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A topology of vector lattices

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Abstract

In [3] we define and study the Denjoy and Henstock-Kurzweil integrals in a vector lattice. Moreover in [5, 6, 7] we show some fixed point theorems in a vector lattice. In those theories we need a topology in a vector lattice and some assumptions. The purpose of this paper is to show criteria for the assumptions.

1 Introduction

There are many fixed point theorems in a topological vector space, for instance, Kirk’s fixed point theorem in a Banach space, and so on; see for example [11].

We consider a derivative and fixed point theorems in a vector lattice. As known well every topological vector space has a linear topology. On the other hand, although every vector lattice does not have a topology, it has two lattice operators, which are the supremum \( \vee \) and the infimum \( \wedge \), and also an order is introduced from these operators; see also [9, 12] about vector lattices. There are some methods how to introduce a topology to a vector lattice. One method is to assume that the vector lattice has a linear topology [1]. On the other hand, there is another method to make up a topology in a vector lattice, for instance, in [3] one method is introduced in case of the vector lattice with unit.

In [3] we define and study the Denjoy and Henstock-Kurzweil integrals in a vector lattice. Moreover in [5, 6, 7] we show some fixed point theorems in a vector lattice. In those theories we need a topology in a vector lattice and some assumptions. The purpose of this paper is to show criteria for the assumptions.

2 Topology in a vector lattice

First we introduce a topology in a vector lattice introduced by [3]; see also [6, 7].

Let \( X \) be a vector lattice. \( e \in X \) is said to be an unit if \( e \wedge x > 0 \) for any \( x \in X \) with \( x > 0 \). Let \( \mathcal{K}_X \) be the class of units of \( X \). In the case where \( X \) is the set of real numbers \( \mathbb{R} \), \( \mathcal{K}_\mathbb{R} \) is the set of positive real numbers. Let \( X \) be a vector lattice with unit and let \( Y \) be a subset of \( X \). \( Y \) is said to be open if for any \( x \in Y \) and for any \( e \in \mathcal{K}_X \) there exists
$\varepsilon \in \mathcal{K}_R$ such that $[x - \varepsilon e, x + \varepsilon e] \subset Y$. Let $\mathcal{O}_X$ be the class of open subsets of $X$. For $e \in \mathcal{K}_X$ and for an interval $[a, b]$ we consider the following subset

$$[a, b]^e = \{x \mid \text{there exists some } \varepsilon \in \mathcal{K}_R \text{ such that } x - a \geq \varepsilon e \text{ and } b - x \geq \varepsilon e\}.$$ 

By the definition of $[a, b]^e$ it is easy to see that $[a, b]^e \subset [a, b]$. Every mapping from $X \times \mathcal{K}_X$ into $(0, \infty)$ is said to be a gauge. Let $\Delta_X$ be the class of gauges in $X$. For $x \in X$ and $\delta \in \Delta_X$, $O(x, \delta)$ is defined by

$$O(x, \delta) = \bigcup_{e \in \mathcal{K}_X} [x - \delta(x, e)e, x + \delta(x, e)e]^e.$$ 

$O(x, \delta)$ is said to be a $\delta$-neighborhood of $x$. Suppose that for any $x \in X$ and for any $\delta \in \Delta_X$ there exists $U \in \mathcal{O}_X$ such that $x \in U \subset O(x, \delta)$.

Let $X$ be a vector lattice with unit and $Y$ a vector lattice. Let $\mathcal{U}^\delta_Y(\mathcal{K}_X, \geq)$ be the class of $\{v_e \mid e \in \mathcal{K}_X\}$ satisfying the following conditions:

(U1) $v_e \in Y$ with $v_e > 0$;

(U2)$^d$ $v_{e_1} \geq v_{e_2}$ if $e_1 \geq e_2$;

(U3)$^*$ For any $e \in \mathcal{K}_X$ there exists $\theta(e) \in \mathcal{K}_R$ such that $v_{\theta(e)e} \leq \frac{1}{2}v_e$.

Let $x_0 \in Z \subset X$ and $f : Z \rightarrow Y$. $f$ is said to be continuous at $x_0$ if there exists $\{v_e\} \in \mathcal{U}^\delta_Y(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists $\delta \in \mathcal{K}_R$ such that for any $x \in Z$ if $|x - x_0| \leq \delta e$, then $|f(x) - f(x_0)| \leq v_e$.

Let $X$ and $Y$ be vector lattices with unit, $Z \subset X$ and $f : Z \rightarrow Y$. Suppose that there exists $P \subset Y$ satisfying the following conditions:

(P1) $P$ is open and convex;

(P2) If $x \in P$ and $x \leq y$, then $y \in P$;

(P3) $0 \notin P$;

(P4) $\{x \mid x > 0\} \subset P$.

