ON THE JAMES CONSTANT OF EXTREME ABSOLUTE NORMS ON $\mathbb{R}^{2}$

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Abstract

The set of all absolute normalized norms on $\mathbb{R}^{2}$ (denoted by $AN_{2}$) and the set of all convex functions $\psi$ on $[0,1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for $t \in [0,1]$ (denoted by $\Psi_{2}$) have convex structures and they are isomorphic by the one to one correspondence $\psi(t) = \|(1-t, t)\|_{\psi}$ ($t \in [0,1]$). In [5], the set of all extreme points of $AN_{2}$ is determined. In this note, we will report the calculation of the James constants of $(\mathbb{R}^{2}, \| \cdot \|_{\psi})$ and its dual space $(\mathbb{R}^{2}, \| \cdot \|_{\psi})^{*}$ where $\psi$ is an arbitrary extreme point of $\Psi_{2}$. Moreover, we will consider the relation of the James constants of these spaces.

1. Preliminaries

A norm $\| \cdot \|$ on $\mathbb{R}^{2}$ is said to be absolute if $\|(x, y)\| = \|(|x|, |y|)\|$ for all $(x, y) \in \mathbb{R}^{2}$, and normalized if $\|(1, 0)\| = \|(0, 1)\| = 1$. The set of all absolute normalized norms on $\mathbb{R}^{2}$ is denoted by $AN_{2}$. Let $\Psi_{2}$ be the set of all convex functions $\psi$ on $[0,1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for $t \in [0,1]$. $\Psi_{2}$ and $AN_{2}$ can be identified by a one to one correspondence $\psi \rightarrow \| \cdot \|_{\psi}$ with the relation

$$\psi(t) = \|(1-t, t)\|_{\psi} \tag{1.1}$$

for $t \in [0,1]$. For $1 \leq p \leq \infty$, we denote

$$\psi_{p}(t) = \begin{cases} (1-t)^{p} + t^{p} \frac{1}{p} & (1 \leq p < \infty) \\ \max\{1-t, t\} & (p = \infty) \end{cases}$$

Then $\psi_{p} \in \Psi_{2}$ ($1 \leq p \leq \infty$), and they correspond to the $l_{p}$-norms $\| \cdot \|_{p}$ on $\mathbb{R}^{2}$.

We call a norm $\| \cdot \| \in AN_{2}$ (resp. $\psi \in \Psi_{2}$) an extreme point of $AN_{2}$ (resp. $\Psi_{2}$) if $\| \cdot \| = \frac{1}{2}(\| \cdot ' \| + \| \cdot '' \|)$ and $\| \cdot ' \|, \| \cdot '' \| \in AN_{2}$ imply $\| \cdot ' \| = \| \cdot '' \|$ (resp. $\psi = \frac{1}{2}(\psi' + \psi'')$ and $\psi', \psi'' \in \Psi_{2}$ imply $\psi' = \psi''$).

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Let $0 \leq \alpha \leq \frac{1}{2} \leq \beta \leq 1$. For the case $(\alpha, \beta) \neq (\frac{1}{2}, \frac{1}{2})$, we define

$$\psi_{\alpha, \beta}(t) = \begin{cases} 
1 - t & (t \in [0, \alpha]) \\
\alpha + \beta - 1 - t & \beta - \alpha t + \beta - 2\alpha\beta \\
\beta - \alpha & (t \in [\alpha, \beta]) \\
t & (t \in [\beta, 1])
\end{cases}$$

$$E = \{\psi_{\alpha, \beta} \mid 0 \leq \alpha \leq \frac{1}{2} < \beta \leq 1\}.$$

**Proposition 1** ([5]). The following conditions are equivalent.

1. $\|\cdot\|_{\psi}$ is an extreme point of $AN_2$.
2. $\psi$ is an extreme point of $\Psi_2$.
3. $\psi \in E$.

