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ON THE JAMES CONSTANT OF EXTREME ABSOLUTE NORMS ON $\mathbb{R}^2$

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Abstract

The set of all absolute normalized norms on $\mathbb{R}^2$ (denoted by AN_2) and the set of all convex functions ψ on $[0,1]$ satisfying $\max\{1-t,t\} \leq \psi(t) \leq 1$ for $t \in [0,1]$ (denoted by Ψ_2) have convex structures and they are isomorphic by the one to one correspondence $\psi(t) = \|(1-t,t)\|_\psi$ $(t \in [0,1])$. In [5], the set of all extreme points of AN_2 is determined. In this note, we will report the calculation of the James constants of $(\mathbb{R}^2, \|\cdot\|_\psi)$ and its dual space $(\mathbb{R}^2, \|\cdot\|_\psi)^*$ where ψ is an arbitrary extreme point of Ψ_2. Moreover, we will consider the relation of the James constants of these spaces.

1. Preliminaries

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(x,y)\| = \|(|x|,|y|)\|$ for all $(x,y) \in \mathbb{R}^2$, and normalized if $\|(1,0)\| = \|(0,1)\| = 1$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by AN_2. Let Ψ_2 be the set of all convex functions ψ on $[0,1]$ satisfying $\max\{1-t,t\} \leq \psi(t) \leq 1$ for $t \in [0,1]$. Ψ_2 and AN_2 can be identified by a one to one correspondence $\psi \to \|\cdot\|_\psi$ with the relation

$$\psi(t) = \|(1-t,t)\|_\psi \tag{1.1}$$

for $t \in [0,1]$. For $1 \leq p \leq \infty$, we denote

$$\psi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{\frac{1}{p}} & (1 \leq p < \infty) \\ \max\{1-t,t\} & (p = \infty). \end{cases}$$

Then $\psi_p \in \Psi_2$ $(1 \leq p \leq \infty)$, and they correspond to the l_p-norms $\|\cdot\|_p$ on $\mathbb{R}^2$.

We call a norm $\|\cdot\| \in AN_2$ (resp. $\psi \in \Psi_2$) an extreme point of AN_2 (resp. Ψ_2) if $\|\cdot\| = \frac{1}{2}(\|\cdot\|' + \|\cdot\|'')$ and $\|\cdot\|', \|\cdot\|'' \in AN_2$ imply $\|\cdot\|' = \|\cdot\|''$ (resp. $\psi = \frac{1}{2}(\psi' + \psi'')$ and $\psi', \psi'' \in \Psi_2$ imply $\psi' = \psi''$).

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Let $0 \le \alpha \le \frac{1}{2} \le \beta \le 1$. For the case $(\alpha, \beta) \ne (\frac{1}{2}, \frac{1}{2})$, we define

$$\psi_{\alpha,\beta}(t) = \begin{cases} 1-t & (t \in [0,\alpha]) \\ \dfrac{\alpha+\beta-1}{\beta-\alpha}t + \dfrac{\beta - 2\alpha\beta}{\beta-\alpha} & (t \in [\alpha,\beta]) \\ t & (t \in [\beta, 1]) \end{cases},$$

$$E = \{\psi_{\alpha,\beta} \mid 0 \le \alpha \le \frac{1}{2} < \beta \le 1\}.$$

Proposition 1([5]). The following conditions are equivalent.

(1) $\|\cdot\|_\psi$ is an extreme point of AN_2.
(2) ψ is an extreme point of Ψ_2.
(3) $\psi \in E$.

Let $\widehat{\Psi}_2 = \{\psi \in \Psi_2 \mid \psi(1-t) = \psi(t) \ (t \in [0,1])\}$. If $\psi \in \Psi_2$, then $\psi \in \widehat{\Psi}_2$ if and only if $\|(x_1, x_2)\|_\psi = \|(x_2, x_1)\|_\psi$ for $(x_1, x_2) \in \mathbb{R}^2$. $\widehat{\Psi}_2$ also has a convex structure, and by an analogy of Proposition 1, we have

Corollary 2. Let $\hat{E} = E \cap \widehat{\Psi}_2 = \{\psi_{\alpha,1-\alpha} \in E \mid 0 \le \alpha \le \frac{1}{2}\}$. Then ψ is an extreme point of $\widehat{\Psi}_2$ if and only if $\psi \in \hat{E}$.

