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Kyoto University
Direct and inverse bifurcation problems for nonlinear Sturm-Liouville problems

1 Introduction

We consider the nonlinear Sturm-Liouville problem

\begin{align}
- u''(t) + f(u(t)) &= \lambda u(t), \quad t \in I := (0, 1), \\
\end{align}

\begin{align}
&u(t) > 0, \quad t \in I, \\
&u(0) = u(1) = 0,
\end{align}

where \( \lambda > 0 \) is a positive parameter. \( f(u) \) is assumed to satisfy the conditions (A.1)- (A.2):

(A.1) \( f(u) \) is \( C^1 \) for \( u \geq 0 \) satisfying \( f(u) > 0 \) for \( u > 0 \). Furthermore, \( f(0) = f'(0) = 0 \).

(A.2) \( f(u)/u \) is strictly increasing for \( u \geq 0 \). Moreover, \( f(u)/u \to \infty \) as \( u \to \infty \).

The following are the typical examples of \( f(u) \) which satisfy (A.1) and (A.2).

\begin{align}
&f(u) = u^p \quad (u \geq 0), \\
&f(u) = u^p + u^m \quad (u \geq 0), \\
&f(u) = u^p \left(1 - \frac{1}{1+u^2}\right) \quad (u \geq 0),
\end{align}

where \( p > m > 1 \) are constants.
Before stating our result, let us briefly recall some known facts (cf. [1]).

(a) For each given $\alpha > 0$, there exists a unique solution $(\lambda, u) = (\lambda_q(\alpha), u_{\alpha}) \in \mathbb{R}_+ \times C^2(\overline{I})$ of (1.1)-(1.3) with $\|u_{\alpha}\|_q = \alpha$. Here, $\|u_{\alpha}\|_q$ is the $L^q$-norm of $u_{\alpha}$, and $\lambda_q(\alpha)$ is called $L^q$-bifurcation curve.

(b) The set $\{ (\lambda_q(\alpha), u_{\alpha}) : \alpha > 0 \}$ gives all solutions of (1.1)-(1.3) and is an unbounded curve of class $C^1$ in $\mathbb{R}_+ \times L^q(I)$ emanating from $(\pi^2, 0)$. Furthermore, $\lambda_q(\alpha)$ is strictly increasing for $\alpha > 0$ and $\lambda_q(\alpha) \rightarrow \infty$ as $\alpha \rightarrow \infty$.

The objective here is to discuss inverse bifurcation problems for nonlinear Sturm-Liouville problems from an asymptotic point of view.

The direct bifurcation problem, that is, for a given nonlinear term $f(u)$, the problem to investigate the local and global behavior of bifurcation curve has a long history and has been studied by many authors. We refer to [1-17] and the references therein. However, it seems that there exists a few works concerning inverse bifurcation problems. We only refer to [21].

Recently, the following basic result was obtained in [20].

**Theorem 1.0 ([20]).** Assume that $f_1(u)$ and $f_2(u)$ are unknown to satisfy (A.1) (A.2). Further, assume that the connected components of the set $V := \{ u \geq 0 : f_1(u) = f_2(u) \}$ are locally finite. Let $\lambda_2(1, \alpha)$ and $\lambda_2(2, \alpha)$ be the $L^2$-bifurcation curves of (1.1)-(1.3) associated with the nonlinear term $f(u) = f_1(u)$ and $f(u) = f_2(u)$, respectively. Assume that $\lambda_2(1, \alpha) = \lambda_2(2, \alpha)$ for any $\alpha > 0$. Then $f_1(u) \equiv f_2(u)$ for $u \geq 0$.

Motivated by the result above, we here introduce an asymptotic approach to inverse bifurcation problem for (1.1)-(1.3). To be more precise, we assume that the nonlinear term $f(u)$ is unknown. Then we show that, if the asymptotic formula for the $L^q$-bifurcation curve $\lambda_q(\alpha)$ as $\alpha \rightarrow \infty$ is known precisely, then we are able to characterize the asymptotic property of $f(u)$ for $u \gg 1$. Here, $1 \leq q < \infty$ is a constant and we fix it throughout this paper. We call this idea *asymptotic approach for inverse bifurcation problems*.