Let $\hat{\Psi}_2 = \{\psi \in \Psi_2 \mid \psi(1-t) = \psi(t) \ (t \in [0,1])\}$. If $\psi \in \Psi_2$, then $\psi \in \hat{\Psi}_2$ if and only if $\|(x_1, x_2)\|_{\psi} = \|(x_2, x_1)\|_{\psi}$ for $(x_1, x_2) \in \mathbb{R}^2$. $\hat{\Psi}_2$ also has a convex structure, and by an analogy of Proposition 1, we have

**Corollary 2.** Let $\hat{E} = E \cap \hat{\Psi}_2 = \{\psi_{\alpha,1-\alpha} \in E \mid 0 \leq \alpha \leq \frac{1}{2}\}$. Then $\psi$ is an extreme point of $\hat{\Psi}_2$ if and only if $\psi \in \hat{E}$.

## 2. Known Facts on James Constant of $(\mathbb{R}^2, \|\cdot\|_{\psi})$

For a Banach space $(X, \|\cdot\|)$, the James constant is defined by

$$J((X, \|\cdot\|)) = \sup\{\min\{\|x+y\|, \|x-y\|\} \mid x, y \in X, \|x\| = \|y\| = 1\}.$$ 

$\sqrt{2} \leq J((X, \|\cdot\|)) \leq 2$ holds and $J((X, \|\cdot\|)) = \sqrt{2}$ if $X$ is a Hilbert space. (The converse is not true.) For $1 \leq p \leq \infty$, $J(L_p) = \max\{2^\frac{1}{p}, 2^\frac{1}{q}\}$ holds where $\frac{1}{p} + \frac{1}{q} = 1$ and $\dim L_p \geq 2$. It is known that $J(X) < 2$ if and only if $X$ is uniformly non-square, that is, there exists $\delta > 0$ such that $\|x+y\|/2 \leq 1 - \delta$ holds whenever $\|(x-y)/2\| \geq 1 - \delta$, $\|x\| \leq 1$, $\|y\| \leq 1$. Moreover, $J(X^{**}) = J(X)$ holds and

$$(2.1) \quad 2J(X) - 2 \leq J(X^*) \leq \frac{J(X)}{2} + 1.$$ 

There are some Banach spaces which do not satisfy $J(X^*) = J(X)$.

For the 2-dimensional spaces with absolute normalized norms, we know the following facts on the James constant.

**Proposition 3** ([8]).

1. If $\psi_2 \leq \psi \in \hat{\Psi}_2$ and $\max_{t \in [0,1]} \frac{\psi(t)}{\psi_2(t)}$ is taken at $t = \frac{1}{2}$, then

$$J((\mathbb{R}^2, \|\cdot\|_{\psi})) = 2\psi(\frac{1}{2}).$$
(2) If \( \psi_2 \geq \psi \in \hat{\Psi}_2 \) and \( \max_{t \in [0,1]} \frac{\psi_2(t)}{\psi(t)} \) is taken at \( t = \frac{1}{2} \), then

\[
J((\mathbb{R}^2, \| \cdot \|_\psi)) = \frac{1}{\psi(\frac{1}{2})}.
\]

(3) For \( \beta \in [\frac{1}{2}, 1] \),

\[
J((\mathbb{R}^2, \| \cdot \|_{\psi_{1-\beta, \beta}})) = \begin{cases} \frac{1}{\beta} & (\beta \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]) \\ 2\beta & (\beta \in [\frac{1}{\sqrt{2}}, 1]) \end{cases}.
\]

The results in Proposition 3 are obtained by the following proposition. Also in [9] and [10], the James constants of 2 dimensional Lorentz sequence spaces and their dual spaces were calculated by using the following proposition.

