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# A Reverse of Generalized Ando-Hiai Inequality

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## §1 Introduction

The Löwner-Heinz inequality says that the function $t^{\alpha}$ is operator monotone for $\alpha \in [0,1]$.

$(LH) \qquad A \geq B \geq 0 \quad \Longrightarrow \quad A^{\alpha} \geq B^{\alpha} \quad for\ \alpha \in [0,1].$

It induces the $\alpha$-geometric operator mean defined for $\alpha \in [0,1]$ as

$$A \sharp_{\alpha} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}.$$

if $A > 0$, i.e., $A$ is invertible, by the Kubo-Ando theory [1] see also [7].

Ando and Hiai [2] proposed a log-majorization inequality, whose essential part is the following operator inequality. We say it the Ando-Hiai inequality, simply (AH).

$(AH) \qquad A\#_{\alpha}B \leq I \quad \Longrightarrow \quad A^{r}\#_{\alpha}B^{r} \leq I \quad (r \geq 1).$

In some sense, (AH) might be motivated by the following operator inequality,which is a super extension of Löwner-Heinz inequality, the case of $r = 0$ in below:

Furuta inequality (FI). If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(A^{\frac{r}{2}}A^{p}A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^{p}B^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

By using the mean theoretic notation,the Furuta inequality has the following expression by virtue of (LH):

(FI) *If* $A \geq B > 0$, then

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^{p} \leq A \quad for \quad p \geq 1 \ and \ r \geq 0.$$

Moreover, to make a simultaneous extension of both (FI) and (AH), Furuta added variables as in the extension of (LH) to (FI). Actually he established so-called grand Furuta inequality, simply (GFI). It is sometimes said to be generalized Furuta inequality, [3], [6] and [9].

Grand Furuta inequality (GFI). If $A \geq B > 0$ and $t \in [0,1]$, then

$$[A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}})^{s}A^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}} \leq A^{1-t+r}$$

holds for $r \geq t$ and $p, s \geq 1$.

It is easily seen that

(GFI) for $t = 1,\ r = s \iff$ (AH)
(GFI) for $t = 0,\ (s = 1) \iff$ (FI) .

Based on an idea of Furuta inequality, we proposed two variables version of Ando-Hiai inequality, [4] and [5].

Generalized Ando-Hiai inequality (GAH). For $A, B > 0$ and $\alpha \in [0,1]$, if $A \natural_{\alpha} B \leq I$, then

$$A^{r} \natural_{\frac{\alpha r}{\alpha r+(1-\alpha)s}} B^{s} \leq I \quad for \quad r, s \geq 1.$$

It is obvious that the case $r = s$ in (GAH) is just Ando-Hiai inequality.

Now we consider two one-sided versions of (GAH):

(1) For $A, B > 0$ and $\alpha \in [0,1]$, if $A \natural_{\alpha} B \leq I$, then

$$A^{r} \natural_{\frac{\alpha r}{\alpha r+1-\alpha}} B \leq I \quad r \geq 1.$$

(2) For $A, B > 0$ and $\alpha \in [0,1]$, if $A \natural_{\alpha} B \leq I$, then

$$A \natural_{\frac{\alpha}{\alpha+(1-\alpha)s}} B^{s} \leq I \quad s \geq 1.$$

It is known in [5] that (1) and (2) are equivalent and they are equivalent to (FI). Furthermore generalized Ando-Hiai inequality (GAH) is understood as the case $t = 1$ in (GFI):

## §2 Generalisation of Seo's result

Recently Seo showed the following reverse inequality of (AH) in [8].

Theorem S . Let A,B be positive invertible operators. Then

$$A\#_{\alpha}B \leq I \Longrightarrow A^{r}\#_{\alpha}B^{r} \leq \left\|(A\#_{\alpha}B)^{-1}\right\|^{1-r} I \quad (0 \leq r \leq 1).$$

Thus we give a generalization of Theorem S. As a matter of fact, we show the following theorem.

