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Existence and non-existence of the nonlinear Schrödinger equations for one and high dimensional case

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0. Introduction

In this report, we will introduce the results of [S] and related results. We consider the following nonlinear Schrödinger equations:

\[- \Delta u + (1 + b(x))u = f(u) \quad \text{in } \mathbb{R}^N,\]
\[u \in H^1(\mathbb{R}^N).\]

We mainly considered the one-dimensional case in [S] but, in this report, we consider not only one-dimensional case but also the high-dimensional case. Here, we assume that the potential \( b(x) \in C(\mathbb{R}, \mathbb{R}) \) satisfies the following assumptions:

(b.1) \( 1 + b(x) \geq 0 \) for all \( x \in \mathbb{R}^N \).
(b.2) \( \lim_{|x| \to \infty} b(x) = 0 \).
(b.3) There exist \( \beta_0 > 2 \) and \( C_0 > 0 \) such that \( b(x) \leq C_0 e^{-\beta_0|x|} \) for all \( x \in \mathbb{R}^N \).

We also assume that the nonlinearity \( f(u) \in C(\mathbb{R}, \mathbb{R}) \) satisfies the following

(f.0) \( f(u) = |u|^{p-1}u \) for \( p \in (1, \frac{N+2}{N-2}) \) when \( N \geq 3 \) and \( p \in (1, \infty) \) when \( N = 2 \).
(f.1) There exists \( \eta_0 > 0 \) such that \( \lim_{|u| \to 0} \frac{f(u)}{|u|^{1+\eta_0}} = 0 \).
(f.2) There exists \( u_0 > 0 \) such that

\[ F(u) < \frac{1}{2} u^2 \quad \text{for all } u \in (0, u_0), \]
\[ F(u_0) = \frac{1}{2} u_0^2, \quad f(u_0) > u_0. \]
(f.3) There exists \( \mu_0 > 2 \) such that \( 0 < \mu_0 F(u) \leq uf(u) \) for all \( u \neq 0 \).
To consider the (⋆), the following equation plays an important roles:

\[-\Delta u + u = f(u) \quad \text{in} \quad \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N). \tag{0.1}\]

From (b.2), the equation \(-\Delta u + u = f(u)\) appears as a limit when \(|x|\) goes to \(\infty\) in (⋆). To show the existence of positive solution of (⋆) in our arguments, the uniqueness (up to translation) of positive solutions of (0.1) is also important. Under the condition (f.0), it is well-known that the uniqueness (up to translation) of the positive solutions of (0.1). When \(N = 1\), it is known that the conditions (f.1) and (f.2) are sufficient conditions for (0.1) to have an unique (up to translation) positive solution:

**Remark 0.1.** In Section 5 of [BeL1], Berestycki-Lions showed that if \(f(u)\) is of locally Lipschitz continuous and \(f(u) = 0\), then (f.2) is a necessary and sufficient condition for the existence of a non-trivial solution of (1.0). Moreover, it also was shown that the uniqueness (up to translation) of positive solutions under the (f.2). In Section 2 of [JT1], Jeanjean-Tanaka showed that when \(f(u)\) is of continuous, (f.1) and (f.2) are sufficient conditions for (0.1) to have an unique positive solution.

The condition (f.3) is so called Ambrosetti-Rabinowitz condition, which guarantees the boundedness of (PS)-sequences for the functional corresponding to the equation (⋆) and (0.1). To state an our result for one-dimensional case, we also need the following assumption for \(b(x)\).

(b.4) When \(N = 1\), there exists \(x_0 \in \mathbb{R}\) such that

\[\int_{-\infty}^{\infty} b(x - x_0)e^{2|x|} \, dx \in [-\infty, 2).\]

Our first theorem is the following.

**Theorem 0.2.** When \(N \geq 2\), we assume that (b.1)–(b.3) and (f.0) hold. Then (⋆) has at least a positive solution. When \(N = 1\), we assume that (b.1)–(b.4) and (f.1)–(f.3) hold. Then (⋆) has at least a positive solution.

In [S], we give a proof of Theorem 0.2 for the one-dimensional case. To prove the theorem 0.2, we developed the arguments of [BaL] and [Sp]. We remark that, for high-dimensional case, the proof of Theorem 0.2 almost are parallel to the proof of [BaL]. However, for the proof of the one-dimensional case, we essentially developed the arguments of [BaL] and [Sp]. Bahri-Li [BaL] showed that there exists a positive solution of

\[-\Delta u + u = (1 - b(x))|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \tag{0.2}\]
where $N \geq 3$, $1 < p < \frac{N+2}{N-2}$ and $b(x) \in C(\mathbb{R}, \mathbb{R})$ satisfies (b.2)-(b.3) and

\[ (b.1)\' \quad 1 - b(x) \geq 0 \text{ for all } x \in \mathbb{R}^N. \]

For one dimensional case, Spradlin [Sp] proved that there exists a positive solution of the equation

\[ -u'' + u = (1 - b(x))f(u) \quad \text{in } \mathbb{R}, \quad u \in H^1(\mathbb{R}). \]  \hspace{1cm} (0.3)

They also assumed that $b(x) \in C(\mathbb{R}, \mathbb{R})$ satisfies $(b.1)'$ and $(b.2)$-$(b.3)$ and $f(u)$ satisfies $(f.1)$-$(f.3)$ and

\[ (f.4) \quad \frac{f(u)}{u} \text{ is an increasing function for all } u > 0. \]

When (f.0) or (f.4) holds, we can consider the Nehari manifold and they argued on Nehari manifold in [BaL] and [Sp]. In our situation, when $N = 1$, we can not argue on Nehari manifold. This was one of the difficulties which had to overcome in [S].

From the above results and Theorem 0.2, it seems that, when $N = 1$, Theorem 0.2 holds without condition (b.4). However (b.4) is an essential assumption for $(*)$ to have non-trivial solutions. In what follows, we will show a result about the non-existence of nontrivial solutions for $(*)$.

In next our result, we will assume that $N = 1$ and $b(x)$ satisfies the following condition:

\[ (b.5) \quad \text{There exist } \mu > 0 \text{ and } m_2 \geq m_1 > 0 \text{ such that} \]

\[ m_1 \mu e^{-\mu|x|} \leq b(x) \leq m_2 \mu e^{-\mu|x|} \quad \text{for all } x \in \mathbb{R}. \]

Here, we remark that, if (b.5) holds for $\mu > 2$, then $b(x)$ satisfies (b.1)-(b.3) and

\[ \frac{2\mu}{\mu - 2} m_1 \leq \int_{-\infty}^{\infty} b(x) e^{2|x|} \, dx \leq \frac{2\mu}{\mu - 2} m_2. \]

Thus, when $m_2 < 1$ and $\mu$ is very large, the condition (b.4) also holds.

