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Direct and inverse bifurcation problems for nonlinear Sturm-Liouville problems

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1 Introduction

We consider the nonlinear Sturm-Liouville problem

$$-u''(t) + f(u(t)) = \lambda u(t), \quad t \in I := (0,1), \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(0) = u(1) = 0, \tag{1.3}$$

where $\lambda > 0$ is a positive parameter. $f(u)$ is assumed to satisfy the conditions (A.1)–(A.2):

(A.1) $f(u)$ is C^1 for $u \geq 0$ satisfying $f(u) > 0$ for $u > 0$. Furthermore, $f(0) = f'(0) = 0$.

(A.2) $f(u)/u$ is strictly increasing for $u \geq 0$. Moreover, $f(u)/u \to \infty$ as $u \to \infty$.

The following are the typical examples of $f(u)$ which satisfy (A.1) and (A.2).

$$f(u) = u^p \quad (u \geq 0), \tag{1.4}$$

$$f(u) = u^p + u^m \quad (u \geq 0), \tag{1.5}$$

$$f(u) = u^p\left(1 - \frac{1}{1+u^2}\right) \quad (u \geq 0), \tag{1.6}$$

where $p > m > 1$ are constants.

Before stating our result, let us briefly recall some known facts (cf. [1]).

(a) For each given $\alpha > 0$, there exists a unique solution $(\lambda, u) = (\lambda_q(\alpha), u_\alpha) \in \mathbf{R}_+ \times C^2(\bar{I})$ of (1.1)-(1.3) with $\|u_\alpha\|_q = \alpha$. Here, $\|u_\alpha\|_q$ is the L^q-norm of u_α, and $\lambda_q(\alpha)$ is called L^q-bifurcation curve.

(b) The set $\{(\lambda_q(\alpha), u_\alpha) : \alpha > 0\}$ gives all solutions of (1.1)–(1.3) and is an unbounded curve of class C^1 in $\mathbf{R}_+ \times L^q(I)$ emanating from $(\pi^2, 0)$. Furthermore, $\lambda_q(\alpha)$ is strictly increasing for $\alpha > 0$ and $\lambda_q(\alpha) \to \infty$ as $\alpha \to \infty$.

The objective here is to discuss inverse bifurcation problems for nonlinear Sturm-Liouville problems from an asymptotic point of view.

The direct bifurcation problem, that is, for a given nopnlinear term $f(u)$, the problem to investigate the local and global behavior of bifurcation curve has a long history and has been studied by many authors. We refer to [1-17] and the references therein. However, it seems that there exists a few works concerning inverse bifurcation problems. We only refer to [21].

Recently, the following basic result was obtained in [20].

Theorem 1.0 ([20]). *Assume that $f_1(u)$ and $f_2(u)$ are unknown to satisfy (A.1) (A.2). Further, assume that the connected components of the set $V := \{u \ge 0 : f_1(u) = f_2(u)\}$ are locally finite. Let $\lambda_2(1, \alpha)$ and $\lambda_2(2, \alpha)$ be the L^2-bifurcation curves of (1.1)–(1.3) associated with the nonlinear term $f(u) = f_1(u)$ and $f(u) = f_2(u)$, respectively. Assume that $\lambda_2(1, \alpha) = \lambda_2(2, \alpha)$ for any $\alpha > 0$. Then $f_1(u) \equiv f_2(u)$ for $u \ge 0$.*

Motivated by the result above, we here introduce an asymptotic approach to inverse bifurcation problem for (1.1)-(1.3). To be more precise, we assume that the nonlinear term $f(u)$ is unknown. Then we show that, if the asymptotic formula for the L^q-bifurcation curve $\lambda_q(\alpha)$ as $\alpha \to \infty$ is known precisely, then we are able to characterize the asymptotic property of $f(u)$ for $u \gg 1$. Here, $1 \le q < \infty$ is a constant and we fix it throughout this paper. We call this idea *asymptotic approach for inverse bifurcation problems.*