Theorem 1 . Let A, B be positive invertible operators and $\alpha \in [0,1]$. Then

$$A\#_{\alpha}B \leq I \Longrightarrow A^{r}\#_{\frac{\alpha r}{\alpha r+(1-\alpha)s}}B^{s} \leq \left\|(A\#_{\alpha}B)^{-1}\right\|^{\frac{\alpha(r-s)+(1-r)s}{\alpha r+(1-\alpha)s}} I \quad (0 \leq r, s \leq 1)$$

To prove this, we need some lemmas.

Lemma 2 . (Araki-Cordes inequality)

$$\|A^pB^pA^p\| \leq \|ABA\|^p \quad for\ A,B \geq 0\ and\ 0 \leq p \leq 1.$$

It is well-known that Lemma 2 is equivalent to (LH), and so the following lemma is regarded as a reverse of both (LH) and Lemma 2.

Lemma 3 . If $A \geq B > 0$, then

$$\left\|A^{\frac{p}{2}}B^{-p}A^{\frac{p}{2}}\right\| B^p \geq A^p \quad (0 \leq p \leq 1).$$

proof. $A \geq B > 0$ implies $A^p \geq B^p$ for all $0 \leq p \leq 1$ and then

$$I \geq A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}} \geq \left\|A^{\frac{p}{2}}B^{-p}A^{\frac{p}{2}}\right\|^{-1}.$$

Hence we have $\left\|A^{\frac{p}{2}}B^{-p}A^{\frac{p}{2}}\right\| B^p \geq A^p$.

Finally, we cite a convenient formula on exponent of operators.

Lemma 4 . Let A, B be an invertible operators. Then

$$(BAB^*)^r = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{r-1}A^{\frac{1}{2}}B^* \quad (r \in \mathbf{R})$$

We prove Theorem 1 in the below.

Proof of Theorem 1 . If we put

$C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ then the assumption $A\#_\alpha B \leq I$ implies $C^\alpha \leq A^{-1}$.

By Lemma 3, we have

$$A^r = A^{\frac{1}{2}}A^{r-1}A^{\frac{1}{2}} \leq \left\|A^{-\frac{1-r}{2}}C^{-\alpha(1-r)}A^{-\frac{1-r}{2}}\right\| A^{\frac{1}{2}}C^{\alpha(1-r)}A^{\frac{1}{2}}.$$

In addition,by Lemma 2, we have

$$A^r \leq \left\|A^{-\frac{1}{2}}C^{-\alpha}A^{-\frac{1}{2}}\right\|^{1-r} A^{\frac{1}{2}}C^{\alpha(1-r)}A^{\frac{1}{2}}.$$

On the other hand, multiplying $C^{-\frac{1}{2}}$ on both sides of $C^\alpha \leq A^{-1}$, we have

$$C^{\alpha-1} \leq \left(C^{\frac{1}{2}}AC^{\frac{1}{2}}\right)^{-1}.$$

We apply this to Lemma 3. Namely we have

$$\begin{aligned} B^s &= \left(A^{\frac12}CA^{\frac12}\right)^s \\ &= A^{\frac12}C^{\frac12}\left(C^{\frac12}AC^{\frac12}\right)^{s-1}C^{\frac12}A^{\frac12} \text{ by Lemma 4} \\ &\le \left\|\left(C^{\frac12}AC^{\frac12}\right)^{-\frac{1-s}{2}}C^{-(\alpha-1)(1-s)}\left(C^{\frac12}AC^{\frac12}\right)^{-\frac{1-s}{2}}\right\| A^{\frac12}C^{\frac12}C^{(\alpha-1)(1-s)}C^{\frac12}A^{\frac12} \\ &\le \left\|\left(C^{\frac12}AC^{\frac12}\right)^{-\frac12}C^{-(\alpha-1)}\left(C^{\frac12}AC^{\frac12}\right)^{-\frac12}\right\|^{1-s} A^{\frac12}C^{\frac12}C^{(\alpha-1)(1-s)}C^{\frac12}A^{\frac12}. \end{aligned}$$

Let r(A) be the spectral radious of A. Then we have

$$\begin{aligned} \left\|\left(C^{\frac12}AC^{\frac12}\right)^{-\frac12}C^{-(\alpha-1)}\left(C^{\frac12}AC^{\frac12}\right)^{-\frac12}\right\| &= r\left((C^{\frac12}AC^{\frac12})^{-1}C^{-(\alpha-1)}\right) \\ &= r\left(A^{-1}C^{-\alpha}\right) \\ &= r\left(A^{-\frac12}C^{-\alpha}A^{-\frac12}\right) \\ &= \left\|A^{-\frac12}C^{-\alpha}A^{-\frac12}\right\|, \end{aligned}$$

so that $B^s \le \left\|A^{-\frac12}C^{-\alpha}A^{-\frac12}\right\|^{1-s} A^{\frac12}C^{(\alpha-1)(1-s)+1}A^{\frac12}$.