Let $\mathcal{P}_Y$ be the class of the above $P$'s. $f$ is said to be upper semi-continuous with respect to $P \in \mathcal{P}_Y$ if $\{x \mid y - f(x) \in P\} \in \mathcal{O}_X \cap Z$ for any $y \in Y$. $f$ is said to be lower semi-continuous with respect to $P \in \mathcal{P}_Y$ if $\{x \mid f(x) - y \in P\} \in \mathcal{O}_X \cap Z$ for any $y \in Y$. $f$ is said to be semi-continuous with respect to $P \in \mathcal{P}_Y$ if it is upper and lower semi-continuous with respect to $P \in \mathcal{P}_Y$.

A vector lattice is said to be Archimedean if it holds that $x = 0$ whenever there exists $y \in X$ with $y \geq 0$ such that $0 \leq rx \leq y$ for any $r \in \mathcal{K}_R$.

Let $X$ be an Archimedean vector lattice. Then there exists a positive homomorphism $f$ from $X$ into $\mathbb{R}$, that is, $f$ satisfies the following conditions:
(H1) \( f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \) for any \( x, y \in X \) and for any \( \alpha, \beta \in \mathbb{R} \);

(H2) \( f(x) \geq 0 \) for any \( x \in X \) with \( x \geq 0 \);

see [7]*Example 3.1. Suppose that there exists a homomorphism \( f \) from \( X \) into \( \mathbb{R} \) satisfying the following condition instead of (H2):

(H2)* \( f(x) > 0 \) for any \( x \in X \) with \( x > 0 \).

The following hold under the topology above; see [6, 7].

**Lemma 2.1.** Let \( X \) be an Archimedean vector lattice with unit and \( \{x_1, \ldots, x_n\} \) a subset of \( X \). Then \( \text{co}\{x_1, \ldots, x_n\} \) is homeomorphic to a compact and convex subset of \( \mathbb{R}^n \).

**Lemma 2.2.** Let \( X \) be an Archimedean vector lattice with unit, \( Y \) a vector lattice with unit, \( Z \subset X \) and \( f \) a mapping from \( Z \) into \( Y \). Suppose that there exists a homomorphism from \( X \) into \( \mathbb{R} \) satisfying condition (H2)* and that \( \mathcal{P}_Y \neq \emptyset \).

Then \( f \) is semi-continuous with respect to any \( P \in \mathcal{P}_Y \) if it is continuous at any \( x \in Z \).

**Lemma 2.3.** Let \( X \) be an Archimedean vector lattice with unit, \( Y \) a vector lattice with unit, \( x_0 \in Z \subset X \) and \( f \) a mapping from \( Z \) into \( Y \). Suppose that there exists a homomorphism from \( X \) into \( \mathbb{R} \) satisfying condition (H2)*.

Then \( f \) is continuous at \( x_0 \) in the sense of topology if it is continuous at \( x_0 \).

## 3 Criteria for the condition (H2)*

**Theorem 3.1.** Let \( X \) be an Archimedean vector lattices with unit.

Then the following are equivalent:

1. \( X \) satisfies the condition (H2)*;
2. \( \mathcal{P}_X \neq \emptyset \);
3. There exists \( O \in \mathcal{O}_X \) such that \( O \neq \emptyset \) and \( \{x \mid x > 0\} \subset \text{co}(O) \neq X \).