**Proposition 4([8]).** If \( \psi \in \hat{\Psi}_2 \), then

\[
J((\mathbb{R}^2, \| \cdot \|_\psi)) = \max_{0 \leq t \leq \frac{1}{2}} \frac{2 - 2t}{\psi(t)} \psi(\frac{1}{2 - 2t}).
\]

We have only few results on the James constants of \((\mathbb{R}^2, \| \cdot \|_\psi)\) when \( \psi \in \Psi_2 \setminus \hat{\Psi}_2 \). In this note we focus our consideration on the James constants of \((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})\) and its dual space \((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})^*\) where \( \psi_{\alpha, \beta} \in E \). There is a unique \( \psi^* \in \Psi_2 \) such that \((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})^* = (\mathbb{R}^2, \| \cdot \|_{\psi^*})\), and it is obvious that \( \psi_{\alpha, \beta}, \ \psi^* \notin \hat{\Psi}_2 \) whenever \( \alpha + \beta \neq 1 \).

### 3. James Constants for Extreme Norms in \( AN_2 \).

In this section we consider \( J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})) \) where \( \| \cdot \|_{\psi_{\alpha, \beta}} \) is the extreme norm of \( AN_2 \). Since \( J((\mathbb{R}^2, \| \cdot \|_{\bar{\psi}})) = J((\mathbb{R}^2, \| \cdot \|_{\psi})) \) where \( \bar{\psi}(t) = \psi(1 - t) \), it is sufficient to calculate James constant in the case that \( \alpha + \beta \leq 1 \).

**Theorem 5([4]).** Suppose that \( \alpha + \beta \leq 1 \), then

\[
J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})) = \begin{cases}
\frac{1}{\psi(1/2)} & (\text{if } \psi(\frac{1}{2}) \leq \frac{1}{2(1 - \alpha)}) \\
1 + \frac{1}{2\psi(1/2) + \gamma} & (\frac{1}{2(1 - \alpha)} \leq \psi(\frac{1}{2}) \leq \frac{1}{4(1 - \alpha)}(1 + \frac{1}{\gamma})) \\
2\psi(1/2) & (\frac{1}{4(1 - \alpha)}(1 + \frac{1}{\gamma}) \leq \psi(\frac{1}{2}))
\end{cases}
\]

where \( \gamma = \frac{2\beta - 1}{\beta - \alpha} \).

**Corollary 6.** If \( \beta \leq 1 - \alpha \leq \frac{1}{\sqrt{2}} \), then

\[
J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha, \beta}})) = \frac{1}{\psi(1/2)}.
\]

We have some other formulations of Theorem 5. Put
\[
\gamma = \gamma(\alpha, \beta) = \begin{cases} 
\frac{2\beta - 1}{\beta - \alpha} & (\alpha + \beta \leq 1) \\
\frac{1 - 2\alpha}{\beta - \alpha} & (\alpha + \beta \geq 1)
\end{cases}
\]

\[
f = f(\gamma) = \frac{1}{4}(1 - \gamma + \sqrt{(1 + \gamma)^2 + 4\gamma}),
\]
\[
g = g(\gamma) = \frac{1}{4}(1 - \gamma + \sqrt{(1 + \gamma)^2 + 4}),
\]
\[
M = 1 + \frac{1}{2\psi(1/2) + \gamma}.
\]

It can be shown by a simple calculation that \(f\) is increasing with respect to \(\gamma\) while \(g\) is decreasing and that
\[
\frac{1}{2} \leq f(\gamma) \leq \frac{1}{\sqrt{2}} \leq g(\gamma) \leq \frac{1 + \sqrt{5}}{4} \quad (\gamma \in [0, 1]).
\]