As for the asymptotic behavior of $\lambda_q(\alpha)$ and $u_{\alpha}$ as $\alpha \rightarrow \infty$, it is known from [1] that

\begin{equation}
\frac{u_{\alpha}(t)}{\|u_{\alpha}\|_\infty} \rightarrow 1
\end{equation}
locally uniformly on $I$ as $\alpha \to \infty$. We set $g(u) := f(u)/u$. Then as $\alpha \to \infty$,

(1.8) \[ \lambda_q(\alpha) = g(\|u_\alpha\|_\infty) + \xi_\alpha, \]

where $\xi_\alpha = O(1)$ is the remainder term. By (1.7), we see that $\|u_\alpha\|_\infty = \alpha (1 + o(1))$ for $\alpha \gg 1$. By this and (1.8), for $\alpha \gg 1$,

(1.9) \[ \lambda_q(\alpha) = g(\alpha) + o(g(\alpha)). \]

Motivated by (1.9), as a direct problem, more precise asymptotic formula for $\lambda_q(\alpha)$ as $\alpha \to \infty$ has been given in [18].

**Theorem 1.1 ([18]).** Let $f(u) = u^p$, where $p > 1$ is a given constant. Then as $\alpha \to \infty$.

(1.10) \[ \lambda_q(\alpha) = \alpha^{p-1} + C_0 \alpha^{(p-1)/2} + C_1 + o(1), \]

where

\[
C_0 = \frac{2(p-1)}{q}C_2, \quad C_1 = \frac{2(p-1)}{q}C_2^2, \\
C_2 = \int_0^1 \frac{1 - s^q}{\sqrt{1 - s^2 - 2(1 - s^{p+1})/(p + 1)}} ds.
\]

The formula (1.10) has been obtained first for $q = 2$ in [15] by using the relationship between $\lambda_2(\alpha)$ and the critical value associated with $\lambda_2(\alpha)$.

From a view point of Theorems 1.1, we consider the following inverse problem.

**Problem 1.** Let $f(u)$ be unknown to satisfy (A.1) and (A.2). Assume that as $\alpha \to \infty$,

(1.11) \[ \lambda_q(\alpha) = g(\alpha) + Ag(\alpha)^{1/2} + O(1), \]

where $A > 0$ is a constant. Then can you conclude that $f(u) = u^p$ for some constant $p > 1$?

To state our results, we assume additional conditions (A.3) and (A.4). We put $f(u) := u^p h(u)$. 
(A.3) $h(u)$ is a $C^1$ function for $u > 0$, and there exists a constant $\delta_0 > 0$ such that $h(u) \geq \delta_0$ for $u > 0$. Furthermore, for an arbitrary fixed constant $0 < \epsilon \ll 1$, as $u \to \infty$,

\begin{equation}
\max_{c \leq s \leq 1} \left| \frac{uh'(us)}{h(u)} \right| = O((u^{p-1}h(u))^{-1/2}),
\end{equation}

\begin{equation}
\max_{0 \leq s \leq \epsilon} s^p \left| \frac{uh'(us)}{h(u)} \right| = O((u^{p-1}h(u))^{-1/2}).
\end{equation}

(A.4) There exists a constant $0 < \delta_1 \ll 1$ such that for $(1 + \delta_1)u > u > v \gg 1$,

\begin{equation}
f(u) = f(v) + f'(v)(u-v) + O(f(v)/v^2)(u-v)^2.
\end{equation}

The typical examples of $h(u)$ (i.e. $f(u)$) satisfying (A.3) and (A.4) are:

\begin{align*}
h(u) &= 1 \quad (f(u) = u^p) \quad \text{and} \quad h(u) = 1 + u^{m-p} \quad (f(u) = u^p + u^m, \ 1 < m \leq \frac{p+1}{2}).
\end{align*}

The answer to Problem 1 is as follows.

