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Remarks on a paper of Zhang and Sun

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ABSTRACT

We are interested in the existence of positive solutions for second order boundary value problem: (E) $y'' + h(t)f(y) = 0$, $0 < t < 1$, subject to multi-point boundary conditions. We prove an extension of a recent result by Zhang and Sun [3] and illustrate with examples.

1. Introduction

We are interested in the existence of positive solutions for the second order nonlinear differential equation

$$y'' + h(t)f(y) = 0, \quad 0 \le t \le 1, \tag{1.1}$$

where $h(t) \in L^1(0,1)$ and $f(y) \in C(\mathbb{R}, \mathbb{R}_+)$ are non-negative functions subject to multi-point boundary condition

$$y(0) = \langle \alpha, y(\xi) \rangle = \sum_{i=1}^{m} \alpha_i \xi_i, y(1) = \langle \beta, y(\xi_i) \rangle = \sum_{i=1}^{m} \beta_i \xi_i \tag{1.2}$$

where $\alpha = (\alpha_1, \cdots, \alpha_m), \beta = (\beta_1, \cdots, \beta_m), \alpha_i, \beta_i$ real, and $0 < \xi_1 < \cdots < \xi_m < 1$. Denote by $\langle \alpha, y(\xi) \rangle$ the scalar product between m-vectors α and $y(\xi) = \big(y(\xi_1), \cdots, y(\xi_m)\big)$. We assume that $\alpha_i, \beta_i \geq 0$ for $i = 1, \cdots, m$, and $h(t)$ may be singular at $t = 0$ or $t = 1$, or both.

In a recent paper [3], Zhang and Sun proved the following generalization of Krasnoselskii Cone Fixed Point Theorem:

Theorem A ([3], p.583, Corollary 2.1). *Let Ω_1 and Ω_2 be two bounded open sets in a Banach space X and $P \subseteq X$ be an ordered cone such that $\theta \in \overline{\Omega_1} \subseteq \Omega_2$. Suppose that $A : P \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \to P$ is completely continuous and $\rho : P \to [0, \infty)$ is a uniformly continuous convex functional with $\rho(\theta) = 0$ and $\rho(x) > 0$, $x \neq \theta$. If one of the two conditions:*

(a) *(Expansion) For $x \in P \cap \partial\Omega_1$, $\rho(Ax) \leq \rho(x)$ and for $x \in P \cap \partial\Omega_2$, $\inf_{x \in \partial\Omega_2} \rho(x) > 0$, $\rho(Ax) \geq \rho(x)$; or*

(b) *(Compression) For $x \in P \cap \partial\Omega_2, \rho(Ax) \leq \rho(x)$ and for $x \in P \cap \partial\Omega_1$, $\inf_{x \in \partial\Omega_1} \rho(x) > 0$, $\rho(Ax) \geq \rho(x)$,*

then A has a fixed point $\hat{x} \in P \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$, i.e. $A\hat{x} = \hat{x}$.

Clearly if $\rho(x) = \|x\|$, where $\|\cdot\|$ denotes the norm of the Banach space X, then Theorem A reduces to the classical Krasnoselskii theorem, see Guo and Lakshmikantham [2]. As an application of Theorem A, Zhang and Sun proved an existence theorem for the multipoint boundary value problem (1.1), (1.2) where $\alpha_i \equiv 0$ for $i = 1, 2, \cdots, m$ in (1.2) subject to the assumptions:

(H_1) $0 < \overline{\beta} = \sum_{i=1}^{m} \beta_i < 1$,

(H_2) $h : (0, 1) \to [0, \infty)$ is continuous, $h \in L^1(0, 1)$ and $h(t) \not\equiv 0$ any subinterval in the open interval $(0, 1)$.