**Theorem 7 ([4]).**

1. If \(\psi(1/2) \leq f(\gamma)\), then
   \[
   2\psi(1/2) \leq M \leq \frac{1}{\psi(1/2)}, \quad \text{and} \quad J(\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}}) = \frac{1}{\psi(1/2)}.
   \]
2. If \(f(\gamma) \leq \psi(1/2) \leq g(\gamma)\), then
   \[
   2\psi(1/2), \frac{1}{\psi(1/2)} \leq M, \quad \text{and} \quad J(\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}}) = M.
   \]
3. If \(g(\gamma) \leq \psi(1/2)\), then
   \[
   \frac{1}{\psi(1/2)} \leq M \leq 2\psi(1/2), \quad \text{and} \quad J(\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}}) = 2\psi(1/2).
   \]

**Theorem 7'.** For \(\psi_{\alpha,\beta}\), put
\[
\gamma = \gamma(\alpha, \beta) = \begin{cases} 
\frac{2\beta - 1}{\beta - \alpha} & (\alpha + \beta \leq 1) \\
\frac{1 - 2\alpha}{\beta - \alpha} & (\alpha + \beta \geq 1)
\end{cases}, \quad \text{then}
\]
\[
J(\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}}) = \max\{\frac{1}{\psi(1/2)}, 1 + \frac{1}{2\psi(1/2) + \gamma}, 2\psi(\frac{1}{2})\}.
\]

It is known that \(J(\mathbb{R}^2, \| \cdot \|_{\psi}) = \sqrt{2}\) holds for \(\psi \in [\psi_2, \psi_{1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}}] = \{(1 - \lambda)\psi_2 + \lambda\psi_{1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}} | \lambda \in [0, 1]\} \). By Theorem 7 or Theorem 7' we can prove that

**Corollary 8.** \(\| \cdot \|_{\psi_{1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}}}\) is the only extreme point of \(\text{AN}_2\) whose James constant is \(\sqrt{2}\), that is,
\[
\{\psi_{\alpha,\beta} \in E | J(\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}}) = \sqrt{2}\} = \{\psi_{1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}}\}.
\]

4. JAMES CONSTANTS FOR THE DUAL NORMS.

Let \(C_{r,s}\) be the convex hull of the set consisting of eight points \((\pm 1, 0), (0, \pm 1), (\pm r, \pm s)\) with \(r, s \in [0, 1], r + s \geq 1\). \(C_{r,s}\) is an octagon whenever \(1 < r + s, r < 1, s < 1\). In the exceptional cases, it is a hexagon or a square. Let \(\psi_{r,s} \in \Psi_2\) be
such that the unit sphere of the norm $\| \cdot \|_{\psi_{r,s}^*}$ is $C_{r,s}$. Then $\psi_{r,s}^*$ and $\| \cdot \|_{\psi_{r,s}}$ are given by:

$$\psi_{r,s}^*(t) = \begin{cases} 1 - \frac{r + s - 1}{s} t & (t \in [0, \frac{s}{r + s}]) \\ 1 - s \frac{r + s - 1}{s} t & (t \in [\frac{s}{r + s}, 1]), \end{cases}$$

$$\|(x_1, x_2)\|_{\psi_{r,s}^*} = \begin{cases} x_1 - \frac{r - 1}{r} x_2 & (0 \leq rx_2 \leq sx_1) \\ 1 - s \frac{r - 1}{s} x_1 + x_2 & (0 \leq sx_1 \leq rx_2). \end{cases}$$

It is easy to find that $\| \cdot \|_{\psi_{r,s}^*}$ is the dual norm of $\| \cdot \|_{\psi_{r,s}}$ if and only if

(4.1) \[
\begin{cases} 
\alpha = \frac{1 - r}{s + 1 - r} \\
\beta = \frac{1 - r}{s + 1 - r}.
\end{cases}
\]

It is easy to see that for each $\psi \in \Psi_2$

$$J((\mathbb{R}^2, \| \cdot \|_{\psi})) = J((\mathbb{R}^2, \| \cdot \|_{\tilde{\psi}}))$$
where $\tilde{\psi}$ is defined by $\tilde{\psi}(t) = \psi(1 - t) \ (t \in [0, 1])$. Since $\tilde{\psi}_{r,s}^* = \psi_{s,r}^*$ holds, it follows that $J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})) = J((\mathbb{R}^2, \| \cdot \|_{\psi_{s,r}^*}))$ for all $r, s \in [0, 1]$ with $r + s \geq 1$. Hence it is sufficient to consider the case that $r \leq s$.

