<table>
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<th>Title</th>
<th>Some inequalities concerning geometric constants of Banach spaces (Banach space theory and related topics)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2011(2011), 1753: 23-28</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2011-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/171184">http://hdl.handle.net/2433/171184</a></td>
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<td>Right</td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Some inequalities concerning geometric constants of Banach spaces

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Let $X$ be a real Banach space with $\dim X \geq 2$. The closed unit ball and unit sphere of $X$ are denoted by $B_X$ and $S_X$, respectively. We shall consider the following constants:

\begin{align}
C_{NJ}(X) &= \sup \left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X \right\}, \quad (1) \\
C'_{NJ}(X) &= \sup \left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X \right\}, \quad (2) \\
C_Z(X) &= \sup \left\{ \frac{\|x+y\|\|x-y\|}{\|x\|^2 + \|y\|^2} : x \in S_X, y \in B_X \right\}, \quad (3) \\
C'_Z(X) &= \sup \left\{ \frac{\|x+y\|\|x-y\|}{2} : x, y \in S_X \right\}, \quad (4) \\
J(X) &= \sup \{ \min(\|x+y\|, \|x-y\|) : x, y \in S_X \}, \quad (5) \\
\rho_X(1) &= \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : x, y \in S_X \right\}. \quad (6)
\end{align}

Properties and relations concerning these constants have been studied by many authors. In particular the James constant $J(X)$ and the von Neumann-Jordan constant $C_{NJ}(X)$ have been most widely treated. Recall that a Banach space $X$ is \textit{uniformly non-square} provided $J(X) < 2$ or equivalently $C_{NJ}(X) < 2$. The constant $C'_{NJ}(X)$ may be considered as the unitary version of $C_{NJ}(X)$ ([2],[6]). The constant $C_Z(X)$ was introduced by Zbąganu [11], who conjectured that $C_{NJ}(X) = C_Z(X)$ for all Banach spaces $X$, but in general these two constants are different.

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The constant $C'_Z(X)$ may be considered as the unitary version of $C_Z(X)$ ([6]). Then we easily have

$$\frac{J(X)^2}{2} \leq C'_Z(X) \leq C_Z(X) \leq C_{NJ}(X), \quad (7)$$

$$\frac{J(X)^2}{2} \leq C'_Z(X) \leq \frac{(1 + \rho_X(1))^2}{2} \leq C'_{NJ}(X) \leq C_{NJ}(X). \quad (8)$$

There are many Banach spaces $X$ for which all the terms in (7) and (8) coincide. On the other hand, there exists a Banach space $X$ such that $C_Z(X) < (1 + \rho_X(1))^2/2$ and $C_Z(X^*) > C'_{NJ}(X^*)$, where $X^*$ is the dual space of $X$ ([6, Example 3]).

We shall start with the estimate of $C_{NJ}(X)$ by $C'_{NJ}(X)$ and $\rho_X(1)$, which was recently proved in Kato and Takahashi [5]. In the following, let $f(u, v) = u - v + \sqrt{(u - v - 1)^2 + v^2}$. Then it is easy to see that for all $u_2 \geq u_1 \geq 1$ and $v_2 \geq v_1 \geq u_1 - 1$,

$$f(u_1, v_1) \leq f(u_2, v_2),$$

where equality holds only when $u_1 = u_2$ and $v_1 = v_2$ ([5, Lemma 3.6]).

**Theorem 1** ([5, Theorem 3.4]). *Let $X$ be a Banach space. Then*

$$C_{NJ}(X) \leq f(C'_{NJ}(X), \rho_X(1)) \quad (9)$$

*or*

$$C_{NJ}(X) \leq C'_{NJ}(X) - \rho_X(1) + \sqrt{(C'_{NJ}(X) - \rho_X(1) - 1)^2 + \rho_X(1)^2}.$$

If $X$ is not uniformly non-square, we have equality in (9) as $C_{NJ}(X) = C'_{NJ}(X) = 2$ and $\rho_X(1) = 1$. The inequality (9) also attains equality with both of the Day-James $\ell_\infty$-$\ell_1$ and $\ell_2$-$\ell_\infty$ spaces, which are uniformly non-square ([5, Remark 3.5]).

Let us mention that Theorem 1 yields some previous results concerning the estimates of $C_{NJ}(X)$ by $C'_{NJ}(X)$ and by $\rho_X(1)$. To see this we need the following inequalities

$$\frac{(1 + \rho_X(1))^2}{2} \leq C'_{NJ}(X) \leq 1 + \rho_X(1)^2, \quad (10)$$

see [5, Proposition 3.2]. Since $\rho_X(1) \leq \sqrt{2C'_{NJ}(X)} - 1$, by (9) we have

**Corollary 1** ([2, Theorem 1]). *Let $X$ be a Banach space. Then*

$$C_{NJ}(X) \leq f(C'_{NJ}(X), \sqrt{2C'_{NJ}(X)} - 1),$$
which is written as
\[ C_{NJ}(X) \leq 1 + \left( \sqrt{2C'_{NJ}(X)} - 1 \right)^2. \tag{11} \]

Since \( C'_{NJ}(X) \leq 1 + \rho_X(1)^2 \), by (9) we also have \( C_{NJ}(X) \leq 1 + \rho_X(1)^2 \).