Our second result is the following:

**Theorem 0.3.** Assume $N = 1$, (b.5) holds and $f(u) = |u|^{p-1}u$ ($p > 1$).

(i) If $m_1 > 1$, there exists $\mu_1 > 0$ such that $(*)$ does not have non-trivial solution for all $\mu \geq \mu_1$.

(ii) If $m_2 < 1$, there exists $\mu_2 > 0$ such that $(*)$ has at least a non-trivial solution for all $\mu \geq \mu_2$.

(iii) There exists $\mu_3 > 0$ such that $(*)$ does not have sign-changing solutions for all $\mu \geq \mu_3$.

From Theorem 0.3, we see that Theorem 0.2 does not hold except for condition (b.4). This is a drastically different situation from the high-dimensional cases. This is one of the interesting points in our results.
We remark that the condition (b.4) implies \( \int_{-\infty}^{\infty} b(x) \, dx < 2 \) and the assumption of (ii) of Theorem 0.3 also means \( \int_{-\infty}^{\infty} b(x) \, dx < 2 \). Thus we expect that the difference from existence and non-existence of non-trivial solutions of (*) depends on the quantity of integrate of \( b(x) \).

We can obtain this expectation from another viewpoint, which is a perturbation problem. Setting \( b_\mu(x) = m\mu e^{-\mu|x|} \), \( b_\mu(x) \) satisfies (b.5) and, when \( \mu \to \infty \), \( b_\mu(x) \) converges to the delta function \( 2m\delta_0 \) in distribution sense. Thus (*) approaches to the equation

\[
-u'' + (1 + 2m\delta_0)u = |u|^{p-1}u \quad \text{in } \mathbb{R}, \quad u \in H^1(\mathbb{R}),
\]

in distribution sense. Here, if \( u \) is a solution of (0.4) in distribution sense, we can see that \( u \) is of \( C^2 \)-function in \( \mathbb{R} \setminus \{0\} \) and continuous in \( \mathbb{R} \) and \( u \) satisfies

\[
u'(+0) = u'(-0) = 2mu(0).
\]

Moreover, since \( u \) is a homoclinic orbit of \(-u'' + u = f(u) \) in \((\infty,0)\) or \((0,\infty)\), respectively, \( u \) satisfies

\[
-\frac{1}{2}u'(x)^2 + \frac{1}{2}u(x)^2 - \frac{1}{p+1}|u(x)|^{p+1} = 0 \quad \text{for } x \neq 0.
\]

When \( x \to \pm 0 \) in (0.6), from (f.1), we find

\[
u'(-0) = -u'(0), \quad |u'(\pm 0)| < |u(0)|.
\]

Thus, from (0.5) and (0.7), it easily see that (0.4) has an unique positive solution when \( |m| < 1 \) and (0.4) has no non-trivial solutions when \( |m| \geq 1 \). Therefore we can regard Theorem 0.3 as results of a perturbation problem of (0.4).

To prove Theorem 0.3, we develop the shooting arguments which used in [BE]. Bianchi and Egnell [BE] argued about the existence and non-existence of radial solutions for

\[
-\Delta u = K(|x|)|u|^{\frac{N+2}{N-2}}, \quad u > 0 \quad \text{in } \mathbb{R}^N, \quad u(x) = O(|x|^{2-N}) \quad \text{as } |x| \to \infty.
\]

Here \( N \geq 3 \) and \( K(|x|) \) is a radial continuous function. Roughly speaking their approach, by setting \( u(r) = u(|x|) \), they reduce (0.8) to an ordinary differential equation and considered solutions of two initial value problems of that ordinary differential equation which have initial conditions \( u(0) = \lambda \) and \( \lim_{r \to \infty} r^{N-2}u(r) = \lambda \). And, examining whether those solutions have suitable matchings at \( r = 1 \), they argued about the existence and non-existence of radial solutions.
In [S], to prove Theorem 0.3, we also consider two initial value problems from \( \pm \infty \), that is, for \( \lambda_1, \lambda_2 > 0 \), we consider the following two problems:

\[
-u'' + (1 + b(x))u = f(u),
\]
\[
\lim_{x \to -\infty} e^{-x}u(x) = \lim_{x \to -\infty} e^{-x}u'(x) = \lambda_1,
\]

and

\[
-u'' + (1 + b(x))u = f(u),
\]
\[
\lim_{x \to \infty} e^{x}u(x) = -\lim_{x \to \infty} e^{x}u(x) = \lambda_2.
\]

Then (0.9) and (0.10) have an unique solution respectively and write those solutions as

\( u_1(x; \lambda_1) \) and \( u_2(x; \lambda_2) \) respectively. We set

\[\Gamma_1 = \{(u_1(0; \lambda_1), u_1'(0; \lambda_1)) \in \mathbb{R}^2 \mid \lambda_1 > 0\},\]
\[\Gamma_2 = \{(u_2(0; \lambda_2), u_2'(0; \lambda_2)) \in \mathbb{R}^2 \mid \lambda_2 > 0\} .\]

Then, \( \Gamma_1 \cap \Gamma_2 = \emptyset \) is equivalent to the non-existence of solutions for \( (*) \). Thus it is important to study shapes of \( \Gamma_1 \) and \( \Gamma_2 \). In respect to the details of proofs of Theorem 0.3, see [S].

In next sections, we state about the outline of the proof of Theorem 0.2. We will consider the one-dimensional case in Section 1 and treat the high-dimensional case in Section 2.

1. The outline of the proof of Theorem 0.2 for \( N = 1 \)

In this section, we consider the case \( N = 1 \). We will developed a variational approach which was used in [BaL] and [Sp].