**Proof.** (1) \( \Rightarrow \) (2): Let \( 0 < \beta < 1 \) and \( \delta(x, e) = \frac{\beta f(x)}{f(e)} \) for any \( x \in X \) with \( x > 0 \) and for any \( e \in K_X \). Put \( P = \bigcup_{x \in X \text{ with } x > 0} \text{int}(O(x, \delta)) \). Then \( P \) is open and \( \{x \mid x > 0\} \subset P \). Note that by condition (H2)* for any \( x_1, x_2 \in X \) with \( x_1, x_2 > 0 \) and \( x_1 \neq x_2 \), \( \frac{x_1}{f(x_1)} \) and \( \frac{x_2}{f(x_2)} \) are incomparable mutually. Therefore \( x - \delta(x, e)e \not\leq 0 \) for any \( x \in X \) with \( x > 0 \) and for any \( e \in K_X \). Assume that \( 0 \in P \). Then there exist \( x \in X \) with \( x > 0 \) and \( e \in K_X \) such that \( 0 \in \text{int}(A) \). It is a contradiction. Therefore \( 0 \not\in P \). Note that \( x \in \text{int}(A) \) if and only if there exists \( \delta \in \Delta_X \) such that \( O(x, \delta) \subset A \). Let \( x \in P \) and \( x \leq y \). Then there exist \( z \in X \) with \( z > 0 \) and \( \delta \in \Delta_X \) such that \( O(x, \delta) \subset O(z, \delta) \). Let \( \delta_y(u, e) = \delta_z(u - y + x, e) \).
Since $\delta(x_2, e) \leq \delta(x_1 + x_2, e)$ for any $x_1, x_2 \in X$ with $x_1, x_2 > 0$, it holds that $x_1 + O(x_2, \delta) \subset O(x_1 + x_2, \delta)$. Therefore
\[
O(y, \delta_y) = y - x + O(x, \delta_x) \subset O(y - x + O(x, \delta_y)) \subset O(x_1 + x_2, \delta),
\]
that is, $y \in \text{int}(O(z + y - x, \delta)) \subset P$. Let $x_0, x_1 \in P$ and $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq 1$. Then for $i = 0, 1$ there exist $y_i \in X$ with $y_i > 0$ such that $O(x_i, \delta_i) \subset O(y_i, \delta)$. Let $\delta(\alpha x, e) = (1 - \alpha)\delta_0(x, e) + \alpha \delta_1(x, e)$. Take $z \in O((1 - \alpha)x_0 + \alpha x_1, \delta)$ arbitrary. Then there exists $e \in \mathcal{K}_X$ such that
\[
(1 - \alpha)x_0 + \alpha x_1 \in \text{int}(O((1 - \alpha)y_0 + \alpha y_1, \delta)) \subset P.
\]
Therefore $O((1 - \alpha)x_0 + \alpha x_1, \delta) \subset O((1 - \alpha)y_0 + \alpha y_1, \delta)$, that is, $(1 - \alpha)x_0 + \alpha x_1 \in \text{int}(O((1 - \alpha)y_0 + \alpha y_1, \delta)) \subset P$.

(2) $\Rightarrow$ (3): $P \in \mathcal{P}_X$ satisfies $P \in \mathcal{O}_X$, $P \neq \emptyset$ and $\{x | x > 0\} \subset \text{co}(P) \neq X$.

(3) $\Rightarrow$ (1): Take $x_0 \in \text{co}(O)$. Let $p$ be a mapping from $X$ into $[0, \infty]$ defined by $p(x) = \inf\{r | r > 0, \frac{1}{r}x \in \text{co}(O) - x_0\}$. $p$ satisfies the following:

1. $p(x) < \infty$;
2. $\forall \alpha \in \mathbb{R}$ with $\alpha > 0$, $p(\alpha x) = \alpha p(x)$;
3. $p(x + y) \leq p(x) + p(y)$;
4. $\{x | p(x) < 1\} = \text{co}(O) - x_0$.

Since $0 \notin \text{co}(O)$, $p(-x_0) \geq 1$. Let $Y = \{\lambda x_0 | \lambda \in \mathbb{R}\}$ and $g$ a mapping from $Y$ into $\mathbb{R}$ defined by $g(\lambda x_0) = -\lambda$. $g$ is linear and $g(\lambda x_0) \leq p(\lambda x_0)$. By Hanh-Banach theorem there exists a mapping $f$ from $X$ into $\mathbb{R}$ satisfying $f \leq p$ and $f|_Y = g$. $-f$ is an answer of the proposition. \qed
4 Criteria for the Hausdorffness

Let $X$ be a vector lattice with unit. Let $|\mathcal{K}_X|$ be the class of $x$ satisfying $|x| \in \mathcal{K}_X$. For any $x \in |\mathcal{K}_X|$ let $x^\perp_+ = \{0 \vee x\}^\perp$, $x^\perp_- = \{0 \vee (-x)\}^\perp$,

$$Q(x) = \{x_1 \mid x_1 \in |\mathcal{K}_X|, (x_1)^+_+ = x^+_+, (x_1)^-_+ = x^-_+\}$$

and

$$\overline{Q}(x) = \left( \bigcup_{x_1, x_2 \in Q(x)} [0 \wedge x_1, 0 \vee x_2] \right) \setminus \{0\}.$$ 

Theorem 4.1. Let $X$ be a complete vector lattice with unit and satisfying $\mathcal{P}_X \neq \emptyset$.

Then $X$ is Hausdorff.

Proof. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. It holds that for any $y \in X$ there exists $x \in |\mathcal{K}_X|$ such that $y \in \overline{Q}(x)$. Let $y = \frac{x_2 - x_1}{2}$. Let $R_x$ be a mapping from $X$ into $X$ defined by $R_x(y_1 + y_2) = -y_1 + y_2$ for any $y_1 \in x^+_+$ and for any $y_2 \in x^-_+$. Let $O_1 = (\frac{x_1 + x_2}{2} - R_x^{-1}(P))$ and $O_2 = (\frac{x_1 + x_2}{2} + R_x^{-1}(P))$. Then $O_1$ and $O_2$ are answers of the proposition. 

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