ON THE JAMES CONSTANT OF EXTREME ABSOLUTE NORMS ON $\mathbb{R}^2$

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Abstract

The set of all absolute normalized norms on $\mathbb{R}^2$ (denoted by $AN_2$) and the set of all convex functions $\psi$ on $[0,1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for $t \in [0,1]$ (denoted by $\Psi_2$) have convex structures and they are isomorphic by the one to one correspondence $\psi(t) = \| (1-t, t) \|_\psi (t \in [0,1])$. In [5], the set of all extreme points of $AN_2$ is determined. In this note, we will report the calculation of the James constants of $(\mathbb{R}^2, \| \cdot \|_\psi)$ and its dual space $(\mathbb{R}^2, \| \cdot \|_\psi)^*$ where $\psi$ is an arbitrary extreme point of $\Psi_2$. Moreover, we will consider the relation of the James constants of these spaces.

1. Preliminaries

A norm $\| \cdot \|$ on $\mathbb{R}^2$ is said to be absolute if $\| (x, y) \| = \| (|x|, |y|) \|$ for all $(x, y) \in \mathbb{R}^2$, and normalized if $\| (1,0) \| = \| (0,1) \| = 1$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by $AN_2$. Let $\Psi_2$ be the set of all convex functions $\psi$ on $[0,1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for $t \in [0,1]$. $\Psi_2$ and $AN_2$ can be identified by a one to one correspondence $\psi \rightarrow \| \cdot \|_\psi$ with the relation

$$
\psi(t) = \| (1-t, t) \|_\psi
$$

for $t \in [0,1]$. For $1 \leq p \leq \infty$, we denote

$$
\psi_p(t) = \begin{cases} 
(1-t)^p + t^p \frac{1}{p} & (1 \leq p < \infty) \\
\max\{1-t, t\} & (p = \infty)
\end{cases}
$$

Then $\psi_p \in \Psi_2$ ($1 \leq p \leq \infty$), and they correspond to the $l_p$-norms $\| \cdot \|_p$ on $\mathbb{R}^2$.

We call a norm $\| \cdot \| \in AN_2$ (resp. $\psi \in \Psi_2$) an extreme point of $AN_2$ (resp. $\Psi_2$) if $\| \cdot \| = \frac{1}{2} (\| \cdot \|' + \| \cdot \|''$) and $\| \cdot \|', \| \cdot \|'' \in AN_2$ imply $\| \cdot \|' = \| \cdot \|''$ (resp. $\psi = \frac{1}{2} (\psi' + \psi''$) and $\psi', \psi'' \in \Psi_2$ imply $\psi' = \psi''$).

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Let $0 \leq \alpha \leq \frac{1}{2} \leq \beta \leq 1$. For the case $(\alpha, \beta) \neq (\frac{1}{2}, \frac{1}{2})$, we define

$$\psi_{\alpha, \beta}(t) = \begin{cases} 
1 - t & (t \in [0, \alpha]) \\
\alpha + \beta - 1 & (t \in [\alpha, \beta]) \\
\frac{\beta - 2\alpha\beta}{\beta - \alpha} & (t \in [\beta, 1])
\end{cases}$$

and $E = \{\psi_{\alpha, \beta} | 0 \leq \alpha \leq \frac{1}{2} < \beta \leq 1\}$.

**Proposition 1** ([5]). The following conditions are equivalent.

1. $\Vert \cdot \Vert_{\psi}$ is an extreme point of $AN_{2}$.
2. $\psi$ is an extreme point of $\Psi_{2}$.
3. $\psi \in E$.

Let $\hat{\Psi}_{2} = \{\psi \in \Psi_{2} | \psi(1-t) = \psi(t) \ (t \in [0,1])\}$. If $\psi \in \Psi_{2}$, then $\psi \in \hat{\Psi}_{2}$ if and only if $\| (x_{1}, x_{2}) \|_{\psi} = \| (x_{2}, x_{1}) \|_{\psi}$ for $(x_{1}, x_{2}) \in \mathbb{R}^{2}$. $\hat{\Psi}_{2}$ also has a convex structure, and by an analogy of Proposition 1, we have

**Corollary 2.** Let $\hat{E} = E \cap \hat{\Psi}_{2} = \{\psi_{\alpha, 1-\alpha} \in E | 0 \leq \alpha \leq \frac{1}{2}\}$. Then $\psi$ is an extreme point of $\hat{\Psi}_{2}$ if and only if $\psi \in \hat{E}$.