Theorem B ([3], p.584, Theorem 3.1). *Suppose that there exist positive constants r, R and τ such that $0 < r < R$ and $\tau \in (0, \frac{1}{2}]$ satisfying one of the two conditions:*

(a) *(Expansion)* $R \geq \tau^{-2}(1-\tau)^{-2}r$; $f(u) \leq \sigma_1^{-1}h_0^{-1}r$ *for* $0 \leq u \leq \tau^{-1}(1-\tau)^{-1}r$ *and* $f(y) \geq \sigma_2 h_\tau^{-1}R$ *for* $R \geq u \geq \tau(1-\tau)R$; *or*

(b) *(Compression)* $R \geq \sigma_1\sigma_2 h_0 h_\tau^{-1}r$; $f(u) \leq \sigma_1^{-1}h_0^{-1}R$ *for* $0 \leq u \leq \tau^{-1}(1-\tau)^{-1}R$ *and* $f(u) \geq \sigma_2 h_\tau^{-1}$ *for* $r \geq u \geq \tau(1-\tau)r$, *where*

$$\sigma_1 = \frac{1}{4} + \frac{\overline{\beta}}{1-\overline{\beta}}, \quad \sigma_2 = \tau^{-2}(1-\tau)^{-1}, \quad h_0 = \int_0^1 h(t)dt \quad \text{and} \quad h_\tau = \int_\tau^{1-\tau} h(t)dt;$$

then the boundary value problem (1.1) with multipoint boundary condition $\overline{\alpha} = 0$ in (1.2), i.e.

$$y(0) = 0, \; y(1) = \sum_{i=1}^{m} \beta_i y(\xi_i) = \langle \beta, y(\xi) \rangle \tag{1.3}$$

has at least one positive solution.

The purpose of this note is to generalize Theorem B to cover the more general boundary condition (1.2). We obtain bounds on the nonlinear function $f(y)$ sharper than those given in Theorem B, and illustrate our results by examples.

2. Main Result

It is easy to verify that a solution to the boundary value problem (1.1), (1.2) is equivalent to the existence of a fixed point of the operator $A : P \to P$ defined by

$$Ay(t) = \int_0^1 K(t,s)h(s)f\big(y(s)\big)ds \tag{2.1}$$

where P is the cone of non-negative functions in $C[0,1]$ and

$$K(t,s) = g(t,s) + \frac{t}{\Lambda}\{(1-\overline{\alpha}\langle \alpha, \xi\rangle)\langle \beta, g(\xi,s)\rangle + \left(\overline{\beta} - \langle \beta, \xi\rangle\right)\langle \alpha, g(\xi,s)\rangle\} \tag{2.2}$$

with $\overline{\alpha} = \sum_{i=1}^{m} \alpha_i, g(\xi, s) = \big(g(\xi_1, s), \cdots, g(\xi_m, s)\big)$ and $\Lambda = (1-\overline{\alpha})(1-\langle \beta, \xi\rangle) + (1-\overline{\beta})\langle \alpha, \beta\rangle$. When $\overline{\alpha} = 0$ in (2.2), $K(t, s)$ reduces to that given in [3] for the simpler BVP (1.1), (1.3), whilst $g(t, s)$ is the usual Green's function for the two point boundary value problem, i.e. (1.1), (1.2) in absence of all interior boundary points ξ_i, $i = 1, 2, \cdots, m$, and is given by

$$g(t, s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1. \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{2.3}$$

Note that $g(t, s) \le s(1-s)$ for all $t, s \in [0, 1]$ and given $\tau \in (0, \frac{1}{2}]$, $g(t, s) \ge \tau g(s, s)$ for $t \in [\tau, 1-\tau]$ and $s \in [0, 1]$. Since $\alpha_i, \beta_i \ge 0$ for $i = 1, \cdots, m$, it is easy to deduce the following estimates:

$$K(t, s) \ge g(t, s) \ge \tau g(s.s) \qquad \tau \le t \le 1-\tau,\ 0 \le s \le 1 \tag{2.4}$$

$$K(t, s) \le \nu g(s, s),\ \nu = \frac{1}{\Lambda}\left\{1 - \langle \beta, \xi\rangle + \langle \alpha, \xi\rangle + \max(\overline{\beta} - \overline{\alpha}, 0)\right\} \tag{2.5}$$

where $\Lambda = (1-\overline{\alpha})(1-\langle \beta, \xi\rangle) + \langle \alpha, \xi\rangle(1-\overline{\beta})$.

