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A topology of vector lattices

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Abstract

In [3] we define and study the Denjoy and Henstock-Kurzweil integrals in a vector lattice. Moreover in [5, 6, 7] we show some fixed point theorems in a vector lattice. In those theories we need a topology in a vector lattice and some assumptions. The purpose of this paper is to show criteria for the assumptions.

1 Introduction

There are many fixed point theorems in a topological vector space, for instance, Kirk's fixed point theorem in a Banach space, and so on; see for example [11].

We consider a derivative and fixed point theorems in a vector lattice. As known well every topological vector space has a linear topology. On the other hand, although every vector lattice does not have a topology, it has two lattice operators, which are the supremum $\vee$ and the infimum $\wedge$, and also an order is introduced from these operators; see also [9, 12] about vector lattices. There are some methods how to introduce a topology to a vector lattice. One method is to assume that the vector lattice has a linear topology [1]. On the other hand, there is another method to make up a topology in a vector lattice, for instance, in [3] one method is introduced in case of the vector lattice with unit.

In [3] we define and study the Denjoy and Henstock-Kurzweil integrals in a vector lattice. Moreover in [5, 6, 7] we show some fixed point theorems in a vector lattice. In those theories we need a topology in a vector lattice and some assumptions. The purpose of this paper is to show criteria for the assumptions.

2 Topology in a vector lattice

First we introduce a topology in a vector lattice introduced by [3]; see also [6, 7].

Let $X$ be a vector lattice. $e \in X$ is said to be an unit if $e \wedge x > 0$ for any $x \in X$ with $x > 0$. Let $\mathcal{K}_X$ be the class of units of $X$. In the case where $X$ is the set of real numbers $\mathbb{R}$, $\mathcal{K}_\mathbb{R}$ is the set of positive real numbers. Let $X$ be a vector lattice with unit and let $Y$ be a subset of $X$. $Y$ is said to be open if for any $x \in Y$ and for any $e \in \mathcal{K}_X$ there exists
\( \epsilon \in \mathcal{K}_R \) such that \([x - \epsilon e, x + \epsilon e] \subset Y\). Let \( \mathcal{O}_X \) be the class of open subsets of \( X \). For \( e \in \mathcal{K}_X \) and for an interval \([a, b]\) we consider the following subset

\[
[a, b]^e = \{x | \text{ there exists some } \epsilon \in \mathcal{K}_R \text{ such that } x - a \geq \epsilon e \text{ and } b - x \geq \epsilon e\}.
\]

By the definition of \([a, b]^e\) it is easy to see that \([a, b]^e \subset [a, b]\). Every mapping from \( X \times \mathcal{K}_X \) into \((0, \infty)\) is said to be a gauge. Let \( \Delta_X \) be the class of gauges in \( X \). For \( x \in X \) and \( \delta \in \Delta_X \), \( O(x, \delta) \) is defined by

\[
O(x, \delta) = \bigcup_{e \in \mathcal{K}_X} [x - \delta(x, e)e, x + \delta(x, e)e]^e.
\]

\( O(x, \delta) \) is said to be a \( \delta \)-neighborhood of \( x \). Suppose that for any \( x \in X \) and for any \( \delta \in \Delta_X \) there exists \( U \in \mathcal{O}_X \) such that \( x \in U \subset O(x, \delta) \).

Let \( X \) be a vector lattice with unit and \( Y \) a vector lattice. Let \( \mathcal{U}_Y^e(\mathcal{K}_X, \geq) \) be the class of \( \{v_e | e \in \mathcal{K}_X\} \) satisfying the following conditions:

(U1) \( v_e \in Y \) with \( v_e > 0 \);

(U2)\(^d \) \( v_{e_1} \geq v_{e_2} \) if \( e_1 \geq e_2 \);

(U3)\(^s \) For any \( e \in \mathcal{K}_X \) there exists \( \theta(e) \in \mathcal{K}_R \) such that \( v_{\theta(e)e} \leq \frac{1}{2}v_e \).

Let \( x_0 \in Z \subset X \) and \( f : Z \rightarrow Y \). \( f \) is said to be continuous at \( x_0 \) if there exists \( \{v_e\} \in \mathcal{U}_Y^e(\mathcal{K}_X, \geq) \) such that for any \( e \in \mathcal{K}_X \) there exists \( \delta \in \mathcal{K}_R \) such that for any \( x \in Z \) if \(|x - x_0| \leq \delta e\), then \(|f(x) - f(x_0)| \leq v_e\).

Let \( X \) and \( Y \) be vector lattices with unit, \( Z \subset X \) and \( f : Z \rightarrow Y \). Suppose that there exists \( P \subset Y \) satisfying the following conditions:

(P1) \( P \) is open and convex;

(P2) If \( x \in P \) and \( x \leq y \), then \( y \in P \);

(P3) \( 0 \notin P \);

(P4) \( \{x | x > 0]\} \subset P \).