**Theorem 1.2.** Assume that all conditions in Problem 1, (A.3) and (A.4) are satisfied. Then $f(u) = u^ph(u)$ with $p = 1 + (qA)/(2C_2)$ and $h(u) = D + d(u)$, where $C_2$ is a constant in Theorem 1.1, $A$ is a constant in (1.11), $D > 0$ is an arbitrary positive constant and $d(u) = O(u^{(1-p)/2})$ for $u \gg 1$.

**Remark 1.3.** (i) The next inverse bifurcation problem we consider in a near future should be to establish the *asymptotic uniqueness* of unknown $f(u)$ from the asymptotic behavior of $\lambda_q(\alpha)$ as $\alpha \to \infty$.

(ii) The condition (A.3) is not technical one. Indeed, if we consider $f(u) = u^{q}e^u$ and $q = 2$, then $g(u) = u^4e^u$ does not satisfy (A.3), and we know from [16] that as $\alpha \to \infty$

\begin{equation}
\lambda_2(\alpha) = \alpha^4e^\alpha + \frac{\pi}{4}\alpha^3e^{\alpha/2} + \frac{\pi}{4}u^2e^{\alpha/2}(1 + o(1)),
\end{equation}

which is different from (1.11). Therefore, (1.11) does not hold without (A.3).

## 2 Sketch of the Proof of Theorem 1.2

In what follows, $C$ denotes various positive constants independent of $\alpha \gg 1$. We write $(\lambda, u_\alpha)$ for a unique solution pair of (1.1)–(1.3) with $\|u_\alpha\|_q = \alpha$. We begin with the fundamental
tools which play important roles in what follows. It is well known that

\begin{align}
(2.1) \quad u_\alpha(t) &= u_\alpha(1-t), \quad t \in I, \quad \|u_\alpha\|_\infty = u_\alpha\left(\frac{1}{2}\right), \\
(2.2) \quad u_\alpha'(t) &> 0, \quad 0 \leq t < \frac{1}{2}.
\end{align}

Multiply (1.1) by $u_\alpha'(t)$. Then

\[(u_\alpha''(t) + \lambda u_\alpha(t) - f(u_\alpha(t)))u_\alpha'(t) = 0.\]

This along with (2.1) implies that

\begin{align}
(2.3) \quad \frac{1}{2}u'_\alpha(t)^2 + \frac{1}{2}\lambda u_\alpha(t)^2 - F(u_\alpha(t)) &= \text{constant} \\
&= \frac{1}{2}\lambda\|u_\alpha\|_\infty^2 - F(\|u_\alpha\|_\infty), \quad \text{(put } t = 1/2) \end{align}

where $F(u) := \int_0^u f(s)ds$. We set

\begin{align}
(2.4) \quad L_\alpha(\theta) &= \lambda(\|u_\alpha\|_\infty^2 - \theta^2) - 2(F(\|u_\alpha\|_\infty) - F(\theta)).
\end{align}

This along with (2.2) and (2.3) implies that for $0 \leq t \leq 1/2$

\begin{align}
(2.5) \quad u_\alpha'(t) &= \sqrt{L_\alpha(u_\alpha(t))}.
\end{align}

By this and (2.1), we obtain

\begin{align}
\|u_\alpha\|_\infty^q - \alpha^q &= 2\int_0^{1/2} \left(\|u_\alpha\|_\infty^q - u_\alpha'(t)^2\right) dt = 2\int_0^{\|u_\alpha\|_\infty} \left(\|u_\alpha\|_\infty^q - \theta^q\right) d\theta \\
&= \frac{2\|u_\alpha\|_\infty^q}{\sqrt{\lambda}} \int_0^{1} \frac{1 - s^q}{\sqrt{B_\alpha(s)}} ds \\
&= \frac{2\|u_\alpha\|_\infty^q}{\sqrt{\lambda}} \left\{ \int_0^{1} \frac{1 - s^q}{\sqrt{J(s)}} ds + \int_0^{1} \left( \frac{1 - s^q}{\sqrt{B_\alpha(s)}} - \frac{1 - s^q}{\sqrt{J(s)}} \right) ds \right\} \\
&= \frac{2\|u_\alpha\|_\infty^q}{\sqrt{\lambda}} (C_2 + M_\alpha),
\end{align}