Therefore, it follows that

$$\begin{aligned} A^r \#_{\frac{\alpha r}{\alpha r+(1-\alpha)s}} B^s &\le \left\{\left\|A^{-\frac12}C^{-\alpha}A^{-\frac12}\right\|^{1-r} A^{\frac12}C^{(1-r)\alpha}A^{\frac12}\right\} \\ &\qquad \#_{\frac{\alpha r}{\alpha r+(1-\alpha)s}} \left\{\left\|A^{-\frac12}C^{-\alpha}A^{-\frac12}\right\|^{1-s} A^{\frac12}C^{(\alpha-1)(1-s)+1}A^{\frac12}\right\} \\ &= \left\|A^{-\frac12}C^{-\alpha}A^{-\frac12}\right\|^{\frac{\alpha(r-s)+(1-r)s}{\alpha r+(1-\alpha)s}} A^{\frac12}\{C^{(1-r)\alpha}\#_{\frac{\alpha r}{\alpha r+(1-\alpha)s}}C^{(\alpha-1)(1-s)+1}\}A^{\frac12} \\ &\qquad \text{by } aX\,\#_\gamma\, bY = a^{1-\gamma}b^\gamma X\,\#\,Y \\ &= \left\|A^{-\frac12}\left(A^{-\frac12}BA^{-\frac12}\right)^{-\alpha}A^{-\frac12}\right\|^{\frac{\alpha(r-s)+(1-r)s}{\alpha r+(1-\alpha)s}} A^{\frac12}C^{\alpha}A^{\frac12} \\ &= \left\|\left\{A^{\frac12}\left(A^{-\frac12}BA^{-\frac12}\right)^{\alpha}A^{\frac12}\right\}^{-1}\right\|^{\frac{\alpha(r-s)+(1-r)s}{\alpha r+(1-\alpha)s}} (A\#_\alpha B) \\ &\le \left\|(A\#_\alpha B)^{-1}\right\|^{\frac{\alpha(r-s)+(1-r)s}{\alpha r+(1-\alpha)s}} I \end{aligned}$$

Hence the proof is complete.

<u>Remark .</u> Theorem S is obtained by putting $r = s$ in Theorem 1.

REFERENCE

1. T.Ando , Topics on Operator Inequalities , Lecture notes (mimeographed), Hokkaido Univ., Sapporo, 1978.

2. T.Ando and F.Hiai, Log-majorization and complementary Golden-Thompson type inequalities , Linear Algebra Appl ., 197,198 (1994), 113-131 .

3. M.Fujii and E.Kamei, Mean theoretic approach to the grand Furuta inequality , Proc. Amer. Math. Soc., 124 (1996), 2751-2756 .

4. M.Fujii and E.Kamei, Ando-Hiai inequality and Furuta inequality , Linear Algebra Appl., 416 (2006), 541-545 .

5. M.Fujii and E.Kamei, Variants of Ando-Hiai inequality, Operator Theory: Adv. Appl., 187 (2008), 169-174 .

6. M.Fujii, A.Matsumoto and R.Nakamoto, A short proof of the best possibility for the grand Furuta inequality, J. Inequal. Appl., 4 (1999), 339-344 .

7. F.Kubo and T.Ando, Means of positive linear operators, Math. Ann. 246 (1980) , 205-224.

8. Y.Seo On a reverse of Ando-Hiai inequality. Banach J. Math. Anal.4(2010), 87-91

9. T.Yamazaki, Simplified proof of Tanahashi's result on the best possibility of generalized Furuta inequality, Math. Inequal. Appl., 2 (1999), 473-477 .