As for the asymptotic behavior of $\lambda_q(\alpha)$ and u_α as $\alpha \to \infty$, it is known from [1] that

$$\frac{u_\alpha(t)}{\|u_\alpha\|_\infty} \to 1 \tag{1.7}$$

locally uniformly on I as $\alpha \to \infty$. We set $g(u) := f(u)/u$. Then as $\alpha \to \infty$,

$$\lambda_q(\alpha) = g(\|u_\alpha\|_\infty) + \xi_\alpha, \tag{1.8}$$

where $\xi_\alpha = O(1)$ is the remainder term. By (1.7), we see that $\|u_\alpha\|_\infty = \alpha(1 + o(1))$ for $\alpha \gg 1$. By this and (1.8), for $\alpha \gg 1$,

$$\lambda_q(\alpha) = g(\alpha) + o(g(\alpha)). \tag{1.9}$$

Motivated by (1.9), as a direct problem, more precise asymptotic formula for $\lambda_q(\alpha)$ as $\alpha \to \infty$ has been given in [18].

Theorem 1.1 **([18]).** *Let $f(u) = u^p$, where $p > 1$ is a given constant. Then as $\alpha \to \infty$.*

$$\lambda_q(\alpha) = \alpha^{p-1} + C_0\alpha^{(p-1)/2} + C_1 + o(1), \tag{1.10}$$

where

$$C_0 = \frac{2(p-1)}{q}C_2, \quad C_1 = \frac{2(p-1)}{q}C_2^2,$$
$$C_2 = \int_0^1 \frac{1 - s^q}{\sqrt{1 - s^2 - 2(1 - s^{p+1})/(p+1)}}ds.$$

The formula (1.10) has been obtained first for $q = 2$ in [15] by using the relationship between $\lambda_2(\alpha)$ and the critical value associated with $\lambda_2(\alpha)$.

From a view point of Theorems 1.1, we consider the following inverse problem.

Problem 1. *Let $f(u)$ be unknown to satisfy (A.1) and (A.2). Assume that as $\alpha \to \infty$,*

$$\lambda_q(\alpha) \quad = \quad g(\alpha) + Ag(\alpha)^{1/2} + O(1), \tag{1.11}$$

where $A > 0$ is a constant. Then can you conclude that $f(u) = u^p$ for some constant $p > 1$?

To state our results, we assume additional conditions (A.3) and (A.4). We put $f(u) := u^p h(u)$.

(A.3) $h(u)$ is a C^1 function for $u > 0$, and there exists a constant $\delta_0 > 0$ such that $h(u) \geq \delta_0$ for $u > 0$. Furthermore, for an arbitrary fixed constant $0 < \epsilon \ll 1$, as $u \to \infty$,

$$\max_{\epsilon \leq s \leq 1} \left| \frac{uh'(us)}{h(u)} \right| = O((u^{p-1}h(u))^{-1/2}), \tag{1.12}$$

$$\max_{0 \leq s \leq \epsilon} s^p \left| \frac{uh'(us)}{h(u)} \right| = O((u^{p-1}h(u))^{-1/2}). \tag{1.13}$$

(A.4) There exists a constant $0 < \delta_1 \ll 1$ such that for $(1 + \delta_1)v > u > v \gg 1$,

$$f(u) = f(v) + f'(v)(u - v) + O(f(v)/v^2)(u - v)^2. \tag{1.14}$$

The typical examples of $h(u)$ (i.e. $f(u)$) satisfying (A.3) and (A.4) are:

$$h(u) = 1 \quad (f(u) = u^p), \qquad h(u) = 1 + u^{m-p} \quad \left(f(u) = u^p + u^m, \ 1 < m \leq \frac{p+1}{2}\right).$$

The answer to Problem 1 is as follows.