Let \( \mathcal{P}_Y \) be the class of the above \( P \)'s. \( f \) is said to be upper semi-continuous with respect to \( P \in \mathcal{P}_Y \) if \( \{x | y - f(x) \in P\} \in \mathcal{O}_X \cap Z \) for any \( y \in Y \). \( f \) is said to be lower semi-continuous with respect to \( P \in \mathcal{P}_Y \) if \( \{x | f(x) - y \in P\} \in \mathcal{O}_X \cap Z \) for any \( y \in Y \). \( f \) is said to be semi-continuous with respect to \( P \in \mathcal{P}_Y \) if it is upper and lower semi-continuous with respect to \( P \in \mathcal{P}_Y \).

A vector lattice is said to be Archimedean if it holds that \( x = 0 \) whenever there exists \( y \in X \) with \( y \geq 0 \) such that \( 0 \leq r x \leq y \) for any \( r \in \mathcal{K}_R \).

Let \( X \) be an Archimedean vector lattice. Then there exists a positive homomorphism \( f \) from \( X \) into \( \mathbb{R} \), that is, \( f \) satisfies the following conditions:
(H1) $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for any $x, y \in X$ and for any $\alpha, \beta \in \mathbb{R}$;

(H2) $f(x) \geq 0$ for any $x \in X$ with $x \geq 0$;

see [7]* Example 3.1. Suppose that there exists a homomorphism $f$ from $X$ into $\mathbb{R}$ satisfying the following condition instead of (H2):

(H2)$^*$ $f(x) > 0$ for any $x \in X$ with $x > 0$.

The following hold under the topology above; see [6, 7].

Lemma 2.1. Let $X$ be an Archimedean vector lattice with unit and \{x\textsubscript{1}, ..., x\textsubscript{n}\} a subset of $X$. Then co\{x\textsubscript{1}, ..., x\textsubscript{n}\} is homeomorphic to a compact and convex subset of $\mathbb{R}^n$.

Lemma 2.2. Let $X$ be an Archimedean vector lattice with unit, $Y$ a vector lattice with unit, $Z \subset X$ and $f$ a mapping from $Z$ into $Y$. Suppose that there exists a homomorphism from $X$ into $\mathbb{R}$ satisfying condition (H2)$^*$ and that $P_Y \neq \emptyset$.

Then $f$ is semi-continuous with respect to any $P \in P_Y$ if it is continuous at any $x \in Z$.

Lemma 2.3. Let $X$ be an Archimedean vector lattice with unit, $Y$ a vector lattice with unit, $x_0 \in Z \subset X$ and $f$ a mapping from $Z$ into $Y$. Suppose that there exists a homomorphism from $X$ into $\mathbb{R}$ satisfying condition (H2)$^*$.

Then $f$ is continuous at $x_0$ in the sense of topology if it is continuous at $x_0$.

3 Criteria for the condition (H2)$^*$.

Theorem 3.1. Let $X$ be an Archimedean vector lattices with unit.

Then the following are equivalent:

(1) $X$ satisfies the condition (H2)$^*$;

(2) $P_X \neq \emptyset$;

(3) There exists $O \in O_X$ such that $O \neq \emptyset$ and \{x $|$ x > 0\} $\subset$ co($O$) $\neq$ X.

Proof. (1) $\Rightarrow$ (2): Let $0 < \beta < 1$ and $\delta(x, e) = \frac{\beta f(x)}{f(e)}$ for any $x \in X$ with $x > 0$ and for any $e \in K_X$. Put $P = \bigcup_{x \in X}$ with $x > 0$ int\($O(x, \delta)$). Then $P$ is open and \{x $|$ x > 0\} $\subset$ P. Note that by condition (H2)$^*$ for any $x_1, x_2 \in X$ with $x_1, x_2 > 0$ and $x_1 \neq x_2$, $\frac{x_1}{f(x_1)}$ and $\frac{x_2}{f(x_2)}$ are incomparable mutually. Therefore $x - \delta(x, e)e \not\leq 0$ for any $x \in X$ with $x > 0$ and for any $e \in K_X$. Assume that $0 \in P$. Then there exist $x \in X$ with $x > 0$ and $e \in K_X$ such that $0 \in [x - \delta(x, e)e, x + \delta(x, e)e]$\textsuperscript{s}. It is a contradiction. Therefore $0 \not\in P$. Note that $x \in$ int\($A$) if and only if there exists $\delta_x \in \Delta_X$ such that $O(x, \delta_x) \subset A$. Let $x \in P$ and $x \leq y$. Then there exist $z \in X$ with $z > 0$ and $\delta_z \in \Delta_X$ such that $O(x, \delta_z) \subset O(z, \delta)$. Let $\delta_y(u, e) = \delta_x(u - y + x, e)$. 