2. Known facts on James constant of $(\mathbb{R}^{2}, \| \cdot \|_{\psi})$

For a Banach space $(X, \| \cdot \|)$, the James constant is defined by

$$J((X, \| \cdot \|)) = \sup \{ \min \{ \| x + y \|, \| x - y \| \} | x, y \in X, \| x \| = \| y \| = 1 \}.$$ 

$\sqrt{2} \leq J((X, \| \cdot \|)) \leq 2$ holds and $J((X, \| \cdot \|)) = \sqrt{2}$ if $X$ is a Hilbert space. (The converse is not true.) For $1 \leq p \leq \infty$, $J(L_{p}) = \max \{2^{\frac{1}{p}}, 2^{\frac{1}{q}}\}$ holds where $\frac{1}{p} + \frac{1}{q} = 1$ and $\dim L_{p} \geq 2$. It is known that $J(X) < 2$ if and only if $X$ is uniformly nonsquare, that is, there exists $\delta > 0$ such that $\| (x + y)/2 \| \leq 1 - \delta$ holds whenever $\| (x - y)/2 \| \geq 1 - \delta$, $\| x \| \leq 1$, $\| y \| \leq 1$. Moreover, $J(X^{**}) = J(X)$ holds and

$$2J(X) - 2 \leq J(X^{*}) \leq \frac{J(X)}{2} + 1.$$ 

There are some Banach spaces which do not satisfy $J(X^{*}) = J(X)$.

For the 2-dimensional spaces with absolute normalized norms, we know the following facts on the James constant.

**Proposition 3** ([8]).

1. If $\psi_{2} \leq \psi \in \hat{\Psi}_{2}$ and $\max_{t \in [0,1]} \frac{\psi(t)}{\psi_{2}(t)}$ is taken at $t = \frac{1}{2}$, then

$$J((\mathbb{R}^{2}, \| \cdot \|_{\psi})) = 2\psi(\frac{1}{2}).$$
If $\psi_2 \geq \psi \in \hat{\Psi}_2$ and $t \in [0,1]$ then $\max_{t \in [0,1]} \frac{\psi_2(t)}{\psi(t)}$ is taken at $t = \frac{1}{2}$, then

$$J((\mathbb{R}^2, \| \cdot \|_\psi)) = \frac{1}{\psi(\frac{1}{2})}.$$  

\begin{equation}
J((\mathbb{R}^2, \| \cdot \|_{\psi_{1-\beta, \beta}})) = \begin{cases} \frac{1}{\beta} & (\beta \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]) \\ 2\beta & (\beta \in [\frac{1}{\sqrt{2}}, 1]) \end{cases}.
\end{equation}

The results in Proposition 3 are obtained by the following proposition. Also in [9] and [10], the James constants of 2 dimensional Lorentz sequence spaces and their dual spaces were calculated by using the following proposition.

**Proposition 4([8]).** If $\psi \in \widehat{\Psi}_2$, then

$$J((\mathbb{R}^2, \| \cdot \|_\psi)) = \max_{0 \leq t \leq \frac{1}{2}} \frac{2 - 2t}{\psi(t)} \psi(\frac{1}{2 - 2t}).$$

We have only few results on the James constants of $(\mathbb{R}^2, \| \cdot \|_\psi)$ when $\psi \in \Psi_2 \setminus \widehat{\Psi}_2$. In this note we focus our consideration on the James constants of $(\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})$ and its dual space $(\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})^*$ where $\psi_{\alpha, \beta} \in E$. There is a unique $\psi^* \in \Psi_2$ such that $(\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})^* = (\mathbb{R}^2, \| \cdot \|_{\psi^*})$, and it is obvious that $\psi_{\alpha, \beta}$, $\psi^* \notin \widehat{\Psi}_2$ whenever $\alpha + \beta \neq 1$.

### 3. James Constants for Extreme Norms in $AN_{2*}$

In this section we consider $J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}}))$ where $\| \cdot \|_{\psi_{\alpha, \beta}}$ is the extreme norm of $AN_{2*}$. Since $J((\mathbb{R}^2, \| \cdot \|_{\bar{\psi}})) = J((\mathbb{R}^2, \| \cdot \|_{\psi}))$ where $\bar{\psi}(t) = \psi(1 - t)$, it is sufficient to calculate James constant in the case that $\alpha + \beta \leq 1$.

**Theorem 5([4]).** Suppose that $\alpha + \beta \leq 1$, then

$$J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})) = \begin{cases} \frac{1}{\psi(1/2)} & (\text{if } \psi(\frac{1}{2}) \leq \frac{1}{2(1 - \alpha)}) \\ \frac{1}{2\psi(1/2)} + \gamma & (\text{if } \frac{1}{2(1 - \alpha)} \leq \psi(\frac{1}{2}) \leq \frac{1}{4(1 - \alpha)}(1 + \frac{1}{\gamma})) \\ 2\psi(1/2) & (\text{if } \frac{1}{4(1 - \alpha)}(1 + \frac{1}{\gamma}) \leq \psi(\frac{1}{2})) \end{cases},$$

where $\gamma = \frac{2\beta - 1}{\beta - \alpha}$.

