s)(x + R) + \gamma_0(t)(x - R).
\]

We remark that \(\overline{\gamma}_R(s, t)\) is a natural path because we can regard \(\overline{\gamma}_R(s, t)\) as one-dimensional version of the path which was used in the proof of the high-dimensional case. (See Proposition 2.2.) Estimating \(\overline{\gamma}_R(s, t)\) by similar way to (1.2), for any \(L > 0\), we have

\[
\lim_{R \to \infty} e^{2R} \left\{ \max_{(s,t) \in [-L, L]^2} I(\overline{\gamma}_R(s, t)) - 2c_0 \right\} \leq \frac{\lambda_0^2}{2} \left( \int_{-\infty}^{\infty} b(x)(e^{2x} + e^{-2x} + 2)\, dx - 4 \right).
\]

We see that, if \(\int_{-\infty}^{\infty} b(x)(e^{2x} + e^{-2x} + 2)\, dx < 4\) holds, then \(\int_{-\infty}^{\infty} b(x)e^{2|x|}\, dx < 2\) also holds. Thus \(\gamma_R(s, t)\) provides a better estimate than \(\overline{\gamma}_R(s, t)\).
2. The outline of the proof of Theorem 0.2 for $N \geq 2$

In this section, we consider the case $N \geq 2$. We remark that, when $N \geq 2$, our proofs almost are parallel to [BaL]. We assume $f(u) = u^p$ for $u \geq 0$ and $f(u) = 0$ for $u < 0$, where $p \in (1, \frac{N+2}{N-2})$ when $N \geq 3$, $p \in (1, \infty)$ when $N = 2$. We set

$$I(u) = \frac{1}{2}||u||^2_{H^1_b(R^N)} - ||u_+||^{p+1}_{L^{p+1}(R^N)} \in C^2(H^1(R^N), \mathbb{R}),$$

where

$$||u||^2_{H^1_b(R^N)} = ||u||^2_{H^1(R^N)} + \int_{R^N} b(x)u^2 dx$$

By the standard ways, we reduce $I_b$ to a functional

$$J(v) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{||v||}{||v_+||}_{H^1_b(R^N)}\right)^{\frac{2(p+1)}{p-1}}$$

which is defined on

$$\Sigma = \{v \in H^1(R^N) ||v||_{H^1_b(R^N)} = 1, v_+ \neq 0\}.$$

Then $J \in C^1(\Sigma, \mathbb{R})$, and, for any critical point $v \in \Sigma$ of $J(v)$, $t vv$ is a non-trivial critical point of $I(u)$ where $t_v = ||v||_{R^N}^{\frac{2}{H^1_b(p-1)}}||v_+||_{L^{p+1}(R^N)}^{-\frac{p+1}{p-1+1}}$. Thus, in what follows, we seek non-trivial critical points of $J(v)$.

Let $\omega(x)$ be an unique radially symmetric positive solution of (0.1) for $f(u) = u^p$ and we set $c_0 = \frac{1}{2}||\omega||^2_{H^1(R^N)} - \frac{1}{p+1}||\omega||_{H^1(R^N)} > 0$. For the (PS)-sequences of $J(u)$, we have the following:

**Proposition 2.1.** Suppose (b.1)-(b.2), (f.0) hold and let $(v_n)_{n=1}^\infty$ be a (PS)-sequence of $J(u)$. Then there exist a subsequence $n_j \to \infty$, $k \in \mathbb{N} \cup \{0\}$, $k$-sequences $(x_j^k)_{j=1}^\infty \subset R^N$, and a critical point $u_0$ of $I(u)$ such that

$$J(v_{n_j}) \to I(u_0) + kc_0 \quad (j \to \infty),$$

$$v_{n_j}(x) - \frac{u_0(x) - \sum_{j=1}^{k} \omega(x-x_j^\ell)}{||u_0(x) - \sum_{j=1}^{k} \omega(x-x_j^\ell)||_{H^1(R^N)}} \to 0 \quad \text{in} \quad H^1(R^N) \quad (j \to \infty),$$

$$|x_j^\ell - x_j^{\ell'}| \to \infty \quad (j \to \infty) \quad (\ell \neq \ell'),$$

