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<th>Title</th>
<th>ON THE JAMES CONSTANT OF EXTREME ABSOLUTE NORMS ON $\mathbb{R}^2$ (Banach space theory and related topics)</th>
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</thead>
<tbody>
<tr>
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Kyoto University
ON THE JAMES CONSTANT OF EXTREME ABSOLUTE NORMS ON $\mathbb{R}^2$

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Abstract

The set of all absolute normalized norms on $\mathbb{R}^2$ (denoted by $AN_{2}$) and the set of all convex functions $\psi$ on $[0,1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for $t \in [0,1]$ (denoted by $\Psi_{2}$) have convex structures and they are isomorphic by the one to one correspondence $\psi(0,1) = \Vert(1-t, t)\Vert_{\psi}$ for $t \in [0,1])$. In [5], the set of all extreme points of $AN_{2}$ is determined. In this note, we will report the calculation of the James constants of $(\mathbb{R}^2, \Vert \cdot \Vert_{\psi})$ and its dual space $(\mathbb{R}^2, \Vert \cdot \Vert_{\psi})^{*}$ where $\psi$ is an arbitrary extreme point of $\Psi_{2}$. Moreover, we will consider the relation of the James constants of these spaces.

1. Preliminaries

A norm $\Vert \cdot \Vert$ on $\mathbb{R}^2$ is said to be absolute if $\Vert (x, y) \Vert = \Vert (|x|, |y|) \Vert$ for all $(x, y) \in \mathbb{R}^2$, and normalized if $\Vert (1,0) \Vert = \Vert (0,1) \Vert = 1$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by $AN_{2}$. Let $\Psi_{2}$ be the set of all convex functions $\psi$ on $[0,1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for $t \in [0,1]$. $\Psi_{2}$ and $AN_{2}$ can be identified by a one to one correspondence $\psi \mapsto \Vert \cdot \Vert_{\psi}$ with the relation

\[ (1.1) \quad \psi(t) = \Vert (1-t, t) \Vert_{\psi} \]

for $t \in [0,1]$. For $1 \leq p \leq \infty$, we denote

\[ \psi_p(t) = \begin{cases} (1-t)^{p} + t^{p} & (1 \leq p < \infty) \\ \max\{1-t, t\} & (p = \infty) \end{cases} \]

Then $\psi_p \in \Psi_{2}$ ($1 \leq p \leq \infty$), and they correspond to the $l_p$-norms $\Vert \cdot \Vert_p$ on $\mathbb{R}^2$.

We call a norm $\Vert \cdot \Vert \in AN_{2}$ (resp. $\psi \in \Psi_{2}$) an extreme point of $AN_{2}$ (resp. $\Psi_{2}$) if $\Vert \cdot \Vert = \frac{1}{2}(\Vert \cdot \Vert' + \Vert \cdot \Vert'')$ and $\Vert \cdot \Vert', \Vert \cdot \Vert'' \in AN_{2}$ imply $\Vert \cdot \Vert' = \Vert \cdot \Vert''$ (resp. $\psi = \frac{1}{2}(\psi' + \psi'')$ and $\psi', \psi'' \in \Psi_{2}$ imply $\psi' = \psi''$).

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Let $0 \leq \alpha \leq \frac{1}{2} \leq \beta \leq 1$. For the case $(\alpha, \beta) \neq (\frac{1}{2}, \frac{1}{2})$, we define

$$
\psi_{\alpha,\beta}(t) = \begin{cases} 
1 - t & (t \in [0, \alpha]) \\
\alpha + \beta - t - \frac{1}{2} & (t \in [\alpha, \beta]) \\
\frac{\beta - 2\alpha\beta}{\beta - \alpha} & (t \in [\beta, 1]) 
\end{cases},
$$

$$
E = \{ \psi_{\alpha,\beta} | 0 \leq \alpha \leq \frac{1}{2} < \beta \leq 1 \}.
$$

**Proposition 1** ([5]). The following conditions are equivalent.

1. $\| \cdot \|_{\psi}$ is an extreme point of $AN_2$.
2. $\psi$ is an extreme point of $\Psi_2$.
3. $\psi \in E$.