**Corollary 6.** If $\beta \leq 1 - \alpha \leq \frac{1}{\sqrt{2}}$, then

$$J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})) = \frac{1}{\psi(1/2)}.$$
\[
\gamma = \gamma(\alpha, \beta) = \begin{cases} 
\frac{2\beta - 1}{\beta - \alpha} (\alpha + \beta \leq 1) \\
\frac{1 - 2\alpha}{\beta - \alpha} (\alpha + \beta \geq 1)
\end{cases},
\]

\[
f = f(\gamma) = \frac{1}{4} \{1 - \gamma + \sqrt{(1 + \gamma)^2 + 4\gamma}\},
\]

\[
g = g(\gamma) = \frac{1}{4} \{1 - \gamma + \sqrt{(1 + \gamma)^2 + 4}\},
\]

\[
M = 1 + \frac{1}{2\psi(1/2) + \gamma}.
\]

It can be shown by a simple calculation that \(f\) is increasing with respect to \(\gamma\) while \(g\) is decreasing and that \(\frac{1}{2} \leq f(\gamma) \leq \frac{1}{\sqrt{2}} \leq g(\gamma) \leq \frac{1 + \sqrt{5}}{4} (\gamma \in [0, 1]).\)

**Theorem 7(4).**

1. If \(\psi(1/2) \leq f(\gamma)\), then
   \[2\psi(1/2) \leq M \leq \frac{1}{\psi(1/2)},\]
   and \(J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \frac{1}{\psi(1/2)}\).

2. If \(f(\gamma) \leq \psi(1/2) \leq g(\gamma)\), then
   \[2\psi(1/2), \frac{1}{\psi(1/2)} \leq M,\]
   and \(J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = M\).

3. If \(g(\gamma) \leq \psi(1/2)\), then
   \[\frac{1}{\psi(1/2)} \leq M \leq 2\psi(1/2),\]
   and \(J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = 2\psi(1/2)\).

**Theorem 7'.** For \(\psi_{\alpha,\beta}\), put \(\gamma = \gamma(\alpha, \beta) = \begin{cases} 
\frac{2\beta - 1}{\beta - \alpha} (\alpha + \beta \leq 1) \\
\frac{1 - 2\alpha}{\beta - \alpha} (\alpha + \beta \geq 1)
\end{cases}\), then

\[
J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \max\{\frac{1}{\psi(1/2)}, \frac{1}{2\psi(1/2) + \gamma}, 2\psi(\frac{1}{2})\}.
\]

It is known that \(J((\mathbb{R}^2, \|\cdot\|_{\psi})) = \sqrt{2}\) holds for \(\psi \in [\psi_2, \psi_{1-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}}] = \{(1 - \lambda)\psi_2 + \lambda\psi_{1-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}} \mid \lambda \in [0, 1]\}.\) By Theorem 7 or Theorem 7' we can prove that

**Corollary 8.** \(\|\cdot\|_{\psi_{1-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}}}\) is the only extreme point of \(\text{AN}_2\) whose James constant is \(\sqrt{2}\), that is,

\[
\{\psi_{\alpha,\beta} \in E \mid J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \sqrt{2}\} = \{\psi_{1-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}}\}.
\]

4. **James constants for the dual norms.**

Let \(C_{r,s}\) be the convex hull of the set consisting of eight points \((\pm 1, 0), (0, \pm 1),\) and \((\pm r, \pm s)\) with \(r, s \in [0, 1], r + s \geq 1.\) \(C_{r,s}\) is an octagon whenever \(1 < r + s, r < 1,\) and \(s < 1.\) In the exceptional cases, it is a hexagon or a square. Let \(\psi_{r,s}^* \in \Psi_2\) be
such that the unit sphere of the norm $\| \cdot \|_{\psi_{r,s}}$ is $C_{r,s}$. Then $\psi_{r,s}^{*}$ and $\| \cdot \|_{\psi_{r,s}}$ are given by:

$$
\psi_{r,s}^{*}(t) = \begin{cases} 
1 - \frac{r + s - 1}{r} t & (t \in [0, \frac{s}{r + s}]) \\
1 - \frac{s}{r + s} r + s - 1 & (t \in [\frac{s}{r + s}, 1]),
\end{cases}
$$

$$
\| (x_1, x_2) \|_{\psi_{r,s}^{*}} = \begin{cases} 
x_1 - \frac{r - 1}{r} s x_2 & (0 \leq r x_2 \leq s x_1) \\
1 - \frac{s}{r} x_1 + x_2 & (0 \leq s x_1 \leq r x_2).
\end{cases}
$$