In what follows, since we seek positive solutions of \( (*) \), without loss of generalities, we assume \( f(u) = 0 \) for \( u < 0 \). To prove Theorem 0.2, we seek non-trivial critical points of the functional

\[
I(u) = \frac{1}{2} ||u||_{H^1(\mathbb{R})}^2 + \frac{1}{2} \int_{-\infty}^{\infty} b(x)u^2 dx - \int_{-\infty}^{\infty} F(u) dx \in C^1(H^1(\mathbb{R}), \mathbb{R}),
\]

whose critical points are positive solutions of \( (*) \). Here we use the following notations:

\[
||u||_{H^1(\mathbb{R})}^2 = ||u'||_{L^2(\mathbb{R})}^2 + ||u||_{L^2(\mathbb{R})}^2,
\]
\[
||u||_{L^p(\mathbb{R})}^p = \int_{\mathbb{R}} |u|^p dx \quad \text{for} \quad p > 1.
\]

From (f.1)–(f.2), we can see that \( I(u) \) satisfies a mountain pass geometry (See Section 3 in [JT2]), that is, \( I(u) \) satisfies
$I(0) = 0$.

(ii) There exist $\delta > 0$ and $\rho > 0$ such that $I(u) \geq \delta$ for all $||u||_{H^1(\mathbb{R})} = \rho$.

(iii) There exists $u_0 \in H^1(\mathbb{R})$ such that $I(u_0) < 0$ and $||u_0||_{H^1(\mathbb{R})} > \rho$.

From the mountain pass geometry (i)–(iii), we can define a standard minimax value $c > 0$ by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

$$\Gamma = \{\gamma(t) \in C([0,1], H^1(\mathbb{R})) | \gamma(0) = 0, I(\gamma(1)) < 0\}.$$ (1.1)

And, by a standard way, we can construct $(PS)_c$-sequence $(u_n)_{n=1}^{\infty}$, that is, $(u_n)_{n=1}^{\infty}$ satisfies

$$I(u_n) \to c \quad (n \to \infty),$$

$$I'(u_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}) \quad (n \to \infty).$$

Moreover, since $(u_n)_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{R})$ from (f.3), $(u_n)_{n=1}^{\infty}$ has a subsequence $(u_{n_j})_{j=1}^{\infty}$ which weakly converges to some $u_0$ in $H^1(\mathbb{R})$. If $(u_{n_j})_{j=1}^{\infty}$ strongly converges to $u_0$ in $H^1(\mathbb{R})$, $c$ is a non-trivial critical value of $I(u)$ and our proof is completed. However, since the embedding $L^p(\mathbb{R}) \subset H^1(\mathbb{R})$ ($p > 1$) is not compact, there may not exist a subsequence $(u_{n_j})_{j=1}^{\infty}$ which strongly converges in $H^1(\mathbb{R})$. Therefore, in our situation, we don’t know $c$ is a critical value.

In our situation, from the lack of the compactness mentioned the above, we must use the concentration-compactness approach as [BaL] and [Sp]. In the concentration-compactness approach, we examine in detail what happens in bounded $(PS)$-sequence. When we state the concentration-compactness argument for the $(PS)$-sequences of $I(u)$, the limit problem (0.1) plays an important role. Setting

$$I_0(u) = \frac{1}{2} ||u||_{H^1(\mathbb{R})}^2 - \int_{-\infty}^{\infty} F(u) \, dx \in C^1(H^1(\mathbb{R}), \mathbb{R}),$$

the critical points of $I_0(u)$ correspond to the solutions of limit problem (0.1). The equation (0.1) has an unique positive solution, identifying ones which obtain by translations. Thus let $\omega(x)$ be an unique positive solution of (0.1) with $\max_{x \in \mathbb{R}} \omega(x) = \omega(0)$ and we set $c_0 = I_0(\omega)$. Since $I_0$ also satisfies the mountain pass geometry (i)–(iii), we see $c_0 > 0$ and $c_0$ is an unique non-trivial critical value.

For the bounded $(PS)$-sequences of $I(u)$, we have the following:

**Proposition 1.1.** Suppose (b.1)–(b.2) and (f.1)–(f.2) hold. If $(u_n)_{n=1}^{\infty}$ is a bounded $(PS)$-sequence of $I(u)$, then there exist a subsequence $n_j \to \infty$, $k \in \mathbb{N} \cup \{0\}$, $k$-sequences
$(x^1_j)_{j=1}^\infty, \ldots, (x^k_j)_{j=1}^\infty \subset \mathbb{R}$, and a critical point $u_0$ of $I(u)$ such that

$$I(u_{n_j}) \to I(u_0) + kc_0 \quad (j \to \infty),$$

$$\left\| u_{n_j}(x) - u_0(x) - \sum_{\ell=1}^{k} \omega(x - x^\ell_j) \right\|_{H^1(\mathbb{R})} \to 0 \quad (j \to \infty),$$

$$|x^\ell_j - x^\ell_j'| \to \infty \quad (j \to \infty) \quad (\ell \neq \ell'),$$

$$|x^\ell_j| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \ldots, k).$$

**Proof.** We can easily get Proposition 1.1 from Theorem 5.1 of [JT1]. Theorem 5.1 of [JT1] required the assumption $\lim_{u\to\infty} f(u)u^{-p} = 0$ $(p > 1)$. However we take off that assumption for one dimensional case by improving Step 2 of Theorem 5.1 of [JT1]. In fact we have only to change $\sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |v^1_n|^2 \, dx \to 0$ in Step 2 to $\|v^1_n\|_{L^\infty(\mathbb{R})} \to 0$.

If the minimax value $c$ satisfies $c \in (0, c_0)$, from Proposition 1.1, we see that $I(u)$ has at least a non-trivial critical point. In fact, let $(u^\infty_n)_{n=1}^\infty$ be a bounded $(PS)_c$-sequence of $I(u)$, from Proposition 1.1, there exists a subsequence $n_j \to \infty$, $k \in \mathbb{N} \cup \{0\}$ and a critical point $u_0$ of $I(u)$ such that

$$I(u_{n_j}) \to I(u_0) + kc_0 \quad (j \to \infty).$$

Here, if $u_0 = 0$, we get $I(u_{n_j}) \to kc_0$ as $j \to \infty$. However this contradicts to the fact that $I(u_n) \to c \in (0, c_0)$ as $n \to \infty$. Thus $u_0 \neq 0$ and $u_0$ is a non-trivial critical point of $I(u)$.

From the above argument, we have the following corollary.

**Corollary 1.2.** Suppose $I(u)$ has no non-trivial critical points and let $(u^\infty_n)_{n=1}^\infty$ be a $(PS)_c$-sequence of $I(u)$. Then, only $kc_0$'s $(k \in \mathbb{N} \cup \{0\})$ can be limit points of $\{I(u_n) \mid n \in \mathbb{N}\}$.