2. Known facts on James constant of $(\mathbb{R}^2, \|\cdot\|_\psi)$

For a Banach space $(X, \|\cdot\|)$, the James constant is defined by

$$J((X, \|\cdot\|)) = \sup\{\min\{\|x+y\|, \|x-y\|\} \mid x, y \in X,\ \|x\| = \|y\| = 1\}.$$

$\sqrt{2} \le J((X, \|\cdot\|)) \le 2$ holds and $J((X, \|\cdot\|)) = \sqrt{2}$ if X is a Hilbert space. (The converse is not true.) For $1 \le p \le \infty$, $J(L_p) = \max\{2^{\frac{1}{p}}, 2^{\frac{1}{q}}\}$ holds where $\frac{1}{p} + \frac{1}{q} = 1$ and $\dim L_p \ge 2$. It is known that $J(X) < 2$ if and only if X is uniformly non-square, that is, there exists $\delta > 0$ such that $\|(x+y)/2\| \le 1 - \delta$ holds whenever $\|(x-y)/2\| \ge 1 - \delta$, $\|x\| \le 1$, $\|y\| \le 1$. Moreover, $J(X^{**}) = J(X)$ holds and

$$2J(X) - 2 \le J(X^*) \le \frac{J(X)}{2} + 1. \tag{2.1}$$

There are some Banach spaces which do not satisfy $J(X^*) = J(X)$.

For the 2-dimensional spaces with absolute normalized norms, we know the following facts on the James constant.

Proposition 3 ([8]).

(1) If $\psi_2 \le \psi \in \widehat{\Psi}_2$ and $\max\limits_{t \in [0,1]} \dfrac{\psi(t)}{\psi_2(t)}$ is taken at $t = \frac{1}{2}$, then

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = 2\psi(\frac{1}{2}).$$

(2) If $\psi_2 \geq \psi \in \widehat{\Psi}_2$ and $\max_{t\in[0,1]} \dfrac{\psi_2(t)}{\psi(t)}$ is taken at $t = \frac{1}{2}$, then

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \frac{1}{\psi(\frac{1}{2})}.$$

(3) For $\beta \in [\frac{1}{2}, 1]$,

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_{1-\beta,\beta}})) = \begin{cases} \frac{1}{\beta} & (\beta \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]) \\ 2\beta & (\beta \in [\frac{1}{\sqrt{2}}, 1]). \end{cases}$$

The results in Proposition 3 are obtained by the following proposition. Also in [9] and [10], the James constants of 2 dimensional Lorentz sequence spaces and their dual spaces were culculated by using the following proposition.

Proposition 4([8]). If $\psi \in \widehat{\Psi}_2$, then

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \max_{0\leq t\leq \frac{1}{2}} \frac{2-2t}{\psi(t)}\psi(\frac{1}{2-2t}).$$

We have only few results on the James constants of $(\mathbb{R}^2, \|\cdot\|_\psi)$ when $\psi \in \Psi_2 \setminus \widehat{\Psi}_2$. In this note we focus our consideration on the James constants of $(\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})$ and its dual space $(\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})^*$ where $\psi_{\alpha,\beta} \in E$. There is a unique $\psi^* \in \Psi_2$ such that $(\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})^* = (\mathbb{R}^2, \|\cdot\|_{\psi^*})$, and it is obvious that $\psi_{\alpha,\beta}$, $\psi^* \notin \widehat{\Psi}_2$ whenever $\alpha + \beta \neq 1$.

3. James constants for extreme norms in AN_2.

In this section we consider $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}}))$ where $\|\cdot\|_{\psi_{\alpha,\beta}}$ is the extreme norm of AN_2. Since $J((\mathbb{R}^2, \|\cdot\|_{\tilde{\psi}})) = J((\mathbb{R}^2, \|\cdot\|_\psi))$ where $\tilde{\psi}(t) = \psi(1-t)$, it is sufficient to culculate James constant in the case that $\alpha + \beta \leq 1$.

Theorem 5([4]). Suppose that $\alpha + \beta \leq 1$, then

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \begin{cases} \dfrac{1}{\psi(1/2)} & (\text{if } \psi(\frac{1}{2}) \leq \dfrac{1}{2(1-\alpha)}) \\ 1 + \dfrac{1}{2\psi(1/2)+\gamma} & (\text{if } \dfrac{1}{2(1-\alpha)} \leq \psi(\frac{1}{2}) \leq \dfrac{1}{4(1-\alpha)}(1+\frac{1}{\gamma})) \\ 2\psi(1/2) & (\text{if } \dfrac{1}{4(1-\alpha)}(1+\frac{1}{\gamma}) \leq \psi(\frac{1}{2})), \end{cases}$$

where $\gamma = \dfrac{2\beta - 1}{\beta - \alpha}$.