where

\begin{align}
(2.7) \quad J(s) &= 1 - s^2 - \frac{2}{p+1}(1 - s^{p+1}), \\
(2.8) \quad B_\alpha(s) &= 1 - s^2 - \frac{2}{\lambda\|u_\alpha\|_\infty^2}(F(\|u_\alpha\|_\infty) - F(\|u_\alpha\|_\infty s)), \\
(2.9) \quad M_\alpha &= \int_0^{1} \left( \frac{1 - s^q}{\sqrt{B_\alpha(s)}} - \frac{1 - s^q}{\sqrt{J(s)}} \right) ds.
\end{align}
Lemma 2.1. $f'(\alpha) \leq C\alpha^{p-1}$ for $\alpha \gg 1$.

Lemma 2.1 is proved by direct calculation. So we omit the proof. By (A.3) and Lemma 2.1, for $\alpha \gg 1$,

\begin{equation}
C^{-1}\alpha^{p-1} \leq \lambda \leq C\alpha^{p-1},
\end{equation}

(2.10)

\begin{equation}
C^{-1}\alpha^{p} \leq f(\alpha) \leq C\alpha^{p},
\end{equation}

(2.11)

\begin{equation}
C^{-1}\alpha^{p-1} \leq g(\alpha) \leq C\alpha^{p-1}.
\end{equation}

(2.12)

The following Lemma 2.2 plays essential roles to prove Theorem 1.2.

Lemma 2.2. $M_{\alpha} = O(g(\alpha)^{-1/2})$ as $\alpha \to \infty$.

We tentatively accept this lemma and prove Theorem 1.2. Lemma 2.2 will be proved in Section 3.

Proof of Theorem 1.2. By Lemma 2.2 and Taylor expansion, for $\alpha \gg 1$,

\begin{equation}
\|u_{\alpha}\|_{\infty} = \alpha \left(1 - \frac{2}{\sqrt{\lambda}}(C_{2} + M_{\alpha})\right)^{-1/q}
\end{equation}

(2.13)

\begin{equation}
= \alpha \left(1 + \frac{2}{q\sqrt{\lambda}}(C_{2} + M_{\alpha}) + \frac{2(q + 1)}{q^{2}\lambda}(C_{2} + M_{\alpha})^{2}(1 + o(1))\right).
\end{equation}

(2.14)

By this, Lemmas 2.1 and 2.2,

\begin{align*}
\lambda &= \frac{f(\|u_{\alpha}\|_{\infty})}{\|u_{\alpha}\|_{\infty}} + \xi_{\alpha} \\
&= \frac{1}{\alpha} \left(1 - \frac{2}{q\sqrt{\lambda}}(C_{2} + M_{\alpha}) + O(\alpha^{1-p})\right) \left(f(\alpha) + \frac{2\alpha}{q\sqrt{\lambda}}f'(\alpha)(C_{2} + M_{\alpha}) + O(\alpha)\right) + \xi_{\alpha} \\
&= \frac{f(\alpha)}{\alpha} + \frac{2C_{2}}{q\sqrt{\lambda}} \left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right) + M_{\alpha} \frac{2C_{2}}{q\sqrt{\lambda}} \left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right) + O(1) \\
&= \frac{f(\alpha)}{\alpha} + \frac{2C_{2}}{q} \left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right)(g(\alpha) + Ag(\alpha)^{1/2} + O(1))^{-1/2} + O(1) \\
&= \frac{f(\alpha)}{\alpha} + \frac{2C_{2}}{q\sqrt{g(\alpha)}} \left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right) + O(1). \\
&= g(\alpha) + Ag(\alpha)^{1/2} + O(1).
\end{align*}

This implies that for $\alpha \gg 1$

\begin{equation}
f'(\alpha) - r\frac{f(\alpha)}{\alpha} = O(\sqrt{g(\alpha)}),
\end{equation}

(2.15)
where $r := 1 + (qA)/(2C_2)$. Then we solve (2.15) directly, and easily obtain that $r = p$, and for $\alpha \gg 1$

\[(2.16) \quad f(\alpha) = D\alpha^p + O(\alpha^{(p+1)/2}),\]

where $D > 0$ is an arbitrary constant. Thus the proof is complete. 