$$|x_j^\ell| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \cdots, k).$$

**Proof.** Let $(v_n)_{n=1}^\infty$ be a (PS)-sequence of $J(v)$. Then $(t_{v_n}v_n)_{n=1}^\infty$ is a (PS)-sequence of $I(u)$. Moreover we remark that the set of the critical points of the functional $\frac{1}{2}||u||^2_{H^1_b(R^N)} - \frac{1}{p+1}||u_+||_{H^1(R^N)} : H^1(R^N) \to \mathbb{R}$ is written by $\{\omega(x+\xi) | \xi \in R^N\} \cup \{0\}$ from the uniqueness of positive solutions of (1.0). Thus Proposition 2.1 easily follows applying Theorem 5.1 of [JT1] to $(t_{v_n}v_n)_{n=1}^\infty$.

By the similar arguments of Section 1, we have the following corollary.
Corollary 2.2. Suppose $I(u)$ has no non-trivial critical points and let $(v_n)_{n=1}^\infty$ be a $(PS)$-sequence of $J(v)$. Then, only $kc_0$'s $(k \in \mathbb{N})$ can be limit points of $\{J(v_n) | n \in \mathbb{N}\}$.

We set
\[ c = \inf_{v \in \Sigma} J(v). \]
Then we can easily see that $0 < c \leq c_0$. From the boundedness of $J(v)$ from below, we get also more strong corollary.

Corollary 2.3. For any $b \in (-\infty, c_0) \cup (c_0, c_0 + c)$, $J(v)$ satisfies $(PS)_b$-condition.

Proof. If $(PS)_b$-condition does not hold for $b \in \mathbb{R}$, then for some $(PS)_b$-sequence $(v_n)_{n=1}^\infty$, it must be $k \neq 0$ in Proposition 2.1. Thus we have
\[ \lim_{n \to \infty} J(v_n) = b = kc_0 \quad \text{or} \quad \lim_{n \to \infty} J(v_n) = b \geq c + kc_0. \]
This implies Corollary 2.3.

When $c < c_0$, from Corollary 2.3, $c$ is a critical value of $J(v)$. Thus this case is easy. Thus we consider the case $c = c_0$. When $c = c_0$, we must define another minimax value. To define another minimax value, the following proposition is important.

Proposition 2.4. Suppose $N \geq 2$, $(b.1)$–$(b.3)$ and $(f.0)$ hold. Then, there exists $R_0 > 0$ such that for any $R \geq R_0$, we have
\[ \max_{(\zeta, \xi, t) \in \partial B_{\frac{1}{2}R} \times \partial B_R \times [0,1]} J\left( \frac{t\omega(x-\zeta) + (1-t)\omega(x-\xi)}{||t\omega(x-\zeta) + (1-t)\omega(x-\xi)||_{H^1(\mathbb{R}^N)}} \right) < 2c_0. \]
(2.1)
Here $B_R = \{ x \in \mathbb{R}^N \mid |x| \leq R \}$.

Proof. To get (2.1), for large $R > 0$, it sufficient to show
\[ \max_{(\zeta, \xi, s, \iota) \in \partial B_{\frac{1}{2}R} \times \partial B_R \times [0,1]} I(s\omega(x-\zeta) + t\omega(x-\xi)) < 2c_0. \]
(2.2)
In many papers [BaL], [A], [H1], [H2], the estimates like (2.2) were obtained. In [A], [H1], [H2], they treated more general $f(u)$ including $u_+^p$. Since we can get (2.2) by similar ways to those calculations, we omit the proof of (2.2).

Remark 2.5. When $N = 1$, the estimate (2.1) does not hold. (See Proposition 1.6 and [S].) We remark that, for some $C_0 > 0$, $\omega(x)$ satisfies
\[ 0 < \omega(x) \leq C_0|x|^{-\frac{N-1}{2}}e^{-|x|} \quad \text{for all} \quad x \in \mathbb{R}^N. \]
(2.3)
Roughly explaining about the difference from $N = 1$ and $N \geq 2$, when $N \geq 2$, we can obtain (2.1) by the effect of $|x|^{-\frac{N-1}{2}}$ in (2.3). On the other hand, when $N = 1$, since the effect of $|x|^{-\frac{N-1}{2}}$ vanishes, (2.1) does not hold.