Let $\hat{\Psi}_2 = \{ \psi \in \Psi_2 | \psi(1 - t) = \psi(t) \ (t \in [0, 1]) \}$. If $\psi \in \Psi_2$, then $\psi \in \hat{\Psi}_2$ if and only if $\|(x_1, x_2)\|_{\psi} = \|\psi(2)\|_\psi$ for $(x_1, x_2) \in \mathbb{R}^2$. $\hat{\Psi}_2$ also has a convex structure, and by an analogy of Proposition 1, we have

**Corollary 2.** Let $\hat{E} = E \cap \hat{\Psi}_2 = \{ \psi_{\alpha,1-\alpha} \in E | 0 \leq \alpha \leq \frac{1}{2} \}$. Then $\psi$ is an extreme point of $\hat{\Psi}_2$ if and only if $\psi \in \hat{E}$.

### 2. Known facts on James constant of $(\mathbb{R}^2, \| \cdot \|_\psi)$

For a Banach space $(X, \| \cdot \|)$, the James constant is defined by

$$
J((X, \| \cdot \|)) = \sup\{\min\{\|x + y\|, \|x - y\|\} | x, y \in X, \|x\| = \|y\| = 1\}.
$$

$\sqrt{2} \leq J((X, \| \cdot \|)) \leq 2$ holds and $J((X, \| \cdot \|)) = \sqrt{2}$ if $X$ is a Hilbert space. (The converse is not true.) For $1 \leq p \leq \infty$, $J(L_p) = \max\{2^\frac{1}{p}, 2^\frac{1}{q}\}$ holds where $\frac{1}{p} + \frac{1}{q} = 1$ and $\dim L_p \geq 2$. It is known that $J(X) < 2$ if and only if $X$ is uniformly nonsquare, that is, there exists $\delta > 0$ such that $\|(x + y)/2\| \leq 1 - \delta$ holds whenever $\|(x - y)/2\| \geq 1 - \delta, \|x\| \leq 1, \|y\| \leq 1$. Moreover, $J(X^{**}) = J(X)$ holds and

$$
2J(X) - 2 \leq J(X^*) \leq \frac{J(X)}{2} + 1.
$$

There are some Banach spaces which do not satisfy $J(X^*) = J(X)$.

For the 2-dimensional spaces with absolute normalized norms, we know the following facts on the James constant.

**Proposition 3** ([8]).

1. If $\psi_2 \leq \psi \in \hat{\Psi}_2$ and $\max_{t \in [0, 1]} \frac{\psi(t)}{\psi_2(t)}$ is taken at $t = \frac{1}{2}$, then

$$
J((\mathbb{R}^2, \| \cdot \|_\psi)) = 2\psi(\frac{1}{2}).
$$
(2) If \( \psi_2 \geq \psi \in \hat{\Psi}_2 \) and \( \max_{t \in [0,1]} \frac{\psi_2(t)}{\psi(t)} \) is taken at \( t = \frac{1}{2} \), then
\[
J((\mathbb{R}^2, \| \cdot \|_\psi)) = \frac{1}{\psi(\frac{1}{2})}.
\]

(3) For \( \beta \in [\frac{1}{2}, 1] \),
\[
J((\mathbb{R}^2, \| \cdot \|_{\psi_{1-\beta, \beta}})) = \begin{cases} \frac{1}{\beta} & (\beta \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]) \\ 2\beta & (\beta \in [\frac{1}{\sqrt{2}}, 1]) \end{cases}.
\]

The results in Proposition 3 are obtained by the following proposition. Also in [9] and [10], the James constants of 2 dimensional Lorentz sequence spaces and their dual spaces were calculated by using the following proposition.