\section{Proof of Lemma 2.2.}

In this section, we prove Lemma 2.2. Let an arbitrary $0 < \epsilon \ll 1$ be fixed. For $0 \leq s \leq 1$, we put

\[(3.1) \quad K_{\alpha}(s) := J(s) - B_{\alpha}(s) = \frac{2}{\lambda \|u_{\alpha}\|_{\infty}^2} \{F(\|u_{\alpha}\|_{\infty}) - F(\|u_{\alpha}\|_{\infty}s)\} - \frac{2}{p + 1} (1 - s^{p+1}).\]

Then

\[(3.2) \quad M_{\alpha} = \int_{0}^{1} \frac{(1 - s^q)K_{\alpha}(s)}{\sqrt{J(s)}\sqrt{B_{\alpha}(s)}} ds = \int_{1-\epsilon}^{1} \frac{(1 - s^q)K_{\alpha}(s)}{\sqrt{J(s)}\sqrt{B_{\alpha}(s)}} ds + \int_{0}^{\epsilon} \frac{(1 - s^q)K_{\alpha}(s)}{\sqrt{J(s)}\sqrt{B_{\alpha}(s)}} ds + \int_{1-\epsilon}^{1-\epsilon} \frac{(1 - s^q)K_{\alpha}(s)}{\sqrt{J(s)}\sqrt{B_{\alpha}(s)}} ds : = M_{1,\alpha} + M_{2,\alpha} + M_{3,\alpha}.\]

\textbf{Lemma 3.1.} For $\alpha \gg 1$

\[(3.3) \quad |M_{1,\alpha}| = O(g(\|u_{\alpha}\|_{\infty})^{-1/2}).\]

\textbf{Proof.} By (3.1),

\[(3.4) \quad \frac{K'_{\alpha}(s)}{2} = -\frac{f(\|u_{\alpha}\|_{\infty}s)}{\lambda \|u_{\alpha}\|_{\infty}} + s^p.\]

This implies that

\[(3.5) \quad \frac{K'_{\alpha}(1)}{2} = \frac{\xi_{\alpha}}{\lambda}.\]
Since \( f(u) = g(u)u \), for \( 1 - \epsilon \leq s \leq 1 \), by Taylor expansion, we obtain

\[
\frac{K''(s)}{2} = -\frac{f'(\|u_\alpha\|_\infty s)}{\lambda} + ps^{p-1}
\]

\[
= -\frac{g'(\|u_\alpha\|_\infty s)\|u_\alpha\|_\infty s + g(\|u_\alpha\|_\infty s)}{g(\|u_\alpha\|_\infty) + \xi_\alpha} + ps^{p-1}
\]

\[
= -\frac{g'(\|u_\alpha\|_\infty s)\|u_\alpha\|_\infty s + g(\|u_\alpha\|_\infty s)}{g(\|u_\alpha\|_\infty)} \left( 1 - \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)}(1 + o(1)) \right) + ps^{p-1}.
\]

We put

\[
(3.7) \quad H(s, u) = ps^{p-1}\frac{h(us)}{h(u)} + us^{p}\frac{h'(us)}{h(u)}.
\]

For \( u \gg 1 \),

\[
(3.8) \quad g'(u) = (p-1)u^{p-2}h(u) + u^{p-1}h'(u).
\]

By this and (3.6), we obtain

\[
(3.9) \quad \frac{K''(s)}{2} = -H(s, \|u_\alpha\|_\infty) \left( 1 - \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)}(1 + o(1)) \right) + ps^{p-1}.
\]

\[
= ps^{p-1} \left( 1 - \frac{h(\|u_\alpha\|_\infty s)}{h(\|u_\alpha\|_\infty)} \right) - \|u_\alpha\|_\infty s\frac{h'(\|u_\alpha\|_\infty s)}{h(\|u_\alpha\|_\infty)}
\]

\[
+ \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)}H(s, \|u_\alpha\|_\infty)(1 + o(1)).
\]