It is easy to find that $\| \cdot \|_{\psi_{r,s}^{*}}$ is the dual norm of $\| \cdot \|_{\psi_{\alpha,\beta}}$ if and only if

$$
\left\{ \begin{array}{l}
\alpha = \frac{1 - r}{1 - r + s} \\
\beta = \frac{1 - r + s}{1 + r - s}.
\end{array} \right.
$$

(4.1)

It is easy to see that for each $\psi \in \Psi_2$

$$
J((\mathbb{R}^2, \| \cdot \|_{\psi})) = J((\mathbb{R}^2, \| \cdot \|_{\tilde{\psi}}))
$$

where $\tilde{\psi}$ is defined by $\tilde{\psi}(t) = \psi(1 - t)$ ($t \in [0, 1]$). Since $\tilde{\psi}_{r,s}^{*} = \psi_{s,r}^{*}$ holds, it follows that $J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^{*}})) = J((\mathbb{R}^2, \| \cdot \|_{\psi_{s,r}^{*}}))$ for all $r, s \in [0, 1]$ with $r + s \geq 1$. Hence it is sufficient to consider the case that $r \leq s$.

**Theorem 9.** Suppose that $r \leq s$, then

$$
J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^{n}})) = \begin{cases} 
1 + \frac{1 - r}{2r(2r - 3s - r + 1)} & (f(r, s) \leq 0) \\
2r^2 - 3r - s + 1 & (f(r, s) \geq 0),
\end{cases}
$$

(4.2)

where $f(r, s) = -4r^2 s^2 - 2r^3 + 4r^2 s + 6rs^2 + 5r^2 - 4rs - s^2 - 4r + 1$.

By a simple calculation we find that there is an implicit function $s = h(r)$ of $f$, such that $h$ is decreasing on $[\frac{1}{2}, \frac{1}{\sqrt{2}}]$ and $h(\frac{1}{2}) = 1$, $h(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}$, and $f(r, h(r)) = 0$ for $r \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]$. Moreover we can see that

$$
f(r, s) \begin{cases} 
\leq 0 & (0 \leq r \leq \frac{1}{2}, \text{ or } \frac{1}{2} \leq r \leq \frac{1}{\sqrt{2}}, s \leq h(r)) \\
\geq 0 & (\frac{1}{2} \leq r \leq \frac{1}{\sqrt{2}}, s \geq h(r), \text{ or } \frac{1}{\sqrt{2}} \leq r \leq 1).
\end{cases}
$$

We have another formulation of (4.2) which is written by the function $\psi_{r,s}^{*}$.

**Theorem 9'.** Suppose that $r \leq s$, then

$$
J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^{*}})) = \begin{cases} 
2\omega (r(r - 2) + \omega + (2r - 1)\omega^2 \leq 0) \\
\frac{2r(r - 2 + \omega)}{(1 - 2\omega)r - 1 + \omega} (r(r - 2) + \omega + (2r - 1)\omega^2 \geq 0),
\end{cases}
$$

(4.2)
where \( \omega = \psi_{r,s}' \left( \frac{1}{2} \right) \). In particular, if \( r = s \), then \( \omega = \frac{1}{2r} \), and

\[
J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})) = \begin{cases} 
2\psi_{r,s}'(1/2) & (\frac{1}{2} \leq r \leq \frac{1}{\sqrt{2}}) \\
\frac{1}{\psi_{r,s}'(1/2)} & (\frac{1}{\sqrt{2}} \leq r).
\end{cases}
\]

As stated in Section 2, \( J(X^*) = J(X) \) does not always hold. We will give a partial result on the relation between \( J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})) \) and \( J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})^*) \). \( (\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})^* \) is given by \( (\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}}) \) where \((\alpha, \beta)\) satisfies (4.1).

**Theorem 10.** Suppose that (4.1) holds, then

1. If \( r = s \) (\( \frac{1}{2} \leq r \leq 1 \)), or \( (r, s) = (\frac{1}{2}, 1) \),
   then \( J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}})) = J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})). \)
2. If \( r \in (0, 1) \setminus \{ \frac{1}{2} \} \), \( s = 1 \), or \( r = \frac{1}{2}, \frac{1}{2} \leq s < 1 \),
   then \( J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}})) \neq J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})). \)

Combining Corollary 2 and Theorem 10, we have

**Corollary 11.** Suppose that \( \psi \in E \cap \widehat{\Psi}_2 \), then \( J((\mathbb{R}^2, \| \cdot \|_{\psi})) = J((\mathbb{R}^2, \| \cdot \|_{\psi})^*). \)

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