We are now ready to state and prove our main result:

Theorem 1 *Suppose that there exist positive constants* r, R *and* τ *such that* $0 < r < R$ *and* $\tau \in (0, \frac{1}{2}]$ *satisfying one of the two conditions:*

(a) *(Expansion)* $R \ge \tau^{-2}(1-\tau)^{-2} r$; $f(u) \le \nu^{-1} h_1^{-1} r$ *for* $0 \le u \le \tau^{-1}(1-\tau)^{-1} r$ *and* $f(u) \ge mR$ *for* $R \ge u \ge \tau(1-\tau)R$, *where* $h_1 = \int_0^1 s(1-s)h(s)ds$; *or*

(b) *(Compression)* $R \ge m\nu h_1 r$; $f(u) \le \nu^{-1} h_1^{-1} R$ *for* $0 \le u \le \tau^{-1}(1-\tau)^{-1} R$ *and* $f(u) \ge mr$ *for* $r \ge u \ge \tau(1-\tau)r$;

where

$$m = \left(\tau \int_\tau^{1-\tau} s(1-s)h(s)ds\right)^{-1},$$

then the boundary value problem (1.1) (1.2) has at least one positive solution.

Proof of Theorem 1 Let $P_1 = \{y \in P : y(t) \text{ concave}, y(t) \ge t(1-t)\|y\|,\ 0 \le t \le 1\}$. We first prove that $A : P_1 \to P_1$. Let $y \in P_1$. Note that $g(t, s) \ge t(1-t)s(1-s)$ for all

$t, s \in [0,1]$. Now use (2.4), (2.5) in (2.1), we observe

$$\begin{aligned} Ay(t) &= \int_0^1 K(t,s)h(s)f\big(y(s)\big)ds \\ &\geq t(1-t)\int_0^1 K(t_0,s)h(s)f\big(y(s)\big)ds \\ &= t(1-t)\nu^{-1}Ay(t_0) \qquad \text{for any} \quad t_0 \in [0,1]. \end{aligned}$$

This shows $Ay(t) \geq t(1-t)\|Ay\|$ proving $A : P_1 \to P_1$.

We only prove part (b) as part (a) is similar. Let $y \in P_1 \cap \partial B_R$ where $B_R = \{y \in P_1 : \rho(y) < R\}$ and $\partial B_R = \{y \in P_1 \cap \overline{B_R} : \rho(y) = R\}$. Observe by (2.5)

$$\rho(Ay) \leq \nu \int_0^1 s(1-s)h(s)f\big(y(s)\big)ds. \tag{2.6}$$

Since $\rho(y) = R$ and $y \in P_1$ imply

$$R \geq y(\hat{t}) = \max_{\tau \leq t \leq 1-\tau} y(t) \geq \nu_1^{-1}\hat{t}(1-\hat{t})\|y\|,$$

which in turn implies for all $s \in [0,1]$

$$0 \leq y(s) \leq \|y\| \leq \nu_1 \left[\hat{t}(1-\hat{t})\right]^{-1} R \leq \nu_1\tau^{-1}(1-\tau)^{-1}R. \tag{2.7}$$

Now (2.7) implies by assumption (a) that $f\big(y(s)\big) \leq M, s \in [0,1]$, which upon using this in (2.6), we find

$$\rho(Ay) \leq \nu \left(\int_0^1 s(1-s)h(s)ds\right) = \nu h_1 M R = R = \rho(y).$$

since $M = \nu^{-1}h_1^{-1}$.