**Proposition 4([8]).** If \( \psi \in \hat{\Psi}_2 \), then
\[
J((\mathbb{R}^2, \| \cdot \|_\psi)) = \max_{0 \leq t \leq \frac{1}{2}} \frac{2 - 2t}{\psi(t)} \psi(\frac{1}{2 - 2t}).
\]

We have only few results on the James constants of \((\mathbb{R}^2, \| \cdot \|_\psi)\) when \( \psi \in \Psi_2 \setminus \hat{\Psi}_2 \). In this note we focus our consideration on the James constants of \((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})\) and its dual space \((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})^*\) where \( \psi_{\alpha, \beta} \in E \). There is a unique \( \psi^* \in \Psi_2 \) such that \((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})^* = (\mathbb{R}^2, \| \cdot \|_{\psi^*})\), and it is obvious that \( \psi_{\alpha, \beta}, \psi^* \notin \hat{\Psi}_2 \) whenever \( \alpha + \beta \neq 1 \).

### 3. James Constants for Extreme Norms in \( \text{AN}_2 \).

In this section we consider \( J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})) \) where \( \| \cdot \|_{\psi_{\alpha, \beta}} \) is the extreme norm of \( \text{AN}_2 \). Since \( J((\mathbb{R}^2, \| \cdot \|_{\psi})) = J((\mathbb{R}^2, \| \cdot \|_{\tilde{\psi}})) \) where \( \tilde{\psi}(t) = \psi(1 - t) \), it is sufficient to calculate James constant in the case that \( \alpha + \beta \leq 1 \).

**Theorem 5([4]).** Suppose that \( \alpha + \beta \leq 1 \), then
\[
J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})) = \begin{cases} \frac{1}{\psi(1/2)} & (\text{if } \psi(\frac{1}{2}) \leq \frac{1}{2(1 - \alpha)}) \\ \frac{1}{2\psi(1/2) + \gamma} & (\text{if } \frac{1}{2(1 - \alpha)} \leq \psi(\frac{1}{2}) \leq \frac{1}{4(1 - \alpha)(1 + \frac{1}{\gamma})}) \\ \frac{1}{2\psi(1/2)} & (\text{if } \frac{1}{4(1 - \alpha)(1 + \frac{1}{\gamma})} \leq \psi(\frac{1}{2})) \end{cases},
\]
where \( \gamma = \frac{2\beta - 1}{\beta - \alpha} \).

**Corollary 6.** If \( \beta \leq 1 - \alpha \leq \frac{1}{\sqrt{2}} \), then
\[
J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})) = \frac{1}{\psi(1/2)}.
\]

We have some other formulations of Theorem 5. Put
\[ \gamma = \gamma(\alpha, \beta) = \begin{cases} \frac{2\beta - 1}{\beta - \alpha} & (\alpha + \beta \leq 1) \\ \frac{1 - 2\alpha}{\beta - \alpha} & (\alpha + \beta \geq 1) \end{cases}, \]

\[ f = f(\gamma) = \frac{1}{4} \{ 1 - \gamma + \sqrt{(1 + \gamma)^2 + 4\gamma} \}, \]

\[ g = g(\gamma) = \frac{1}{4} \{ 1 - \gamma + \sqrt{(1 + \gamma)^2 + 4} \}, \]

\[ M = 1 + \frac{1}{2\psi(1/2) + \gamma}. \]

It can be shown by a simple calculation that \( f \) is increasing with respect to \( \gamma \) while \( g \) is decreasing and that \( \frac{1}{2} \leq f(\gamma) \leq \frac{1}{\sqrt{2}} \leq g(\gamma) \leq \frac{1 + \sqrt{5}}{4} \) \((\gamma \in [0, 1])\).

**Theorem 7** ([4]).

1. If \( \psi(1/2) \leq f(\gamma) \), then \( 2\psi(1/2) \leq M \leq \frac{1}{\psi(1/2)} \), and \( J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}})) = \frac{1}{\psi(1/2)} \).

2. If \( f(\gamma) \leq \psi(1/2) \leq g(\gamma) \), then \( 2\psi(1/2), \frac{1}{\psi(1/2)} \leq M \), and \( J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}})) = M \).