Theorem 1.2. *Assume that all conditions in Problem 1, (A.3) and (A.4) are satisfied. Then $f(u) = u^p h(u)$ with $p = 1 + (qA)/(2C_2)$ and $h(u) = D + d(u)$, where C_2 is a constant in Theorem 1.1, A is a constant in (1.11), $D > 0$ is an arbitrary positive constant and $d(u) = O(u^{(1-p)/2})$ for $u \gg 1$.*

Remark 1.3. (i) The next inverse bifurcation problem we consider in a near future should be to establish the *asymptotic uniqueness* of unknown $f(u)$ from the asymptotic behavior of $\lambda_q(\alpha)$ as $\alpha \to \infty$.
(ii) The condition (A.3) is not technical one. Indeed, if we consider $f(u) = u^5 e^u$ and $q = 2$, then $g(u) = u^4 e^u$ does not satisfy (A.3), and we know from [16] that as $\alpha \to \infty$

$$\lambda_2(\alpha) \quad = \quad \alpha^4 e^\alpha + \frac{\pi}{4}\alpha^3 e^{\alpha/2} + \frac{\pi}{4}u^2 e^{\alpha/2}(1 + o(1)), \tag{1.15}$$

which is different from (1.11). Therefore, (1.11) does not hold without (A.3).

2 Sketch of the Proof of Theorem 1.2

In what follows, C denotes various positive constants independent of $\alpha \gg 1$. We write (λ, u_α) for a unique solution pair of (1.1)–(1.3) with $\|u_\alpha\|_q = \alpha$. We begin with the fundamental tools which play important roles in what follows. It is well known that

$$u_\alpha(t) = u_\alpha(1-t), \ \ t \in I, \quad \|u_\alpha\|_\infty = u_\alpha\left(\frac{1}{2}\right), \tag{2.1}$$

$$u_\alpha'(t) > 0, \ \ 0 \le t < \frac{1}{2}. \tag{2.2}$$

Multiply (1.1) by $u_\alpha'(t)$. Then

$$(u_\alpha''(t) + \lambda u_\alpha(t) - f(u_\alpha(t)))\, u_\alpha'(t) = 0.$$

This along with (2.1) implies that

$$\begin{aligned} \frac{1}{2}u_\alpha'(t)^2 + \frac{1}{2}\lambda u_\alpha(t)^2 - F(u_\alpha(t)) &= \text{constant} \\ &= \frac{1}{2}\lambda\|u_\alpha\|_\infty^2 - F(\|u_\alpha\|_\infty), \quad (\text{put } t = 1/2) \end{aligned} \tag{2.3}$$

where $F(u) := \int_0^u f(s)ds$. We set

$$L_\alpha(\theta) = \lambda(\|u_\alpha\|_\infty^2 - \theta^2) - 2(F(\|u_\alpha\|_\infty) - F(\theta)). \tag{2.4}$$

This along with (2.2) and (2.3) implies that for $0 \le t \le 1/2$

$$u_\alpha'(t) = \sqrt{L_\alpha(u_\alpha(t))}. \tag{2.5}$$

By this and (2.1), we obtain

$$\begin{aligned} \|u_\alpha\|_\infty^q - \alpha^q &= 2\int_0^{1/2} \frac{(\|u_\alpha\|_\infty^q - u_\alpha^q(t))u_\alpha'(t)}{\sqrt{L_\alpha(u_\alpha(t))}}dt = 2\int_0^{\|u_\alpha\|_\infty} \frac{(\|u_\alpha\|_\infty^q - \theta^q)}{\sqrt{L_\alpha(\theta)}}d\theta \\ &= \frac{2\|u_\alpha\|_\infty^q}{\sqrt{\lambda}} \int_0^1 \frac{1-s^q}{\sqrt{B_\alpha(s)}}ds \\ &= \frac{2\|u_\alpha\|_\infty^q}{\sqrt{\lambda}} \left\{ \int_0^1 \frac{1-s^q}{\sqrt{J(s)}}ds + \int_0^1 \left(\frac{1-s^q}{\sqrt{B_\alpha(s)}} - \frac{1-s^q}{\sqrt{J(s)}} \right) ds \right\} \\ &= \frac{2\|u_\alpha\|_\infty^q}{\sqrt{\lambda}}(C_2 + M_\alpha), \end{aligned} \tag{2.6}$$

where

$$J(s) := 1 - s^2 - \frac{2}{p+1}(1 - s^{p+1}), \tag{2.7}$$

$$B_\alpha(s) := 1 - s^2 - \frac{2}{\lambda\|u_\alpha\|_\infty^2}(F(\|u_\alpha\|_\infty) - F(\|u_\alpha\|_\infty s)), \tag{2.8}$$

$$M_\alpha := \int_0^1 \left(\frac{1-s^q}{\sqrt{B_\alpha(s)}} - \frac{1-s^q}{\sqrt{J(s)}} \right) ds. \tag{2.9}$$