To prove the Theorem 0.2, we also define a map $m : H^1(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ which is an expansion of $m$ defined in Section 1. That is, for any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, we consider a map

$$T_u(\xi) = \left( \int_{\mathbb{R}^N} \tan^{-1}(x_1 - \xi_1)|u(x)|^2 \, dx, \ldots, \int_{\mathbb{R}^N} \tan^{-1}(x_N - \xi_N)|u(x)|^2 \, dx \right)$$

$$: \mathbb{R}^N \to \mathbb{R}^N.$$  

Then we can see that $T_u(\xi)$ has an unique $\xi_u \in \mathbb{R}^N$ such that $T_u(\xi_u) = 0$ because

$$DT_u = \left[ \begin{array}{ccc} \int_{\mathbb{R}^N} \frac{1}{1+(x_1-\xi_1)^2} |u(x)|^2 \, dx & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \int_{\mathbb{R}^N} \frac{1}{1+(x_N-\xi_N)^2} |u(x)|^2 \, dx \end{array} \right].$$

Thus for any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, we define $m(u) = \xi_u$. We also find that $m(u)$ is of continuous by the implicit function theorem to $(u, \xi) \mapsto T_u(\xi)$. Since $\omega(x)$ is a radially symmetric function, from the definition of $m(u)$, we can easily see that

$$m(\omega(x-\xi)) = \xi \quad \text{for all } \xi \in \mathbb{R}^N. \quad (2.4)$$

In what follows, we will complete the proof of Theorem 0.2.

**Proof of Theorem 0.2 for $N \geq 2$.** We set

$$c = \inf_{v \in \Sigma} J(v).$$

When $c < c_0$, from Corollary 2.3, $c$ is a critical point of $J(v)$ and our proof is completed. Thus we must consider the case $c = c_0$. For $a \in \mathbb{R}^N$ we defined a minimax value $c_a > 0$ by

$$c_a = \inf_{v \in \Sigma_a} J(v),$$

$$\Sigma_a = \{ v \in \Sigma | m(v) = a \}. $$

Noting $\Sigma_a \subset \Sigma$ and $c = c_0$, we have

$$0 < c_0 \leq c_a.$$  

We will show that $I(u)$ has at least a non-trivial critical point for the following both cases:
(i) For some $a \in \mathbb{R}^N$, $c_0 = c_a$.
(ii) For some $a \in \mathbb{R}^N$, $c_0 < c_a$.

**Proof of Theorem 0.2 for the case (i).** For any $\epsilon > 0$, there exists $\tilde{v}_\epsilon \in \Sigma_a$ such that

$$c_0 \leq J(\tilde{v}_\epsilon) < c_0 + \epsilon.$$ 

Since $\tilde{v}_\epsilon \in \Sigma_a \subset \Sigma$ and $\Sigma$ is an invariant set by standard deformation flows of $J(v)$, by a standard Ekland principle, there exists $v_\epsilon \in \Sigma$ such that

$$c_0 \leq J(v_\epsilon) \leq J(\tilde{v}_\epsilon) < c_0 + \epsilon,$$

$$||J'(v_\epsilon)|| < 2\sqrt{\epsilon},$$

$$||v_\epsilon - \tilde{v}_\epsilon||_{H^1(\mathbb{R})} < \epsilon.$$ (2.5)

Then, from Proposition 2.1, there exist a subsequence $\epsilon_j \to 0$, $k \in \mathbb{N} \cup \{0\}$, $k$-sequences $(x_j^1)_{j=1}^\infty, \cdots, (x_j^k)_{j=1}^\infty \subset \mathbb{R}^N$, and a critical point $u_0$ of $I(u)$ such that

$$J(v_{\epsilon_j}) \to I(u_0) + kc_0 \quad (j \to \infty),$$

$$v_{\epsilon_j}(x) - \frac{u_0(x) - \sum_{\ell=1}^k \omega(x-x_j^\ell)}{||u_0(x) - \sum_{\ell=1}^k \omega(x-x_j^\ell)||_{H^1(\mathbb{R}^N)}} \to 0 \quad \text{in} \quad H^1(\mathbb{R}^N) \quad (j \to \infty),$$

$$|x_j^\ell - x_j^{\ell'}| \to \infty \quad (j \to \infty) \quad (\ell \neq \ell'),$$

$$|x_j^\ell| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \cdots, k).$$ (2.6)