**Remark 1.3.** Corollary 1.2 essentially depends on the uniqueness of the positive solution of (0.1).

As mentioned the above, when $c \in (0, c_0)$, $I(u)$ has at least a non-trivial critical point. However, unfortunately, under the condition (b.1)–(b.4), it may be $c = c_0$. Thus we need consider another minimax value. To define another minimax value, we use a path $\gamma_0(t) \in C(\mathbb{R}, H^1(\mathbb{R}))$ which is defined as follows: for small $\epsilon_0 > 0$, we set

$$h(x) = \begin{cases} \omega(x) & x \in [0, \infty], \\ x^4 + u_0 & x \in [-\epsilon_0, 0], \\ \epsilon_0^4 + u_0 & x \in (-\infty, -\epsilon_0), \end{cases}$$

$$\gamma_0(t)(x) = \begin{cases} h(x - t) & x \geq 0, \\ h(-x - t) & x < 0. \end{cases}$$
Here, we remark that $u_0$ was given in (f.2). This path $\gamma_0(t)$ was introduced in [JT2]. Choosing a proper $\epsilon_0 > 0$ sufficiently small, $\gamma_0(t)$ achieves the mountain pass value of $I_0(u)$ and satisfies the followings:

**Lemma 1.4.** Suppose (f.1)-(f.2) hold. Then $\gamma_0(t)$ satisfies

(i) $\gamma_0(0)(x) = \omega(x)$.
(ii) $I_0(\gamma_0(t)) < I_0(\omega) = c_0$ for all $t \neq 0$.
(iii) $\lim_{t \rightarrow -\infty} ||\gamma_0(t)||_{H^1(R)} = 0$, $\lim_{t \rightarrow \infty} ||\gamma_0(t)||_{H^1(R)} = \infty$.

**Proof.** See Section 3 in [JT2].

**Remark 1.5.** When $f(u)/u$ is a increasing function, we can use a simpler path than $\gamma_0(t)$. In fact, setting $\tilde{\gamma}_0(t) = t\omega : [0, \infty) \rightarrow H^1(R)$, we also have

(i) $\tilde{\gamma}_0(1)(x) = \omega(x)$.
(ii) $I_0(\tilde{\gamma}_0(t)) < I_0(\omega) = c_0$ for all $t \neq 1$.
(iii) $\tilde{\gamma}_0(0) = 0$, $\lim_{t \rightarrow \infty} ||\tilde{\gamma}_0(t)||_{H^1(R)} = \infty$.

Moreover, if $f(u)/u$ is a increasing function, in what follows, we can also construct a simpler proofs by arguing on Nehari manifold $N = \{u \in H^1(R) \setminus \{0\} | I'(u)u = 0\}$. (See [Sp].)

Now, for $R > 0$, we consider a path $\gamma_R \in C(R^2, H^1(R))$ which is defined by

$$\gamma_R(s, t)(x) = \max\{\gamma_0(s)(x + R), \gamma_0(t)(x - R)\}.$$ 

In our proof of Theorem 0.2 in [S], the following proposition is a key proposition.

**Proposition 1.6.** Suppose (b.1)-(b.3) and (f.1)-(f.2) hold. Then, for any $L > 0$, we have

$$\lim_{R \rightarrow \infty} e^{2R} \left\{ \max_{(s,t) \in [-L,L]^2} I(\gamma_R(s, t)) - 2c_0 \right\} \leq \frac{\lambda_0^2}{2} \left( \int_{-\infty}^{\infty} b(x)e^{2|x|} dx - 2 \right).$$

(1.2)

Here $\lambda_0 = \lim_{x \rightarrow \pm \infty} \omega(x)e^{|x|}$.

**Proof.** See [S].

By using a translation, without loss of generalities, we assume $x_0 = 0$ in (b.4). If (b.4) with $x_0 = 0$ holds, from Proposition 1.6, for any $L > 0$, there exists $R_0 > 0$ such that

$$\max_{(s,t) \in [-L,L]^2} I(\gamma_{R_0}(s, t)) < 2c_0.$$
To prove the Theorem 0.2, we also need a map $m : H^1(R) \setminus \{0\} \rightarrow R$ which is defined by the following: for any $u \in H^1(R) \setminus \{0\}$, a function

$$T_u(s) = \int_{-\infty}^{\infty} \tan^{-1}(x-s)|u(x)|^2 \, dx : R \rightarrow R$$

is strictly decreasing and $\lim_{s \to \infty} T_u(s) = -||u||_{L^2(R)}^2 < 0$ and $\lim_{s \to -\infty} T_u(s) = ||u||_{L^2(R)}^2 > 0$. Thus, from the theorem of the intermediate value, $T_u(s)$ has an unique $s = m(u)$ such that $T_u(m(u)) = 0$. We also find that $m(u)$ is of continuous by the implicit function theorem to $(u, s) \mapsto T_u(s)$. The map $m(u)$ was introduced in [Sp]. We remark that $m(u)$ is regarded as a kind of center of mass of $|u(x)|^2$ and we can check the followings.

**Lemma 1.7.** We have

(i) $m(\gamma_0(t)) = 0$ for all $t \in R$.

(ii) $m(\gamma_R(s, t)) > 0$ for all $-R < s < t < R$.

(iii) $m(\gamma_R(s, t)) < 0$ for all $-R < t < s < R$.

**Proof.** Since $\gamma_0(t)(x)$ is a even function, we have (i). We Note that

$$\gamma_R(s, t)(x) = \begin{cases} 
\gamma_0(s)(x + R) & \text{for } x \in (-\infty, \frac{s-t}{2}], \\
\gamma_0(t)(x - R) & \text{for } x \in (\frac{s-t}{2}, \infty). 
\end{cases}$$

Since $\gamma_R(s, s)(x)$ is also a even function, we have

$$m(\gamma_R(s, s)) = 0 \quad \text{for all } s \in R,$$

and we get (ii)–(iii). \qed

In what follows, we will complete the proof of Theorem 0.2 for $N = 1$.