By this and mean value theorem, for \( 1 - \epsilon < s < s_1 < s_2 < 1 \), we obtain

\[
(3.10) \quad \frac{K''_\alpha(s_1)}{2} = ps^{p-1}_1 \left( 1 - \frac{h(\|u_\alpha\|_\infty s_1)}{h(\|u_\alpha\|_\infty)} \right) - \|u_\alpha\|_\infty s_1\frac{h'(\|u_\alpha\|_\infty s_1)}{h(\|u_\alpha\|_\infty)}
\]

\[
+ \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)}H(s_1, \|u_\alpha\|_\infty)(1 + o(1))
\]

\[
= ps^{p-1}_1 \left( \frac{h'(\|u_\alpha\|_\infty s_2)}{h(\|u_\alpha\|_\infty)} \right) \|u_\alpha\|_\infty(1 - s_1) - \|u_\alpha\|_\infty s_1\frac{h'(\|u_\alpha\|_\infty s_1)}{h(\|u_\alpha\|_\infty)}
\]

\[
+ \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)}H(s_1, \|u_\alpha\|_\infty)(1 + o(1))
\]

\[
= O(g(\|u_\alpha\|_\infty)^{-1/2})) + O\left( \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)} \right)
\]

\[
= O(g(\|u_\alpha\|_\infty)^{-1/2})).
\]

Since \( K_\alpha(1) = 0 \), by (3.5), (3.10) and Taylor expansion, for \( 1 - \epsilon \leq s \leq 1 \),

\[
(3.11) \quad \frac{K_\alpha(s)}{2} = \frac{1}{2} \left( K_\alpha(1) + K'_\alpha(1)(s - 1) + \frac{1}{2} K''_\alpha(s)(s - 1)^2 \right)
\]

\[
= \frac{\xi_\alpha}{2\lambda}(s - 1) + O(g(\|u_\alpha\|_\infty)^{-1/2}))(s - 1)^2.
\]
By this, (3.1) and Taylor expansion, for $1 - \epsilon \leq s \leq 1$,

\begin{align}
J(s) \geq (p - 1 - \delta_1)(1 - s)^2, \\
B_\alpha(s) = J(s) - K_\alpha(s) \geq \frac{\xi_\alpha}{\lambda}(1 - s) + \frac{\delta_1}{2}(1 - s)^2.
\end{align}

Then we obtain

\begin{align}
|M_{1,\alpha}| \leq \int_{1-\epsilon}^{1} \frac{(1 - s^p)|K_\alpha(s)|}{J(s)\sqrt{B_\alpha(s)}} ds \\
\leq C \int_{1-\epsilon}^{1} \left( \frac{\xi_\alpha}{\lambda} + O(g(\|u_\alpha\|_\infty^{-1/2}) \right) (1 - s) \sqrt{\frac{1}{\delta_1/2}(1 - s)^2} ds \\
= C \left( \frac{\xi_\alpha}{\lambda} + O(g(\|u_\alpha\|_\infty^{-1/2})) \right) = O(g(\|u_\alpha\|_\infty^{-1/2}).
\end{align}

Thus the proof is complete.

**Lemma 3.2.** $M_{2,\alpha} = O(g(\|u_\alpha\|_\infty^{-1/2})$ as $\alpha \to \infty$.

**Proof.** Since $f(u) = u^p h(u)$, for $0 \leq s \leq 1 - \epsilon$,

\begin{align}
K_\alpha(s) = \frac{1}{\lambda \|u_\alpha\|_\infty^2} \int_{\|u_\alpha\|_\infty}^{\|u_\alpha\|_\infty^p} t^p h(t) dt - \frac{1}{p+1}(1 - s^{p+1}) \\
= \frac{1}{(p+1)\lambda \|u_\alpha\|_\infty^2} \left\{ \int_{\|u_\alpha\|_\infty}^{\|u_\alpha\|_\infty^p} t^{p+1} h(t) dt \right\} - \frac{1}{p+1}(1 - s^{p+1}).
\end{align}