Now, if $u_0 \neq 0$, our proof is completed. So we suppose $u_0 = 0$. Then, from (2.6), it must be $k = 1$. Thus, we have

$$\left\|v_{\epsilon_j}(x) - \frac{\omega(x-x_j)}{||\omega||_{H^1(\mathbb{R}^N)}}\right\|_{H^1(\mathbb{R}^N)} \to 0 \quad (j \to \infty),$$

$$|x_j^1| \to \infty \quad (j \to \infty).$$ (2.7)

From (2.4), (2.5) and (2.7), we see that

$$|m(\tilde{v}_{\epsilon_j})| \to \infty \quad \text{as} \quad j \to \infty.$$ 

This contradicts to $m(\tilde{v}_{\epsilon_j}) = a$. Therefore $u_0 \neq 0$ and $I(u)$ has at least a non-trivial critical point. \[\blacksquare\]
Proof of the Theorem 0.2 for the case (ii). From Proposition 2.4, we set $\zeta_0 = (\frac{1}{2} R_0, 0, \ldots, 0)$ and $\delta = \frac{1}{2} (c_a - c_0) > 0$ and choose a large $R_0 > |a|$ such that

$$\max_{\xi \in \partial B_{R_0}} J(\omega(x - \xi)) < c_0 + \delta < c_a, \quad (2.8)$$

$$\max_{(\xi, t) \in \partial B_{R_0} \times [0, 1]} J \left( \frac{t \omega(x - \zeta_0) + (1 - t) \omega(x - \xi)}{||t \omega(x - \zeta_0) + (1 - t) \omega(x - \xi)||_{H^1(\mathbb{R}^N)}} \right) < 2c_0. \quad (2.9)$$

Here we define the following minimax value:

$$c_2 = \inf_{\gamma \in \Gamma} \max_{\xi \in B_{R_0}} J(\gamma(\xi)), \quad \Gamma = \left\{ \gamma(\xi) \in C(B_{R_0}, \Sigma) \mid \gamma(\xi)(x) = \frac{\omega(x + \xi)}{||\omega||_{H^1(\mathbb{R}^N)}} \text{ for all } \xi \in \partial B_{R_0} \right\}. \quad (2.10)$$

Then we have the following lemma.

**Lemma 2.6.** We have

$$0 < c_0 < c_a \leq c_2 < 2c_0.$$

We postpone the proof of Lemma 2.6 to end of this section. If Lemma 2.6 is true, then $\Gamma$ is an invariant set by the deformation flows of $J(v)$. Thus $J(v)$ has a (PS)-sequence $(v_n)_{n=1}^{\infty}$ such that

$$J(v_n) \rightarrow c_2 \in (c_0, 2c_0) \quad (n \rightarrow \infty).$$

From Corollary 2.3, $J(u)$ satisfies $(PS)_{c_2}$-conditions. Thus $c_2$ is a critical value of $J(v)$. That is, $I(u)$ has at least a non-trivial critical point. Combining the proofs of the cases (i)-(ii), we complete a proof of Theorem 0.2.

Finally we show Lemma 2.6.

**Proof of Lemma 2.6.** The inequality $c_0 < c_a$ is an assumption of the case (ii). From (2.9), $c_2 < 2c_0$ is obvious. Thus we show $c_a \leq c_2$. For any $\gamma \in \Gamma$, from (2.10), we have

$$m(\gamma(\xi)) = \xi \quad \text{for all } \xi \in \partial B_{R_0}. \quad (2.11)$$

Thus we can see

$$\deg(m \circ \gamma, B_{R_0}, a) = 1. \quad (2.10)$$

From (2.10), there exists $\xi_0 \in B_{R_0}$ such that $m(\gamma(\xi_0)) = a$. Therefore, since $\gamma(\xi_0) \in \Sigma_a$, we find that

$$c_a \leq \inf_{v \in \Sigma_a} J(v) \leq J(\gamma(\xi_0))) \leq \max_{\xi \in B_{R_0}} I(\gamma(\xi)). \quad (2.11)$$
Since $\gamma \in \Gamma$ is arbitrary, from (2.11), we have

$$c_a \leq c_2.$$ 

Thus we get Lemma 2.6.

References