Corollary 6. If $\beta \leq 1 - \alpha \leq \frac{1}{\sqrt{2}}$, then

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \frac{1}{\psi(1/2)}.$$

We have some other formulations of Theorem 5. Put

$$\gamma = \gamma(\alpha,\beta) = \begin{cases} \dfrac{2\beta-1}{\beta-\alpha} & (\alpha+\beta \le 1) \\ \dfrac{1-2\alpha}{\beta-\alpha} & (\alpha+\beta \ge 1) \end{cases},$$

$$f = f(\gamma) = \tfrac{1}{4}\{1-\gamma+\sqrt{(1+\gamma)^2+4\gamma}\},$$

$$g = g(\gamma) = \tfrac{1}{4}\{1-\gamma+\sqrt{(1+\gamma)^2+4}\},$$

$$M = 1 + \frac{1}{2\psi(1/2)+\gamma}.$$

It can be shown by a simple calculation that f is increasing with respect to γ while g is decreasing and that $\frac{1}{2} \le f(\gamma) \le \frac{1}{\sqrt{2}} \le g(\gamma) \le \frac{1+\sqrt{5}}{4}$ $(\gamma \in [0,1])$.

Theorem 7([4]).

(1) If $\psi(1/2) \le f(\gamma)$, then
$2\psi(1/2) \le M \le \frac{1}{\psi(1/2)}$, and $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \frac{1}{\psi(1/2)}$.

(2) If $f(\gamma) \le \psi(1/2) \le g(\gamma)$, then
$2\psi(1/2),\ \frac{1}{\psi(1/2)} \le M$, and $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = M$.

(3) If $g(\gamma) \le \psi(1/2)$, then
$\frac{1}{\psi(1/2)} \le M \le 2\psi(1/2)$, and $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = 2\psi(1/2)$.

Theorem 7'. For $\psi_{\alpha,\beta}$, put $\gamma = \gamma(\alpha,\beta) = \begin{cases} \dfrac{2\beta-1}{\beta-\alpha} & (\alpha+\beta \le 1) \\ \dfrac{1-2\alpha}{\beta-\alpha} & (\alpha+\beta \ge 1) \end{cases}$, then

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \max\{\frac{1}{\psi(1/2)},\ 1+\frac{1}{2\psi(1/2)+\gamma},\ 2\psi(\frac{1}{2})\}.$$

It is known that $J((\mathbb{R}^2, \|\cdot\|_\psi)) = \sqrt{2}$ holds for $\psi \in [\psi_2, \psi_{1-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}}] = \{(1-\lambda)\psi_2 + \lambda\psi_{1-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}} \mid \lambda \in [0,1]\}$. By Theorem 7 or Theorem 7' we can prove that

Corollary 8. $\|\cdot\|_{\psi_{1-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}}}$ is the only extreme point of AN_2 whose James constant is $\sqrt{2}$, that is,

$$\{\psi_{\alpha,\beta} \in E \mid J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \sqrt{2}\} = \{\psi_{1-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}}\}.$$

4. James constants for the dual norms.

Let $C_{r,s}$ be the convex hull of the set consisting of eight points $(\pm 1, 0)$, $(0, \pm 1)$, and $(\pm r, \pm s)$ with $r, s \in [0,1], r+s \ge 1$. $C_{r,s}$ is an octagon whenever $1 < r+s$, $r < 1$, and $s < 1$. In the exceptional cases, it is a hexagon or a square. Let $\psi^*_{r,s} \in \Psi_2$ be such that the unit sphere of the norm $\|\cdot\|_{\psi^*_{r,s}}$ is $C_{r,s}$. Then $\psi^*_{r,s}$ and $\|\cdot\|_{\psi^*_{r,s}}$ are given by:

$$\psi^*_{r,s}(t) = \begin{cases} 1 - \dfrac{r+s-1}{s}t & (t \in [0, \dfrac{s}{r+s}]) \\ \dfrac{1-s}{r} + \dfrac{r+s-1}{r}t & (t \in [\dfrac{s}{r+s}, 1]), \end{cases}$$

$$\|(x_1, x_2)\|_{\psi^*_{r,s}} = \begin{cases} x_1 - \dfrac{r-1}{s}x_2 & (0 \le rx_2 \le sx_1) \\ \dfrac{1-s}{r}x_1 + x_2 & (0 \le sx_1 \le rx_2). \end{cases}$$

It is easy to find that $\|\cdot\|_{\psi^*_{r,s}}$ is the dual norm of $\|\cdot\|_{\psi_{\alpha,\beta}}$ if and only if

$$\begin{cases} \alpha &= \dfrac{1-r}{1-r+s} \\ \beta &= \dfrac{r}{1+r-s}. \end{cases} \tag{4.1}$$