\[ \delta_y(u, e) = \delta_x(u - y + x, e). \]
Since $\delta(x_2, e) \leq \delta(x_1 + x_2, e)$ for any $x_1, x_2 \in X$ with $x_1, x_2 > 0$, it holds that $x_1 + O(x_2, \delta) \subset O(x_1 + x_2, \delta)$. Therefore

\[
O(y, \delta_y) = y - x + O(x, \delta_x) \subset y - x + O(z, \delta) \subset O(z + y - x, \delta),
\]

that is, $y \in \text{int}(O(z + y - x, \delta)) \subset P$. Let $x_0, x_1 \in P$ and $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq 1$. Then for $i = 0, 1$ there exist $y_i \in X$ with $y_i > 0$ and $\delta_i \in \Delta_X$ such that $O(x_i, \delta_i) \subset O(y_i, \delta)$. Let $\delta_a(z, e) = (1 - \alpha)\delta_0(x_0, e) + \alpha \delta_1(x_1, e)$. Take $z \in O((1 - \alpha)x_0 + \alpha x_1, \delta_{\alpha})$ arbitrary. Then there exists $e \in K_X$ such that

\[
\begin{align*}
\delta(z_0, e_0) + \delta(z_1, e_1) &\leq \delta(z_0 + z_1, (\frac{f(z_0)}{f(e_0)}e_0 + \frac{f(z_1)}{f(e_1)}e_1)(\frac{f(z_0)}{f(e_0)}e_0 + \frac{f(z_1)}{f(e_1)}e_1)) \\
&= \delta((1 - \alpha)x_0 + \alpha x_1, \delta_{\alpha}) \\
&\subset \text{int}(O((1 - \alpha)y_0 + \alpha y_1, \delta)}) \subset P.
\end{align*}
\]

Since $\delta(\alpha x, e) = \alpha \delta(x, e)$ for any $x \in X$ with $x > 0$ and for any $\alpha \in \mathcal{K}_R$, it holds that $O(\alpha x, \delta) = \alpha O(x, \delta)$. Therefore

\[
O((1 - \alpha)x_0 + \alpha x_1, \delta_{\alpha}) \subset O((1 - \alpha)y_0 + \alpha y_1, \delta),
\]

that is, $(1 - \alpha)x_0 + \alpha x_1 \in \text{int}(O((1 - \alpha)y_0 + \alpha y_1, \delta)) \subset P$.

(2) $\Rightarrow$ (3): $P \in \mathcal{P}_X$ satisfies $P \in \mathcal{O}_X$, $P \neq \emptyset$ and $\{x \mid x > 0\} \subset \text{co}(P) \neq X$.

(3) $\Rightarrow$ (1): Take $x_0 \in \text{co}(O)$. Let $p$ be a mapping from $X$ into $[0, \infty]$ defined by $p(x) = \inf\{r \mid r > 0, \frac{1}{r}x \in \text{co}(O) - x_0\}$. $p$ satisfies the following:

1. $p(x) < \infty$;
2. $\forall \alpha \in \mathbb{R}$ with $\alpha > 0$, $p(\alpha x) = \alpha p(x)$;
3. $p(x + y) \leq p(x) + p(y)$;
4. $\{x \mid p(x) < 1\} = \text{co}(O) - x_0$.

Since $0 \notin \text{co}(O)$, $p(-x_0) \geq 1$. Let $Y = \{\lambda x_0 \mid \lambda \in \mathbb{R}\}$ and $g$ a mapping from $Y$ into $\mathbb{R}$ defined by $g(\lambda x_0) = -\lambda$. $g$ is linear and $g(\lambda x_0) \leq p(\lambda x_0)$. By Hanh-Banach theorem there exists a mapping $f$ from $X$ into $\mathbb{R}$ satisfying $f \leq p$ and $f|_Y = g$. $-f$ is an answer of the proposition. \qed
4 Criteria for the Hausdorffness

Let $X$ be a vector lattice with unit. Let $|\mathcal{K}_X|$ be the class of $x$ satisfying $|x| \in \mathcal{K}_X$. For any $x \in |\mathcal{K}_X|$ let $x^+_\perp = \{0 \vee x\}^\perp$, $x^-_\perp = \{0 \vee (-x)\}^\perp$.

$$Q(x) = \{x_1 \mid x_1 \in |\mathcal{K}_X|, (x_1)^+_\perp = x^+_\perp, (x_1)^-_\perp = x^-_\perp\}$$

and

$$\overline{Q}(x) = \left( \bigcup_{x_1, x_2 \in Q(x)} [0 \wedge x_1, 0 \vee x_2] \right) \setminus \{0\}.$$

**Theorem 4.1.** Let $X$ be a complete vector lattice with unit and satisfying $\mathcal{P}_X \neq \emptyset$. Then $X$ is Hausdorff.

**Proof.** Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. It holds that for any $y \in X$ there exists $x \in |\mathcal{K}_X|$ such that $y \in \overline{Q}(x)$. Let $y = \frac{x_2 - x_1}{2}$. Let $R_x$ be a mapping from $X$ into $X$ defined by $R_x(y_1 + y_2) = -y_1 + y_2$ for any $y_1 \in x^+_\perp$ and for any $y_2 \in x^-_\perp$. Let $O_1 = \left(\frac{x_1 + x_2}{2} - R^{-1}_x(P)\right)$ and $O_2 = \left(\frac{x_1 + x_2}{2} + R^{-1}_x(P)\right)$. Then $O_1$ and $O_2$ are answers of the proposition. \qed

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**References**