**Proof of Theorem 0.2 for $N = 1$.** First of all, we defined a minimax value $c_1 > 0$ by

$$c_1 = \inf_{\gamma \in \Gamma_1} \max_{t \in [0,1]} I(\gamma(t)),$$

$$\Gamma_1 = \{ \gamma(t) \in C([0,1], H^1(R)) | \gamma(0) = 0, \ I(\gamma(1)) < 0, \ |m(\gamma(t))| < 1 \}.$$  

Noting $\Gamma_1 \subset \Gamma$, we have

$$0 < c \leq c_1.$$ 

Since $\Gamma_1$ is not invariant by standard deformation flows of $I(u)$, $c_1$ may not be a critical point of $I(u)$. We will use $c_1$ to divide the case. We divide the case into the following three cases:

(i) $c_1 < c_0$. 

(ii) $c_1 = c_0$.

(iii) $c_1 > c_0$.

**Proof of Theorem 0.2 for the case (i).** Since the inequality $c_1 < c_0$ implies $0 < c < c_0$, from Corollary 1.2, we can see $I(u)$ has at least a non-trivial critical point.

**Proof of Theorem 0.2 for the case (ii).** In this case, if $c < c_1 = c_0$, then $I(u)$ has at least a non-trivial critical point from Corollary 1.2. Thus we may consider the case $c = c_1 = c_0$. In this case, for any $\epsilon > 0$, there exists $\gamma(t) \in \Gamma_1$ such that

$$c \leq \max_{t \in [0,1]} I(\gamma(t)) < c + \epsilon.$$ 

Since $\gamma(t) \in \Gamma_1 \subset \Gamma$ and $\Gamma$ is an invariant set by standard deformation flows of $I(u)$, by a standard Ekland principle, there exists $u_\epsilon \in H^1(\mathbb{R})$ such that

$$c \leq I(u_\epsilon) \leq \max_{t \in [0,1]} I(\gamma(t)) < c + \epsilon,$$

$$||I'(u_\epsilon)|| < 2\sqrt{\epsilon},$$

$$\inf_{t \in [0,1]} ||u_\epsilon - \gamma(t)||_{H^1(\mathbb{R})} < \epsilon. \tag{1.3}$$

Then, from Proposition 1.1, there exist a subsequence $\epsilon_j \to 0$, $k \in \mathbb{N} \cup \{0\}$, $k$-sequences $(x_j^1)_{j=1}^{\infty}, \ldots, (x_j^k)_{j=1}^{\infty} \subset \mathbb{R}$, and a critical point $u_0$ of $I(u)$ such that

$$I(u_{\epsilon_j}) \to I(u_0) + kc_0 \quad (j \to \infty),$$

$$||u_{\epsilon_j}(x) - u_0(x) - \sum_{\ell=1}^{k} \omega(x - x_j^\ell)||_{H^1(\mathbb{R})} \to 0 \quad (j \to \infty),$$

$$|x_j^\ell - x_j^{\ell'}| \to \infty \quad (j \to \infty) \quad (\ell \neq \ell'),$$

$$|x_j^\ell| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \ldots, k). \tag{1.4}$$

Now, if $u_0 \neq 0$, our proof is completed. So we suppose $u_0 = 0$. Then, from (1.4), it must be $k = 1$. Thus, we have

$$||u_{\epsilon_j}(x) - \omega(x - x_j^1)||_{H^1(\mathbb{R})} \to 0 \quad (j \to \infty),$$

$$|x_j^1| \to \infty \quad (j \to \infty) \quad \tag{1.5}$$

On the other hand, we remark that, since $m(\omega) = 0$ and $m$ is of continuous, there exists $\delta > 0$ such that

$$|m(u)| < 1 \quad \text{for all} \quad u \in B_\delta(\omega) = \{v \in H^1(\mathbb{R}) | ||v - \omega||_{H^1(\mathbb{R})} < \delta\}.$$
Thus, from (1.3) and (1.5), for some $\epsilon_0 \in (0, \frac{\delta}{2})$ and $t_0 \in [0, 1]$, we have

$$|m(\gamma_{\epsilon_0}(t_0)) - x_j^1| < 1.$$  

This contradicts to $\gamma_{\epsilon_0} \in \Gamma_1$. Therefore $u_0 \neq 0$ and $I(u)$ has at least a non-trivial critical point.

**Proof of the Theorem 0.2 for the case (iii).** First of all, we set $\delta = \frac{c_1 - c_0}{2} > 0$ and choose $L_0 > 0$ such that

$$\max_{(s,t) \in D_{2L_0} \setminus D_{L_0}} I(\gamma_R(s, t)) < c_0 + \delta < c_1 \quad \text{for all} \quad R > 3L_0. \quad (1.6)$$

Here we set $D_L = [L, L] \times [L, L] \subset \mathbb{R}^2$. Next, from Proposition 1.6, we can choose $R_0 > 3L_0$ such that

$$\max_{(s,t) \in D_{L_0}} I(\gamma_{R_0}(s, t)) < 2c_0. \quad (1.7)$$

Here we fix $\gamma_{R_0}(s, t)$ and define the following minimax value:

$$c_2 = \inf_{\gamma \in \Gamma_2} \max_{(s,t) \in D_{2L_0}} I(\gamma(s, t)),$$

$$\Gamma_2 = \{ \gamma(s, t) \in C(D_{2L_0}, H^1(\mathbb{R})) | \gamma(s, t) = \gamma_{R_0}(s, t) \text{ for all } (s, t) \in D_{2L_0} \setminus D_{L_0} \}.$$

Then we have the following lemma.

**Lemma 1.8.** We have

$$0 < c_0 < c_1 \leq c_2 < 2c_0.$$  

We postpone the proof of Lemma 1.8 to end of this section. If Lemma 1.8 is true, then $\Gamma_2$ is an invariant set by the deformation flows of $I(u)$. Thus $I(u)$ has a (PS)-sequence $(u_n)_{n=1}^\infty$ such that

$$I(u_n) \rightarrow c_2 \in (c_0, 2c_0) \quad (n \rightarrow \infty).$$

From Corollary 1.2, we can see that $I(u)$ must have at least a non-trivial critical point. Combining the proofs of the cases (i)--(iii), we complete a proof of Theorem 0.2.  

Finally we show Lemma 1.8.