Lemma 2.1. *$f'(\alpha) \le C\alpha^{p-1}$ for $\alpha \gg 1$.*

Lemma 2.1 is proved by direct calculation. So we omit the proof. By (A.3) and Lemma 2.1, for $\alpha \gg 1$,

$$C^{-1}\alpha^{p-1} \le \lambda \le C\alpha^{p-1}, \tag{2.10}$$

$$C^{-1}\alpha^{p} \le f(\alpha) \le C\alpha^{p}, \tag{2.11}$$

$$C^{-1}\alpha^{p-1} \le g(\alpha) \le C\alpha^{p-1}. \tag{2.12}$$

The following Lemma 2.2 plays essential roles to prove Theorem 1.2.

Lemma 2.2. *$M_\alpha = O(g(\alpha)^{-1/2})$ as $\alpha \to \infty$.*

We tentatively accept this lemma and prove Theorem 1.2. Lemma 2.2 will be proved in Section 3.

Proof of Theorem 1.2. By Lemma 2.2 and Taylor expansion, for $\alpha \gg 1$,

$$\|u_\alpha\|_\infty = \alpha\left(1 - \frac{2}{\sqrt{\lambda}}(C_2 + M_\alpha)\right)^{-1/q} \tag{2.13}$$

$$= \alpha\left(1 + \frac{2}{q\sqrt{\lambda}}(C_2 + M_\alpha) + \frac{2(q+1)}{q^2\lambda}(C_2 + M_\alpha)^2(1 + o(1))\right). \tag{2.14}$$

By this, Lemmas 2.1 and 2.2,

$$\begin{aligned}
\lambda &= \frac{f(\|u_\alpha\|_\infty)}{\|u_\alpha\|_\infty} + \xi_\alpha \\
&= \frac{1}{\alpha}\left(1 - \frac{2}{q\sqrt{\lambda}}(C_2 + M_\alpha) + O(\alpha^{1-p})\right)\left(f(\alpha) + \frac{2\alpha}{q\sqrt{\lambda}}f'(\alpha)(C_2 + M_\alpha) + O(\alpha)\right) + \xi_\alpha \\
&= \frac{f(\alpha)}{\alpha} + \frac{2C_2}{q\sqrt{\lambda}}\left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right) + M_\alpha\frac{2C_2}{q\sqrt{\lambda}}\left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right) + O(1) \\
&= \frac{f(\alpha)}{\alpha} + \frac{2C_2}{q}\left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right)(g(\alpha) + Ag(\alpha)^{1/2} + O(1))^{-1/2} + O(1) \\
&= \frac{f(\alpha)}{\alpha} + \frac{2C_2}{q\sqrt{g(\alpha)}}\left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right) + O(1). \\
&= g(\alpha) + Ag(\alpha)^{1/2} + O(1).
\end{aligned}$$

This implies that for $\alpha \gg 1$

$$f'(\alpha) - r\frac{f(\alpha)}{\alpha} = O(\sqrt{g(\alpha)}), \tag{2.15}$$

where $r := 1 + (qA)/(2C_2)$. Then we solve (2.15) directly, and easily obtain that $r = p$, and for $\alpha \gg 1$

$$f(\alpha) = D\alpha^p + O(\alpha^{(p+1)/2}), \tag{2.16}$$

where $D > 0$ is an arbitrary constant. Thus the proof is complete. ∎

3 Proof of Lemma 2.2.

In this section, we prove Lemma 2.2. Let an arbitrary $0 < \epsilon \ll 1$ be fixed. For $0 \le s \le 1$, we put

$$\begin{aligned} K_\alpha(s) &:= J(s) - B_\alpha(s) \\ &= \frac{2}{\lambda \|u_\alpha\|_\infty^2}\{F(\|u_\alpha\|_\infty) - F(\|u_\alpha\|_\infty s)\} - \frac{2}{p+1}(1 - s^{p+1}). \end{aligned} \tag{3.1}$$