Since $\xi_\alpha > 0$, for $\epsilon \leq s \leq 1 - \epsilon$,

\begin{align}
\frac{1}{\lambda \|u_\alpha\|_\infty^2} \int_{\|u_\alpha\|_\infty}^{\|u_\alpha\|_\infty^p} t^{p+1} h'(t) dt \leq \frac{1}{h(\|u_\alpha\|_\infty)\|u_\alpha\|_\infty^{p+1}} \int_{\|u_\alpha\|_\infty}^{\|u_\alpha\|_\infty^p} t^{p+1} h'(t) dt \\
\leq \max_{\|u_\alpha\|_\infty^{s_1}} \frac{h(\|u_\alpha\|_\infty) h'(\|u_\alpha\|_\infty^{s_1})}{h(\|u_\alpha\|_\infty)}(1 - s) \\
= O(g(\|u_\alpha\|_\infty^{-1/2}).
\end{align}

By this and mean value theorem, for $\epsilon \leq s < s_1 < 1 - \epsilon$,

\begin{align}
\frac{|K_\alpha(s)|}{2} \leq \frac{1}{(p+1)\lambda \|u_\alpha\|_\infty^2} \left\{ \|u_\alpha\|_\infty^{p+1} h(\|u_\alpha\|_\infty) - \|u_\alpha\|_\infty^{p+1} s^{p+1} h(\|u_\alpha\|_\infty^{s_1}) \right\}
\end{align}
\begin{align}
&+O(g(\|u_\alpha\|_\infty)^{-1/2}) - \frac{1}{p+1}(1-s^{p+1}) \\
&\leq \frac{1}{p+1}(1-s^{p+1}) \left( \frac{\|u_\alpha\|_\infty^{p-1}h(\|u_\alpha\|_\infty)}{\lambda} - 1 \right) \\
&\quad + \frac{\|u_\alpha\|_\infty^{p-1}s^{p+1}}{\lambda(p+1)} \left( h(\|u_\alpha\|_\infty) - h(\|u_\alpha\|_\infty s) + O(g(\|u_\alpha\|_\infty)^{-1/2}) \right) \\
&\leq \frac{\xi_\alpha}{(p+1)\lambda} (1-s^{p+1}) + \frac{\|u_\alpha\|_\infty^{p-1}s^{p+1}}{\lambda(p+1)} \left( h(\|u_\alpha\|_\infty) - h(\|u_\alpha\|_\infty s) \right) + O(g(\|u_\alpha\|_\infty)^{-1/2}) \\
&= O(g(\|u_\alpha\|_\infty)^{-1/2}).
\end{align}

Note that for $0 \leq s \leq 1 - \epsilon$,
\begin{equation}
J(s) \geq \delta_2 > 0.
\end{equation}

By this and (3.14), for $\epsilon \leq s \leq 1 - \epsilon$ and $\alpha \gg 1$,
\begin{equation}
B_\alpha(s) \geq J(s) - K_\alpha(s) \geq \frac{\delta_2}{2} > 0.
\end{equation}

Then by this and direct calculation, we obtain
\[ |M_{2,\alpha}| \leq C \int_{\epsilon}^{1-\epsilon} |K_\alpha(s)|(1-s^q)ds = O(g(\|u_\alpha\|_\infty)^{-1/2}). \]

Thus the proof is complete.

**Lemma 3.3.** $M_{3, \alpha} = O(g(\|u_\alpha\|_\infty)^{-1/2})$ as $\alpha \to \infty$.

The proof of Lemma 3.3 is similar to that of Lemma 3.2. So we omit the proof. Since $\alpha = \|u_\alpha\|_\infty(1+o(1))$, Lemma 2.2 follows from Lemmas 3.1–3.3. Thus the proof is complete.

**References**