Next let $y \in P_1 \cap \partial\Omega_r$ and $\rho(y) = \max_{\tau \leq t \leq 1-\tau} y(t) = y(\bar{t})$ for some $\bar{t} \in [\tau, 1-\tau]$. Since $y \in P_1$ so

$$r = \rho(y) \geq y(\bar{t}) \geq \bar{t}(1-\bar{t})\|y\| \geq \tau(1-\tau)\|y\| \geq \tau(1-\tau)\rho(y) = \tau(1-\tau)r.$$

For $r \geq y(s) \geq \tau(1-\tau)r$ we have by assumption (a) $f(y(s)) \geq mr$ for $s \in [\tau, 1-\tau]$. Now by (2.4) and $g(s,s) = s(1-s)$, we obtain

$$\rho(Ay) \geq \int_0^1 K(t,s)h(s)f(y(s))ds \geq \tau\left(\int_\tau^{1-\tau} h(s)ds\right) m\, r = r = \rho(y).$$

since $m = \left(\tau \int_\tau^{1-\tau} h(s)ds\right)^{-1}$. This completes the proof.

Remark 1. When $\overline{\alpha} = 0$, $\nu = 1 + \overline{\beta}(1 - \langle \beta, \xi \rangle)^{-1}$ by (2.5). Note that $h_1 \leq \frac{1}{4}h_0$, so

$$\nu^{-1}h_1^{-1} \geq \left(1 + \frac{\overline{\beta}}{1-\overline{\beta}}\right)^{-1} 4h_0^{-1} \geq \sigma_1 h_0^{-1}.$$

Also

$$m = \left(\tau \int_\tau^{1-\tau} s(1-s)h(s)ds\right)^{-1} \leq \tau^{-2}(1-\tau)^{-1}h_\tau^{-1} = \sigma_2 h_\tau^{-1}.$$

This shows that when Theorem 1 is applied to the boundary value problem (1.1), (1.3) studied in Zhang and Sun [3], we in fact can obtain sharper bounds on the nonlinear function $f(y)$.

3. Discussion

We discuss two examples given in [3; p.585, Example 3.1] for a special case of boundary value problem (1.1), (1.3):

$$\begin{cases} y'' + h(t)f(y(t)) = 0, \quad 0 < t < 1 \\ y(0) = 0, \ y(1) = \frac{1}{2}y(\eta), \ 0 < \eta < 1, \end{cases} \tag{3.1}$$

where $h(t) = [t(1-t)]^{1/2} \in L_1(0,1)$. Two nonlinear functions are exhibited to illustrate Theorem B as follows:

$$\text{(Expansion)} \quad f_1(u) = \begin{cases} (3/20\pi)u & u \leq 16/3 \\ (4/35\pi)(57576u - 307065) & u > \frac{16}{3} \end{cases}$$

with $r = 1, R = 30$;

$$\text{(Compression)} \quad f_2(u) = \begin{cases} \left(\frac{1024}{3\pi}\right)u & u \leq 3/16 \\ \left(\frac{8}{7677\pi}\right)(16u + 61413) & u > 3/16 \end{cases}$$

with $r = 1, R = 90$.

Using improved upper and lower bounds on $f(u)$ as given in Theorem 1, we give the following alternative examples for the boundary value problem (3.1):

$$\text{(Expansion)} \quad \hat{f}_1(u) = \begin{cases} (3/5\pi)\, u & u \leq 16/3 \\ \left(\frac{4}{35\pi}\right)(47376u - 252644) & u > 16/3 \end{cases}$$

with $r = 1, R = 30$;

$$\text{(Compression)} \quad \hat{f}_2(u) = \frac{7}{80}u + \frac{53}{\pi}, \qquad u \geq 0$$

with $r = 1, R = 15$.

Remark 2. We note that the example $\hat{f}_1(u)$ differs only by small margin with $f_1(u)$ but for the Compression part of Theorem 1, $\hat{f}_2(u)$ is considerably simpler than $f_2(u)$.

Remark 3. Both Theorem B and Theorem 1 impose a significant distance between constants r and R. In another recent paper by Avery, Henderson and O'Regan [1], there is an example of a two point boundary value problem where it is only required that $0 < r < R$. In other words, R can be as close to r as one pleases. No example is known for multipoint boundary value problem when only $0 < r < R$ is assumed even for the three point problem.

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