**Proof of Lemma 1.8.** The inequality $c_0 < c_1$ is an assumption of the case (iii). From $\gamma_{R_0} \in \Gamma_2$ and (1.6)--(1.7), $c_2 < 2c_0$ is obvious. Thus we show $c_1 \leq c_2$. For any $\gamma(s, t) \in \Gamma_2$, we have

$$m(\gamma(s, t)) > 0 \quad \text{for all} \quad (s, t) \in D_1, \quad (1.8)$$

$$m(\gamma(s, t)) < 0 \quad \text{for all} \quad (s, t) \in D_2. \quad (1.9)$$
Here we set $D_1 = \{(s, t) \in D_{2L_0} \setminus D_{L_0} | s < t\}$ and $D_2 = \{(s, t) \in D_{2L_0} \setminus D_{L_0} | s > t\}$.

From (1.8)–(1.9), a set $\{(s, t) \in D_{2L_0} \setminus D_{L_0} | m(\gamma(s, t)) < 1\}$ have a connected component which contains a path joining two points $\gamma_{R_0}(-2L_0, -2L_0)$ and $\gamma_{R_0}(2L_0, 2L_0)$. Thus we construct a path $\gamma_1(t) \in \Gamma_1$ such that

$$\{\gamma_1(t) | t \in [1/3, 2/3]\} \subset \{\gamma(s, t) | (s, t) \in D_{2L_0}\},$$

$$\max_{t \in [0, 1/3] \cup [2/3, 1]} I(\gamma_1(t)) \leq c_0.$$ 

Thus we see

$$c_1 \leq \max_{t \in [0, 1]} I(\gamma_1(t)) \leq \max_{(s, t) \in D_{2L_0}} I(\gamma(s, t)).$$ (1.10)

Since $\gamma(s, t) \in \Gamma_2$ is arbitrary, from (1.10), we have

$$c_1 \leq c_2.$$ 

Thus we get Lemma 1.8.

**Remark 1.9.** In our proofs of Theorem 0.2, the path $\gamma_R(s, t)$ played an important role. In particular, the estimate (1.2) was important. However, we don’t know that $\gamma_R(s, t)$ is the best path to show the existence of positive solutions of $(*)$. Using other path, we might be able to get better estimate than (1.2). Instead of $\gamma_R(s, t)$, we can consider another path $\tilde{\gamma}_R \in C(R^2, H^1(R))$ which is defined by

$$\tilde{\gamma}_R(s, t)(x) = \gamma_0(s)(x + R) + \gamma_0(t)(x - R).$$

We remark that $\tilde{\gamma}_R(s, t)$ is a natural path because we can regard $\tilde{\gamma}_R(s, t)$ as one-dimensional version of the path which was used in the proof of the high-dimensional case. (See Proposition 2.2.) Estimating $\tilde{\gamma}_R(s, t)$ by similar way to (1.2), for any $L > 0$, we have

$$\lim_{R \to \infty} e^{2R} \left\{ \max_{(s, t) \in [-L, L]^2} I(\tilde{\gamma}_R(s, t)) - 2c_0 \right\} \leq \frac{\lambda_0^2}{2} \left( \int_{-\infty}^{\infty} b(x)(e^{2x} + e^{-2x} + 2) dx - 4 \right).$$

We see that, if $\int_{-\infty}^{\infty} b(x)(e^{2x} + e^{-2x} + 2) dx < 4$ holds, then $\int_{-\infty}^{\infty} b(x)e^{2|x|} dx < 2$ also holds. Thus $\gamma_R(s, t)$ provides a better estimate than $\tilde{\gamma}_R(s, t)$. 
2. The outline of the proof of Theorem 0.2 for $N \geq 2$

In this section, we consider the case $N \geq 2$. We remark that, when $N \geq 2$, our proofs almost are parallel to $[BaL]$. We assume $f(u) = u^p$ for $u \geq 0$ and $f(u) = 0$ for $u < 0$, where $p \in (1, \frac{N+2}{N-2})$ when $N \geq 3$, $p \in (1, \infty)$ when $N = 2$. We set

$$I(u) = \frac{1}{2}||u||^2_{H^1_b(R^N)} - ||u_+||^{p+1}_{L^{p+1}(R^N)} \in C^2(H^1(R^N), \mathbb{R}),$$

where

$$||u||^2_{H^1_b(R^N)} = ||u||^2_{H^1(R^N)} + \int_{R^N} b(x)u^2 \, dx$$

By the standard ways, we reduce $I_b$ to a functional

$$J(v) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{||v||_{H^1_b(R^N)}}{||v_+||_{L^{p+1}(R^N)}}\right)^{\frac{2(p+1)}{p-1}}.$$

which is defined on

$$\Sigma = \{v \in H^1(R^N) \mid ||v||_{H^1(R^N)} = 1, v_+ \neq 0\}.$$

Then $J \in C^1(\Sigma, \mathbb{R})$ and, for any critical point $v \in \Sigma$ of $J(v)$, $t_v v$ is a non-trivial critical point of $I(u)$ where $t_v = ||v||_{H^1_b(R^N)}^{\frac{2}{p-1}}||v_+||_{L^{p+1}(R^N)}^{-\frac{p+1}{p-1}}$. Thus, in what follows, we seek non-trivial critical points of $J(v)$.

Let $\omega(x)$ be an unique radially symmetric positive solution of (0.1) for $f(u) = u^p$ and we set $c_0 = \frac{1}{2}||\omega||^2_{H^1(R^N)} - \frac{1}{p+1}||\omega||_{H^1(R^N)} > 0$. For the (PS)-sequences of $J(u)$, we have the following:

**Proposition 2.1.** Suppose (b.1)–(b.2), (f.0) hold and let $(v_n)_{n=1}^\infty$ be a (PS)-sequence of $J(v)$. Then there exist a subsequence $n_j \to \infty$, $k \in \mathbb{N} \cup \{0\}$, $k$-sequences $(x^j)_{j=1}^\infty \subset R^N$, and a critical point $u_0$ of $I(u)$ such that

$$J(v_{n_j}) \to I(u_0) + kc_0 \quad (j \to \infty),$$

$$v_{n_j}(x) - \frac{u_0(x) - \sum_{\ell=1}^k \omega(x - x^j_\ell)}{||u_0(x) - \sum_{\ell=1}^k \omega(x - x^j_\ell)||_{H^1(R^N)}} \to 0 \quad \text{in} \quad H^1(R^N) \quad (j \to \infty),$$

$$|x^j_\ell - x^{j'}_\ell| \to \infty \quad (j \to \infty) \quad (\ell \neq \ell'),$$

$$|x^j_\ell| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \ldots, k).$$

**Proof.** Let $(v_n)_{n=1}^\infty$ be a (PS)-sequence of $J(v)$. Then $(t_{v_n} v_n)_{n=1}^\infty$ is a (PS)-sequence of $I(u)$. Moreover we remark that the set of the critical points of the functional $\frac{1}{2}||u||^2_{H^1(R^N)} - \frac{1}{p+1}||u_+||_{H^1(R^N)} : H^1(R^N) \to \mathbb{R}$ is written by $\{\omega(x+\xi) \mid \xi \in R^N\} \cup \{0\}$ from the uniqueness of positive solutions of (1.0). Thus Proposition 2.1 easily follows applying Theorem 5.1 of [JT1] to $(t_{v_n} v_n)_{n=1}^\infty$. 