**Theorem 9.** Suppose that $r \leq s$, then

(4.2) \[
J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})) = \begin{cases} 
1 + \frac{1 - r}{2r(2rs - 3s - r + 1)} & (f(r, s) \leq 0) \\
\frac{2r(2rs - 3s - r + 1)}{2r^2 - 3r - s + 1} & (f(r, s) \geq 0),
\end{cases}
\]
where $f(r, s) = -4r^2s^2 - 2r^3 + 4r^2s + 6rs^2 + 5r^2 - 4rs - s^2 - 4r + 1$.

By a simple calculation we find that there is an implicit function $s = h(r)$ of $f$, such that $h$ is decreasing on $[\frac{1}{2}, \frac{1}{\sqrt{2}}]$ and $h(\frac{1}{2}) = 1$, $h(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}$, and $f(r, h(r)) = 0$ for $r \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]$. Moreover we can see that

$$f(r, s) \begin{cases} 
\leq 0 & (0 \leq r \leq \frac{1}{2}, \text{ or } \frac{1}{2} \leq r \leq \frac{1}{\sqrt{2}}, s \leq h(r)) \\
\geq 0 & (\frac{1}{2} \leq r \leq \frac{1}{\sqrt{2}}, s \geq h(r), \text{ or } \frac{1}{\sqrt{2}} \leq r \leq 1).
\end{cases}$$

We have another formulation of (4.2) which is written by the function $\psi_{r,s}^*$.

**Theorem 9'.** Suppose that $r \leq s$, then

$$J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})) = \begin{cases} 
2\omega & (r(r - 2) + \omega + (2r - 1)\omega^2 \leq 0) \\
\frac{2r(r - 2 + \omega)}{(1 - 2\omega)r - 1 + \omega} & (r(r - 2) + \omega + (2r - 1)\omega^2 \geq 0),
\end{cases}$$
where $\omega = \psi_{s,\alpha}^*(1/2)$. In particular, if $r = s$, then $\omega = \frac{1}{2r}$, and

$$J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})) = \begin{cases} \frac{2\psi_{r,s}^*(1/2)}{\psi_{r,s}^*(1/2)} & (\frac{1}{2} \leq r \leq \frac{1}{\sqrt{2}}) \\ \frac{1}{\psi_{r,s}^*(1/2)} & (\frac{1}{\sqrt{2}} \leq r) \end{cases}$$

As stated in Section 2, $J(X^*) = J(X)$ does not always hold. We will give a partial result on the relation between $J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*}))$ and $J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})^*)$. $(\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}^*})^*$ is given by $(\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}})$ where $(\alpha, \beta)$ satisfies (4.1).

**Theorem 10.** Suppose that (4.1) holds, then

1. If $r = s$ ($\frac{1}{2} \leq r \leq 1$), or $(r, s) = (\frac{1}{2}, 1)$,
then $J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}})) = J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}}))$.

2. If $r \in (0, 1) \setminus \{\frac{1}{2}\}$, $s = 1$, or $r = \frac{1}{2}, \frac{1}{2} \leq s < 1$,
then $J((\mathbb{R}^2, \| \cdot \|_{\psi_{\alpha,\beta}})) \neq J((\mathbb{R}^2, \| \cdot \|_{\psi_{r,s}}))$.

Combining Corollary 2 and Theorem 10, we have

**Corollary 11.** Suppose that $\psi \in E \cap \hat{\Psi}_2$, then $J((\mathbb{R}^2, \| \cdot \|_{\psi})) = J((\mathbb{R}^2, \| \cdot \|_{\psi})^*)$.

**References**

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