Then

$$\begin{aligned} M_\alpha &= \int_0^1 \frac{(1-s^q)K_\alpha(s)}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)} + \sqrt{B_\alpha(s)})} ds \\ &= \int_{1-\epsilon}^1 \frac{(1-s^q)K_\alpha(s)}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)} + \sqrt{B_\alpha(s)})} ds \\ &\quad + \int_\epsilon^{1-\epsilon} \frac{(1-s^q)K_\alpha(s)}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)} + \sqrt{B_\alpha(s)})} ds \\ &\quad + \int_0^\epsilon \frac{(1-s^q)K_\alpha(s)}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)} + \sqrt{B_\alpha(s)})} ds \\ &:= M_{1,\alpha} + M_{2,\alpha} + M_{3,\alpha}. \end{aligned} \tag{3.2}$$

Lemma 3.1. *For $\alpha \gg 1$*

$$|M_{1,\alpha}| = O(g(\|u_\alpha\|_\infty)^{-1/2}). \tag{3.3}$$

Proof. By (3.1),

$$\frac{K'_\alpha(s)}{2} = -\frac{f(\|u_\alpha\|_\infty s)}{\lambda \|u_\alpha\|_\infty} + s^p. \tag{3.4}$$

This implies that

$$\frac{K'_\alpha(1)}{2} = \frac{\xi_\alpha}{\lambda}. \tag{3.5}$$

Since $f(u) = g(u)u$, for $1-\epsilon \le s \le 1$, by Taylor expansion, we obtain

$$
\begin{aligned}
\frac{K_\alpha''(s)}{2} &= -\frac{f'(\|u_\alpha\|_\infty s)}{\lambda} + ps^{p-1} \\
&= -\frac{g'(\|u_\alpha\|_\infty s)\|u_\alpha\|_\infty s + g(\|u_\alpha\|_\infty s)}{g(\|u_\alpha\|_\infty) + \xi_\alpha} + ps^{p-1} \qquad (3.6)\\
&= -\frac{g'(\|u_\alpha\|_\infty s)\|u_\alpha\|_\infty s + g(\|u_\alpha\|_\infty s)}{g(\|u_\alpha\|_\infty)}\left(1 - \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)}(1+o(1))\right) + ps^{p-1}.
\end{aligned}
$$

We put

$$
H(s,u) = ps^{p-1}\frac{h(us)}{h(u)} + us^p\frac{h'(us)}{h(u)}. \qquad (3.7)
$$

For $u \gg 1$,

$$
g'(u) = (p-1)u^{p-2}h(u) + u^{p-1}h'(u). \qquad (3.8)
$$

By this and (3.6), we obtain

$$
\begin{aligned}
\frac{K_\alpha''(s)}{2} &= -H(s,\|u_\alpha\|_\infty)\left(1 - \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)}(1+o(1))\right) + ps^{p-1}. \qquad (3.9)\\
&= ps^{p-1}\left(1 - \frac{h(\|u_\alpha\|_\infty s)}{h(\|u_\alpha\|_\infty)}\right) - \|u_\alpha\|_\infty s^p \frac{h'(\|u_\alpha\|_\infty s)}{h(\|u_\alpha\|_\infty)} \\
&\quad + \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)} H(s,\|u_\alpha\|_\infty)(1+o(1)).
\end{aligned}
$$