3. If \( g(\gamma) \leq \psi(1/2) \), then \( \frac{1}{\psi(1/2)} \leq M \leq 2\psi(1/2) \), and \( J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}})) = 2\psi(1/2) \).

**Theorem 7'.** For \( \psi_{\alpha,\beta} \), put \( \gamma = \gamma(\alpha, \beta) = \begin{cases} \frac{2\beta - 1}{\beta - \alpha} & (\alpha + \beta \leq 1) \\ \frac{1 - 2\alpha}{\beta - \alpha} & (\alpha + \beta \geq 1) \end{cases} \), then

\[ J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}})) = \max\{ \frac{1}{\psi(1/2)}, 1 + \frac{1}{2\psi(1/2) + \gamma}, 2\psi(\frac{1}{2}) \}. \]

It is known that \( J((\mathbb{R}^2, \| \cdot \|_{\psi})) = \sqrt{2} \) holds for \( \psi \in [\psi_2, \psi_{1-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}}] = \{(1-\lambda)\psi_2 + \lambda\psi_{1-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}} \mid \lambda \in [0, 1]\} \). By Theorem 7 or Theorem 7' we can prove that

**Corollary 8.** \( \| \cdot \|_{\psi_{1-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}}} \) is the only extreme point of \( AN_2 \) whose James constant is \( \sqrt{2} \), that is,

\[ \{ \psi_{\alpha,\beta} \in E \mid J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}})) = \sqrt{2} \} = \{ \psi_{1-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}} \}. \]

4. **James constants for the dual norms.**

Let \( C_{r,s} \) be the convex hull of the set consisting of eight points \((\pm 1, 0), (0, \pm 1), \) and \((\pm r, \pm s)\) with \( r, s \in [0, 1], r + s \geq 1 \). \( C_{r,s} \) is an octagon whenever \( 1 < r + s, r < 1, \) and \( s < 1 \). In the exceptional cases, it is a hexagon or a square. Let \( \psi_{r,s}^* \in \Psi_2 \) be
such that the unit sphere of the norm $\| \cdot \|_{\psi_{r,s}^*}$ is $C_{r,s}$. Then $\psi_{r,s}^*$ and $\| \cdot \|_{\psi_{r,s}^*}$ are given by:

$$
\psi_{r,s}^*(t) = \begin{cases} 
1 - \frac{r + s - 1}{s} t & (t \in [0, \frac{s}{r + s}]) \\
1 - \frac{s}{r} + \frac{s - 1}{r} t & (t \in [\frac{s}{r + s}, 1])
\end{cases}
$$

$$
\| (x_1, x_2) \|_{\psi_{r,s}^*} = \begin{cases} 
x_1 - \frac{r - 1}{s} x_2 & (0 \leq r x_2 \leq s x_1) \\
1 - \frac{s}{r} x_1 + x_2 & (0 \leq s x_1 \leq r x_2)
\end{cases}
$$

It is easy to find that $\| \cdot \|_{\psi_{r,s}^*}$ is the dual norm of $\| \cdot \|_{\psi_{r,s}}$ if and only if

$$
\begin{align*}
\alpha &= \frac{1 - r}{1 - \frac{r}{s} + \frac{1}{s}} \\
\beta &= \frac{1}{1 + r - s}
\end{align*}
$$

It is easy to see that for each $\psi \in \Psi_2$

$$
J((\mathbb{R}^2, \| \cdot \|_{\tilde{\psi}})) = J((\mathbb{R}^2, \| \cdot \|_{\psi}))
$$

where $\tilde{\psi}$ is defined by $\tilde{\psi}(t) = \psi(1 - t)$ ($t \in [0, 1]$). Since $\tilde{\psi}_{r,s}^{*} = \psi_{s,r}^{*}$ holds, it follows that $J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})) = J((\mathbb{R}^2, \| \cdot \|_{\psi_{l,r}^*}))$ for all $r, s \in [0, 1]$ with $r + s \geq 1$. Hence it is sufficient to consider the case that $r \leq s$.