By the similar arguments of Section 1, we have the following corollary.
Corollary 2.2. Suppose $I(u)$ has no non-trivial critical points and let $(v_n)_{n=1}^{\infty}$ be a (PS)-sequence of $J(v)$. Then, only $kc_0$'s ($k \in \mathbb{N}$) can be limit points of $\{J(v_n) | n \in \mathbb{N}\}$.

We set
\[ c = \inf_{v \in \Sigma} J(v). \]

Then we can easily see that $0 < c \leq c_0$. From the boundedness of $J(v)$ from below, we get also more strong corollary.

Corollary 2.3. For any $b \in (-\infty, c_0) \cup (c_0, c_0 + c)$, $J(v)$ satisfies (PS)$_b$-condition.

**Proof.** If (PS)$_b$-condition does not hold for $b \in \mathbb{R}$, then for some (PS)$_b$-sequence $(v_n)_{n=1}^{\infty}$, it must be $k \neq 0$ in Proposition 2.1. Thus we have
\[ \lim_{n \to \infty} J(v_n) = b = kc_0 \quad \text{or} \quad \lim_{n \to \infty} J(v_n) = b \geq c + kc_0. \]

This implies Corollary 2.3.

When $c < c_0$, from Corollary 2.3, $c$ is a critical value of $J(v)$. Thus this case is easy. Thus we consider the case $c = c_0$. When $c = c_0$, we must define another minimax value. To define another minimax value, the following proposition is important.

**Proposition 2.4.** Suppose $N \geq 2$, (b.1)-(b.3) and (f.0) hold. Then, there exists $R_0 > 0$ such that for any $R \geq R_0$, we have
\[ \max_{(\zeta, \xi, s) \in \partial B \times \partial B \times [0,1]} \int J \left( \frac{t\omega(x-\zeta) + (1-t)\omega(x-\xi)}{|t\omega(x-\zeta) + (1-t)\omega(x-\xi)|_{H^1(\mathbb{R}^N)}} \right) < 2c_0. \tag{2.1} \]

Here $B_R = \{x \in \mathbb{R}^N | |x| \leq R\}$.

**Proof.** To get (2.1), for large $R > 0$, it sufficient to show
\[ \max_{(\zeta, \xi, s, t) \in \partial B \times \partial B \times \mathbb{R} \times \mathbb{R}^2} I(s\omega(x-\zeta) + t\omega(x-\xi)) < 2c_0. \tag{2.2} \]

In many papers [BaL], [A], [H1], [H2], the estimates like (2.2) were obtained. In [A], [H1], [H2], they treated more general $f(u)$ including $u_n^p$. Since we can get (2.2) by similar ways to those calculations, we omit the proof of (2.2).

**Remark 2.5.** When $N = 1$, the estimate (2.1) does not hold. (See Proposition 1.6 and [S].) We remark that, for some $C_0 > 0$, $\omega(x)$ satisfies
\[ 0 < \omega(x) \leq C_0|x|^{-\frac{N-1}{2}} e^{-|x|} \quad \text{for all} \quad x \in \mathbb{R}^N. \tag{2.3} \]
Roughly explaining about the difference from $N = 1$ and $N \geq 2$, when $N \geq 2$, we can obtain (2.1) by the effect of $|x|^{-\frac{N-1}{2}}$ in (2.3). On the other hand, when $N = 1$, since the effect of $|x|^{-\frac{N-1}{2}}$ vanishes, (2.1) does not hold.

To prove the Theorem 0.2, we also define a map $m : H^1(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ which is an expansion of $m$ defined in Section 1. That is, for any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, we consider a map

$$T_u(\xi) = \left( \int_{\mathbb{R}^N} \tan^{-1}(x_1 - \xi_1)|u(x)|^2 \, dx, \ldots, \int_{\mathbb{R}^N} \tan^{-1}(x_N - \xi_N)|u(x)|^2 \, dx \right) : \mathbb{R}^N \to \mathbb{R}^N.$$ 

Then we can see that $T_u(\xi)$ has an unique $\xi_u \in \mathbb{R}^N$ such that $T_u(\xi_u) = 0$ because

$$DT_u = \begin{bmatrix} \int_{\mathbb{R}^N} \frac{1}{1+(x_1-\xi_1)^2} |u(x)|^2 \, dx & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \int_{\mathbb{R}^N} \frac{1}{1+(x_N-\xi_N)^2} |u(x)|^2 \, dx \end{bmatrix}.$$ 

Thus for any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, we define $m(u) = \xi_u$. We also find that $m(u)$ is of continuous by the implicit function theorem to $(u, \xi) \mapsto T_u(\xi)$. Since $\omega(x)$ is a radially symmetric function, from the definition of $m(u)$, we can easily see that

$$m(\omega(x - \xi)) = \xi \quad \text{for all } \xi \in \mathbb{R}^N. \quad (2.4)$$

In what follows, we will complete the proof of Theorem 0.2.

**Proof of Theorem 0.2 for $N \geq 2$.** We set

$$c = \inf_{v \in \Sigma} J(v).$$

When $c < c_0$, from Corollary 2.3, $c$ is a critical point of $J(v)$ and our proof is completed. Thus we must consider the case $c = c_0$. For $a \in \mathbb{R}^N$ we defined a minimax value $c_a > 0$ by

$$c_a = \inf_{v \in \Sigma_a} J(v),$$

$$\Sigma_a = \{ v \in \Sigma \mid m(v) = a \}.$$ 

Noting $\Sigma_a \subset \Sigma$ and $c = c_0$, we have

$$0 < c_0 \leq c_a.$$ 

We will show that $I(u)$ has at least a non-trivial critical point for the following both cases:
(i) For some $a \in \mathbb{R}^N$, $c_0 = c_a$.
(ii) For some $a \in \mathbb{R}^N$, $c_0 < c_a$.