By this and mean value theorem, for $1-\epsilon < s < s_1 < s_2 < 1$, we obtain

$$
\begin{aligned}
\frac{K_\alpha''(s_1)}{2} &= ps_1^{p-1}\left(1 - \frac{h(\|u_\alpha\|_\infty s_1)}{h(\|u_\alpha\|_\infty))}\right) - \|u_\alpha\|_\infty s_1^p \frac{h'(\|u_\alpha\|_\infty s_1)}{h(\|u_\alpha\|_\infty)} \qquad (3.10)\\
&\quad + \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)} H(s_1,\|u_\alpha\|_\infty)(1+o(1)) \\
&= ps_1^{p-1}\left(\frac{h'(\|u_\alpha\|_\infty s_2)}{h(\|u_\alpha\|_\infty)}\right)\|u_\alpha\|_\infty(1-s_1) - \|u_\alpha\|_\infty s_1^p \frac{h'(\|u_\alpha\|_\infty s_1)}{h(\|u_\alpha\|_\infty)} \\
&\quad + \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)} H(s_1,\|u_\alpha\|_\infty)(1+o(1)) \\
&= O(g(\|u_\alpha\|_\infty)^{-1/2})) + O\left(\frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)}\right) \\
&= O(g(\|u_\alpha\|_\infty)^{-1/2})).
\end{aligned}
$$

Since $K_\alpha(1) = 0$, by (3.5), (3.10) and Taylor expansion, for $1-\epsilon \le s \le 1$,

$$
\begin{aligned}
\frac{K_\alpha(s)}{2} &= \frac{1}{2}\left(K_\alpha(1) + K_\alpha'(1)(s-1) + \frac{1}{2}K_\alpha''(s_1)(s-1)^2\right) \qquad (3.11)\\
&= \frac{\xi_\alpha}{2\lambda}(s-1) + O(g(\|u_\alpha\|_\infty)^{-1/2}))(s-1)^2.
\end{aligned}
$$

By this, (3.1) and Taylor expansion, for $1-\epsilon \le s \le 1$,

$$J(s) \ge (p-1-\delta_1)(1-s)^2, \tag{3.12}$$

$$B_\alpha(s) = J(s) - K_\alpha(s) \ge \frac{\xi_\alpha}{\lambda}(1-s) + \frac{\delta_1}{2}(1-s)^2. \tag{3.13}$$

Then we obtain

$$\begin{aligned} |M_{1,\alpha}| &\le \int_{1-\epsilon}^1 \frac{(1-s^q)|K_\alpha(s)|}{J(s)\sqrt{B_\alpha(s)}} ds \\ &\le C\int_{1-\epsilon}^1 \frac{(\xi_\alpha/(\lambda) + O(g(\|u_\alpha\|_\infty^{-1/2}))(1-s)}{\sqrt{(\xi_\alpha/\lambda)(1-s) + (\delta_1/2)(1-s)^2}} ds \\ &= C\int_{1-\epsilon}^1 \sqrt{\frac{\xi_\alpha}{\lambda}} \frac{1}{\sqrt{1-s}} ds + O(g(\|u_\alpha\|_\infty)^{-1/2}) \int_{1-\epsilon}^1 \frac{1-s}{\sqrt{(\delta_1/2)(1-s)^2}} ds \\ &\le C\left(\sqrt{\frac{\xi_\alpha}{\lambda}} + O(g(\|u_\alpha\|_\infty)^{-1/2})\right) = O(g(\|u_\alpha\|_\infty)^{-1/2}). \end{aligned} \tag{3.14}$$

Thus the proof is complete. ∎

Lemma 3.2. *$M_{2,\alpha} = O(g(\|u_\alpha\|_\infty)^{-1/2})$ as $\alpha \to \infty$.*

Proof. Since $f(u) = u^p h(u)$, for $0 \le s \le 1-\epsilon$,

$$\begin{aligned} K_\alpha(s) &= \frac{1}{\lambda\|u_\alpha\|_\infty^2} \int_{\|u_\alpha\|_\infty s}^{\|u_\alpha\|_\infty} t^p h(t)dt - \frac{1}{p+1}(1-s^{p+1}) \\ &= \frac{1}{(p+1)\lambda\|u_\alpha\|_\infty^2} \left\{ \Big[t^{p+1}h(t)\Big]_{\|u_\alpha\|_\infty s}^{\|u_\alpha\|_\infty} - \int_{\|u_\alpha\|_\infty s}^{\|u_\alpha\|_\infty} t^{p+1}h'(t)dt \right\} \\ &\quad - \frac{1}{p+1}(1-s^{p+1}). \end{aligned} \tag{3.15}$$