It is easy to see that for each $\psi \in \Psi_2$

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = J((\mathbb{R}^2, \|\cdot\|_{\tilde{\psi}}))$$

where $\tilde{\psi}$ is defined by $\tilde{\psi}(t) = \psi(1-t)$ $(t \in [0,1])$. Since $\tilde{\psi}^*_{r,s} = \psi^*_{s,r}$ holds, it follows that $J((\mathbb{R}^2, \|\cdot\|_{\psi^*_{r,s}})) = J((\mathbb{R}^2, \|\cdot\|_{\psi^*_{s,r}}))$ for all $r, s \in [0,1]$ with $r+s \ge 1$. Hence it is sufficient to consider the case that $r \le s$.

Theorem 9. Suppose that $r \le s$, then

$$J((\mathbb{R}^2, \|\cdot\|_{\psi^*_{r,s}})) = \begin{cases} 1 + \dfrac{1-r}{s} & (f(r,s) \le 0) \\ \dfrac{2r(2rs - 3s - r + 1)}{2r^2 - 3r - s + 1} & (f(r,s) \ge 0), \end{cases} \tag{4.2}$$

where $f(r,s) = -4r^2s^2 - 2r^3 + 4r^2s + 6rs^2 + 5r^2 - 4rs - s^2 - 4r + 1$.

By a simple calculation we find that there is an implicit function $s = h(r)$ of f, such that h is decreasing on $[\frac{1}{2}, \frac{1}{\sqrt{2}}]$ and $h(\frac{1}{2}) = 1$, $h(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}$, and $f(r, h(r)) = 0$ for $r \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]$. Moreover we can see that

$$f(r,s) \begin{cases} \le 0 & (0 \le r \le \frac{1}{2}, \text{ or } \frac{1}{2} \le r \le \frac{1}{\sqrt{2}}, s \le h(r)) \\ \ge 0 & (\frac{1}{2} \le r \le \frac{1}{\sqrt{2}}, s \ge h(r), \text{ or } \frac{1}{\sqrt{2}} \le r \le 1). \end{cases}$$

We have another formulation of (4.2) which is written by the function $\psi^*_{r,s}$.

Theorem 9'. Suppose that $r \le s$, then

$$J((\mathbb{R}^2, \|\cdot\|_{\psi^*_{r,s}})) = \begin{cases} 2\omega & (r(r-2) + \omega + (2r-1)\omega^2 \le 0) \\ \dfrac{2r(r-2+\omega)}{(1-2\omega)r - 1 + \omega} & (r(r-2) + \omega + (2r-1)\omega^2 \ge 0), \end{cases}$$

where $\omega = \psi^*_{r,s}(\frac{1}{2})$. In particular, if $r = s$, then $\omega = \frac{1}{2r}$, and

$$J((\mathbb{R}^2, \|\cdot\|_{\psi^*_{r,s}})) = \begin{cases} 2\psi^*_{r,s}(1/2) & (\frac{1}{2} \le r \le \frac{1}{\sqrt{2}}) \\ \dfrac{1}{\psi^*_{r,s}(1/2)} & (\frac{1}{\sqrt{2}} \le r). \end{cases}$$

As stated in Section 2, $J(X^*) = J(X)$ does not always hold. We will give a partial result on the relation between $J((\mathbb{R}^2, \|\cdot\|_{\psi^*_{r,s}}))$ and $J((\mathbb{R}^2, \|\cdot\|_{\psi^*_{r,s}})^*)$. $(\mathbb{R}^2, \|\cdot\|_{\psi^*_{r,s}})^*$ is given by $(\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})$ where (α, β) satisfies (4.1).

Theorem 10. Suppose that (4.1) holds, then

(1) If $r = s$ $(\frac{1}{2} \le r \le 1)$, or $(r, s) = (\frac{1}{2}, 1)$,

then $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = J((\mathbb{R}^2, \|\cdot\|_{\psi^*_{r,s}}))$.

(2) If $r \in (0, 1) \setminus \{\frac{1}{2}\}$, $s = 1$, or $r = \frac{1}{2}, \frac{1}{2} \le s < 1$,

then $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) \ne J((\mathbb{R}^2, \|\cdot\|_{\psi^*_{r,s}}))$.

Combining Corollary 2 and Theorem 10, we have

Corollary 11. Suppose that $\psi \in E \cap \widehat{\Psi}_2$, then $J((\mathbb{R}^2, \|\cdot\|_\psi)) = J((\mathbb{R}^2, \|\cdot\|_\psi)^*)$.

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