**Corollary 2** ([7, Theorem 2]). Let \( X \) be a Banach space. Then
\[ C_{NJ}(X) \leq f(1 + \rho_X(1)^2, \rho_X(1)), \]
which is written as
\[ C_{NJ}(X) \leq 1 + \rho_X(1) \left( \sqrt{(1-\rho_X(1))^2 + 1} - (1-\rho_X(1)) \right). \tag{12} \]

We shall present the estimate of \( C_{NJ}(X) \) by \( C'_{Z}(X) \). To do this we need the estimates of \( \rho_X(1) \) and \( C'_{NJ}(X) \) by \( C'_{Z}(X) \).

**Proposition 1.** For any Banach space \( X \),
\[ \rho_X(1) \leq \frac{C'_{Z}(X)}{2}, \tag{13} \]
\[ C'_{NJ}(X) \leq 1 + \frac{C'_{Z}(X)^2}{4}. \tag{14} \]

In (13) and (14) equalities are attained if \( X \) is not uniformly non-square. If \( X \) is the \( \ell_2 - \ell_1 \) space, we also have equalities in (13) and (14) as \( \rho_X(1) = 1/\sqrt{2} \), \( C'_{NJ}(X) = 3/2 \) and \( C'_{Z}(X) = \sqrt{2} \) ([6, Example 3]).

By (9), (13) and (14) we have

**Theorem 2.** Let \( X \) be a Banach space. Then
\[ C_{NJ}(X) \leq f(1 + C'_{Z}(X)^2/4, C'_{Z}(X)/2), \]
which is written as
\[ C_{NJ}(X) \leq \frac{C'_{Z}(X)^2}{4} + 1 + \frac{C'_{Z}(X)}{4} \left( \sqrt{C'_{Z}(X)^2 - 4C'_{Z}(X) + 8} - 2 \right). \tag{15} \]

Now we shall present the estimate of \( C_{Z}(X) \) by \( C'_{Z}(X) \) and \( \rho_X(1) \), which is similar to Theorem 1.

**Theorem 3.** Let \( X \) be a Banach space. Then
\[ C_{Z}(X) \leq f(C'_{Z}(X), \rho_X(1)) \tag{16} \]
or
\[ C_Z(X) \leq C'_Z(X) - \rho_X(1) + \sqrt{(C'_Z(X) - \rho_X(1) - 1)^2 + \rho_X(1)^2}. \]

Since \( C'_Z(X) \leq (1 + \rho_X(1))^2/2 \), by (16) we have

**Corollary 3.** For any Banach space \( X \)
\[ C_Z(X) \leq f((1 + \rho_X(1))^2/2, \rho_X(1)) = 1 + \rho_X(1)^2. \] (17)

**Remark 1.** The estimates given by Theorem 3 and Corollary 3 are sharp. It should be noted that if \( C_Z(X) = 1 + \rho_X(1)^2 \), we have equality in (16). It is clear that if \( X \) is not uniformly non-square, we have equality in (17) as \( C_Z(X) = 2 \) and \( \rho_X(1) = 1 \). The inequality (17) also attains equality with both of the Day-James \( \ell_\infty-\ell_1 \) and \( \ell_2-\ell_\infty \) spaces, which are uniformly non-square. In fact, \( C_Z(X) = 5/4, \rho_X(1) = 1/2 \) if \( X \) is the \( \ell_\infty-\ell_1 \) space ([6, Example 4]), and \( C_Z(X) = 3/2, \rho_X(1) = 1/\sqrt{2} \) if \( X \) is the \( \ell_2-\ell_\infty \) space ([6, Example 3]).

By (13) and (16) we have

**Theorem 4.** Let \( X \) be a Banach space. Then
\[ C_Z(X) \leq f(C'_Z(X), C'_Z(X)/2), \]
which is written as
\[ C_Z(X) \leq \frac{1}{2} \left\{ C'_Z(X) + \sqrt{(C'_Z(X) - 2)^2 + C'_Z(X)^2} \right\}. \] (18)

Finally we shall present the estimates of \( C'_Z(X) \) and \( C_Z(X) \) by \( J(X) \).

**Theorem 5.** Let \( X \) be a Banach space. Then
\[ C'_Z(X) \leq 4(1 - 1/J(X)). \] (19)

The estimate (19) yields that \( \rho_X(1) \leq C'_Z(X)/2 \leq 2(1 - 1/J(X)) \) ([7, Theorem 1]).

By Theorems 2 and 5 we shall have the quite simple inequality \( C_{NJ}(X) \leq J(X) \), which was proved in 2009 by Takahashi and Kato [7] (see also [5, 8, 10]).

**Corollary 4.** Let \( X \) be a Banach space. Then
\[ C_{NJ}(X) \leq \frac{4}{4 - C'_Z(X)} \leq J(X). \] (20)
By (14), (15), (18) and (19) we have

**Theorem 6.** Let $X$ be a Banach space and let $J = J(X)$. Then

$$C_{NJ}(X) \leq \left( \frac{\sqrt{2}(J - 1) + \sqrt{(J - 1)^2 + 1}}{J} \right)^2,$$

(21)

$$C'_{NJ}(X) \leq 1 + 4 \left( \frac{1 - 1}{J} \right)^2,$$

(22)

$$C_Z(X) \leq \frac{2(J - 1) + \sqrt{5J^2 - 12J + 8}}{J}.$$

(23)

**Remark 2.** Let $J = J(X) < 2$. Then it is easy to see that

$$\frac{2(J - 1) + \sqrt{5J^2 - 12J + 8}}{J} < 1 + 4(1 - 1/J)^2 < \left( \frac{\sqrt{2}(J - 1) + \sqrt{(J - 1)^2 + 1}}{J} \right)^2.$$

The estimates (21) and (22) was proved by Kato and Takahashi [5], see also Wang [8]. On the other hand, the estimate (23) is new.

**References**