Since $\xi_\alpha > 0$, for $\epsilon \le s \le 1-\epsilon$,

$$\begin{aligned} \frac{1}{\lambda\|u_\alpha\|_\infty^2}\left|\int_{\|u_\alpha\|_\infty s}^{\|u_\alpha\|_\infty} t^{p+1}h'(t)dt\right| &\le \frac{1}{h(\|u_\alpha\|_\infty)\|u_\alpha\|_\infty^{p+1}} \int_{\|u_\alpha\|_\infty s}^{\|u_\alpha\|_\infty} |t^{p+1}h'(t)|dt \\ &\le \max_{\epsilon\le s\le 1} \left|\frac{\|u_\alpha\|_\infty h'(\|u_\alpha\|_\infty s)}{h(\|u_\alpha\|_\infty)}\right|(1-s) \\ &= O(g(\|u_\alpha\|_\infty)^{-1/2}). \end{aligned} \tag{3.16}$$

By this and mean value theorem, for $\epsilon \le s < s_1 < 1-\epsilon$,

$$\left|\frac{K_\alpha(s)}{2}\right| \le \frac{1}{(p+1)\lambda\|u_\alpha\|_\infty^2}\left\{\|u_\alpha\|_\infty^{p+1}h(\|u_\alpha\|_\infty) - \|u_\alpha\|_\infty^{p+1}s^{p+1}h(\|u_\alpha\|_\infty s)\right\}$$

$$
\begin{aligned}
(3.17) \qquad & +O(g(\|u_\alpha\|_\infty)^{-1/2}) - \frac{1}{p+1}(1-s^{p+1}) \\
\le\ & \frac{1}{p+1}(1-s^{p+1})\left(\frac{\|u_\alpha\|_\infty^{p-1}h(\|u_\alpha\|_\infty)}{\lambda} - 1\right) \\
& + \frac{\|u_\alpha\|_\infty^{p-1}s^{p+1}}{\lambda(p+1)}(h(\|u_\alpha\|_\infty) - h(\|u_\alpha\|_\infty s) + O(g(\|u_\alpha\|_\infty)^{-1/2}) \\
\le\ & \frac{\xi_\alpha}{(p+1)\lambda}(1-s^{p+1}) + \left|\frac{\|u_\alpha\|_\infty h'(\|u_\alpha\|_\infty s_1)}{h(\|u_\alpha\|_\infty)}\right| + O(g(\|u_\alpha\|_\infty)^{-1/2}) \\
=\ & O(g(\|u_\alpha\|_\infty)^{-1/2}).
\end{aligned}
$$

Note that for $0 \le s \le 1-\epsilon$,

$$
J(s) \ge \delta_2 > 0. \tag{3.18}
$$

By this and (3.14), for $\epsilon \le s \le 1-\epsilon$ and $\alpha \gg 1$,

$$
B_\alpha(s) \ge J(s) - K_\alpha(s) \ge \frac{\delta_2}{2} > 0. \tag{3.19}
$$

Then by this and direct calculation, we obtain

$$
|M_{2,\alpha}| \le C\int_\epsilon^{1-\epsilon} |K_\alpha(s)|(1-s^q)ds = O(g(\|u_\alpha\|_\infty)^{-1/2}).
$$

Thus the proof is complete. ∎

Lemma 3.3. *$M_{3,\alpha} = O(g(\|u_\alpha\|_\infty)^{-1/2})$ as $\alpha \to \infty$.*

The proof of Lemma 3.3 is similar to that of Lemma 3.2. So we omit the proof. Since $\alpha = \|u_\alpha\|_\infty(1+o(1))$, Lemma 2.2 follows from Lemmas 3.1–3.3. Thus the proof is complete.

∎

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