**Proof of Theorem 0.2 for the case (i).** For any $\epsilon > 0$, there exists $\tilde{v}_\epsilon \in \Sigma_a$ such that

$$c_0 \leq J(\tilde{v}_\epsilon) < c_0 + \epsilon.$$ 

Since $\tilde{v}_\epsilon \in \Sigma_a \subset \Sigma$ and $\Sigma$ is an invariant set by standard deformation flows of $J(v)$, by a standard Ekland principle, there exists $v_\epsilon \in \Sigma$ such that

$$c_0 \leq J(v_\epsilon) \leq J(\tilde{v}_\epsilon) < c_0 + \epsilon,$$

$$||J'(v_\epsilon)|| < 2\sqrt{\epsilon},$$

$$||v_\epsilon - \tilde{v}_\epsilon||_{H^1(\mathbb{R}^N)} < \epsilon.$$ 

(2.5)

Then, from Proposition 2.1, there exist a subsequence $\epsilon_j \to 0$, $k \in \mathbb{N} \cup \{0\}$, $k$-sequences $(x^1_j)_{j=1}^\infty, \cdots, (x^k_j)_{j=1}^\infty \subset \mathbb{R}^N$, and a critical point $u_0$ of $I(u)$ such that

$$J(v_{\epsilon_j}) \to I(u_0) + kc_0 \quad (j \to \infty),$$

(2.6)

$$v_{\epsilon_j}(x) - \frac{u_0(x) - \sum_{\ell=1}^k \omega(x - x^\ell_j)}{||u_0(x) - \sum_{\ell=1}^k \omega(x - x^\ell_j)||_{H^1(\mathbb{R}^N)}} \to 0 \quad \text{in} \quad H^1(\mathbb{R}^N) \quad (j \to \infty),$$

$$|x^\ell_j - x^{\ell'}_j| \to \infty \quad (j \to \infty) \quad (\ell \neq \ell'),$$

$$|x^\ell_j| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \cdots, k).$$

Now, if $u_0 \neq 0$, our proof is completed. So we suppose $u_0 = 0$. Then, from (2.6), it must be $k = 1$. Thus, we have

$$\left| \left| v_{\epsilon_j}(x) - \frac{\omega(x - x^1_j)}{||\omega||_{H^1(\mathbb{R}^N)}} \right| \right|_{H^1(\mathbb{R}^N)} \to 0 \quad (j \to \infty),$$

(2.7)

$$|x^1_j| \to \infty \quad (j \to \infty).$$

From (2.4), (2.5) and (2.7), we see that

$$|m(\tilde{v}_{\epsilon_j})| \to \infty \quad \text{as} \quad j \to \infty.$$ 

This contradicts to $m(\tilde{v}_{\epsilon_j}) = a$. Therefore $u_0 \neq 0$ and $I(u)$ has at least a non-trivial critical point. 

\[\blacksquare\]
Proof of the Theorem 0.2 for the case (ii). From Proposition 2.4, we set \( \zeta_0 = (\frac{1}{2}R_0, 0, \ldots, 0) \) and \( \delta = \frac{1}{2}(c_a - c_0) > 0 \) and choose a large \( R_0 > |a| \) such that

\[
\max_{\xi \in \partial B_{R_0}} J(\omega(x - \xi)) < c_0 + \delta < c_a, \tag{2.8}
\]

\[
\max_{(\xi, t) \in \partial B_{R_0} \times [0,1]} J \left( \frac{t\omega(x - \zeta_0) + (1-t)\omega(x - \xi)}{||t\omega(x - \zeta_0) + (1-t)\omega(x - \xi)||_{H^1(\mathbb{R}^N)}} \right) < 2c_0. \tag{2.9}
\]

Here we define the following minimax value:

\[ c_2 = \inf_{\gamma \in \Gamma} \max_{\xi \in B_{R_0}} J(\gamma(\xi)), \]

\[ \Gamma = \left\{ \gamma(\xi) \in C(B_{R_0}, \Sigma) \mid \gamma(\xi)(x) = \frac{\omega(x + \xi)}{||\omega||_{H^1(\mathbb{R}^N)}} \text{ for all } \xi \in \partial B_{R_0} \right\}. \]

Then we have the following lemma.

**Lemma 2.6.** We have

\[ 0 < c_0 < c_a \leq c_2 < 2c_0. \]

We postpone the proof of Lemma 2.6 to end of this section. If Lemma 2.6 is true, then \( \Gamma \) is an invariant set by the deformation flows of \( J(v) \). Thus \( J(v) \) has a (PS)-sequence \( (v_n)_{n=1}^{\infty} \) such that

\[ J(v_n) \rightarrow c_2 \in (c_0, 2c_0) \quad (n \rightarrow \infty). \]

From Corollary 2.3, \( J(u) \) satisfies (PS)\( c_2 \)-conditions. Thus \( c_2 \) is a critical value of \( J(v) \). That is, \( I(u) \) has at least a non-trivial critical point. Combining the proofs of the cases (i)-(ii), we complete a proof of Theorem 0.2. \[ \blacksquare \]

Finally we show Lemma 2.6.

**Proof of Lemma 2.6.** The inequality \( c_0 < c_a \) is an assumption of the case (ii). From (2.9), \( c_2 < 2c_0 \) is obvious. Thus we show \( c_a \leq c_2 \). For any \( \gamma \in \Gamma \), from (2.10), we have

\[ m(\gamma(\xi)) = \xi \quad \text{for all } \xi \in \partial B_{R_0}. \]

Thus we can see

\[ \deg(m \circ \gamma, B_{R_0}, a) = 1. \tag{2.10} \]

From (2.10), there exists \( \xi_0 \in B_{R_0} \) such that \( m(\gamma(\xi_0)) = a \). Therefore, since \( \gamma(\xi_0) \in \Sigma_a \), we find that

\[
c_a \leq \inf_{v \in \Sigma_a} J(v) \leq J(\gamma(\xi_0))) \leq \max_{\xi \in B_{R_0}} I(\gamma(\xi)). \tag{2.11}
\]
Since \( \gamma \in \Gamma \) is arbitrary, from (2.11), we have
\[
c_a \leq c_2.
\]
Thus we get Lemma 2.6.

References