**Theorem 9.** Suppose that $r \leq s$, then

$$
J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})) = \begin{cases} 
1 + \frac{1 - r}{2r(2r^2 - 3r - s + 1)} & (f(r, s) \leq 0) \\
2r(2r^2 - 3r - s + 1) & (f(r, s) \geq 0)
\end{cases}
$$

where $f(r, s) = -4r^2s^2 - 2r^3 + 4r^2s + 6rs^2 + 5r^2 - 4rs - s^2 - 4r + 1$.

By a simple calculation we find that there is an implicit function $s = h(r)$ of $f$, such that $h$ is decreasing on $[\frac{1}{2}, \frac{1}{\sqrt{2}}]$ and $h(\frac{1}{2}) = 1$, $h(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}$, and $f(r, h(r)) = 0$ for $r \in \left[\frac{1}{2}, \frac{1}{\sqrt{2}}\right]$. Moreover we can see that

$$
f(r, s) \begin{cases} 
\leq 0 & (0 \leq r \leq \frac{1}{2}, \text{ or } \frac{1}{2} \leq r \leq \frac{1}{\sqrt{2}}, s \leq h(r)) \\
\geq 0 & (\frac{1}{2} \leq r \leq \frac{1}{\sqrt{2}}, s \geq h(r), \text{ or } \frac{1}{\sqrt{2}} \leq r \leq 1)
\end{cases}
$$

We have another formulation of (4.2) which is written by the function $\psi_{r,s}^*$.

**Theorem 9'.** Suppose that $r \leq s$, then

$$
J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})) = \begin{cases} 
2\omega & (r(r - 2) + \omega + (2r - 1)\omega^2 \leq 0) \\
2r(r - 2 + \omega)(1 - 2\omega)r - 1 + \omega & (r(r - 2) + \omega + (2r - 1)\omega^2 \geq 0)
\end{cases}
$$
where \( \omega = \psi^{*}_{r,s}(\frac{1}{2}) \). In particular, if \( r = s \), then \( \omega = \frac{1}{2r} \), and

\[
J((\mathbb{R}^2, \| \cdot \|_{\psi^{*}_{r,s}})) = \begin{cases} 
2\psi^{*}_{r,s}(1/2) & (\frac{1}{2} \leq r \leq \frac{1}{\sqrt{2}}) \\
\frac{1}{\psi^{*}_{r,s}(1/2)} & (\frac{1}{\sqrt{2}} \leq r).
\end{cases}
\]

As stated in Section 2, \( J(X^*) = J(X) \) does not always hold. We will give a partial result on the relation between \( J((\mathbb{R}^2, \| \cdot \|_{\psi^{*}_{r,s}})) \) and \( J((\mathbb{R}^2, \| \cdot \|_{\psi^{*}_{r,s}})^*) \). \( (\mathbb{R}^2, \| \cdot \|_{\psi^{*}_{r,s}})^* \) is given by \((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}})\) where \((\alpha, \beta)\) satisfies (4.1).

**Theorem 10.** Suppose that (4.1) holds, then

1. If \( r = s \) \((\frac{1}{2} \leq r \leq 1)\), or \((r, s) = (\frac{1}{2}, 1)\),
   \[
   J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}})) = J((\mathbb{R}^2, \| \cdot \|_{\psi^{*}_{r,s}})).
   \]
2. If \( r \in (0, 1) \setminus \{\frac{1}{2}\} \), \( s = 1 \), or \( r = \frac{1}{2}, \frac{1}{2} \leq s < 1 \),
   \[
   J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}})) \neq J((\mathbb{R}^2, \| \cdot \|_{\psi^{*}_{r,s}})).
   \]

Combining Corollary 2 and Theorem 10, we have

**Corollary 11.** Suppose that \( \psi \in E \cap \widehat{\psi}_2 \), then \( J((\mathbb{R}^2, \| \cdot \|_{\psi})) = J((\mathbb{R}^2, \| \cdot \|_{\psi})^*) \).

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