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A NEW PROOF OF THE HAHN BANACH THEOREM IN A PARTIALLY ORDERED VECTOR SPACE AND ITS APPLICATIONS

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1. Introduction

The Hahn-Banach theorem is one of the most fundamental theorems in the functional analysis theory and the separation theorem is one of the most fundamental theorems in the optimization theory. These theorems are known well in the case where the range space is the real number system. The following is the Hahn-Banach theorem:

Let p be a sublinear mapping from a vector space X to the real number system R, Y a subspace of X and q a linear mapping from Y to R such that $q\leq p$ on Y. Then q can be extended to a linear mapping g defined on the whole space X such that $g\leq p$.

Moreover, the following is the separation theorem:

Let X be a normed space, X^{*} its dual space, A, B subsets of X such that A is closed convex and B is compact convex subset with $A\cap B=\emptyset$. Then there exists a $f\in X^{*}\setminus\{0\}$ such that $\inf\{f(y)\mid y\in B\}\geq\sup\{f(x)\mid x\in A\}$.

It is known that Hahn-Banach theorem establishes in the case where the range space is a Dedekind complete Riesz space; see [3, 14, 16] and the separation theorem establishes in a Cartesian product space of a vector space and a Dedekind complete ordered vector space; see [6, 13].

The Hahn-Banach theorem is proved often using the Zorn lemma. For the proof of the Hahn-Banach theorem, there exist several approaches. For instance, Hirano, Komiya, and Takahashi [7] showed the Hahn-Banach theorem by using the Markov-Kakutani fixed point theorem [8] in the case where the range space is the real number system.

In this paper, in Section 3, using the Bourbaki-Kneser fixed point theorem, we give a new proof of the Hahn-Banach theorem and the Mazur-Orlicz theorem in the case where the range space is a Dedekind complete partially ordered vector space (Theorems 4 and 5). In Section 4, we give a new proof of the separation theorem in a Cartesian product of the vector space and Dedekind complete partially ordered vector space (Theorem 6); see [5, 6, 13].

2. Preliminaries

Let R be the set of real numbers, N the set of natural numbers, I an indexed set, E a partially ordered set and F a subset of E. The set F is called a *chain* if any two elements are comparable, that is, $x\leq y$ or $y\leq x$ for any x, $y\in F$. An element $x\in E$ is called a *lower bound* of F if $x\leq y$ for any $y\in F$. An element $x\in E$ is called the *minimum* of F if x is a lower bound of F and $x\in F$. If there exists a lower bound of F, then F is said to be *bounded from below*. An element $x\in E$ is called an *upper bound* of F if $y\leq x$ for any $y\in F$. An element $x\in E$ is called the *maximum* of F if x is an upper bound and $x\in F$. If there exists an upper bound of F, then F is said to be *bounded from above*. If the set of all lower bounds of F has the maximum, then the maximum is called an *infimum* of F and denoted by $\inf F$. If the set of all upper bounds of F has the minimum, then the minimum is called a *supremum* of F and denoted by $\sup F$. A partially ordered set E is said to be *complete* if every nonempty chain of E has an infimum; E is said to be *chain complete* if every nonempty chain of E which is bounded from below has an infimum; E is said to be *Dedekind complete* if every nonempty subset of E which is bounded from below has an infimum. A mapping f from E to E is called *decreasing* if $f(x) \leq x$ for every $x \in E$. For the further information of a partially ordered set, see [1, 3, 4, 12, 14].

In a complete partially ordered set, the following theorem is obtained [2, 9, 10].

Theorem 1 (Bourbaki-Kneser). *Let E be a complete partially ordered set. Let f be a decreasing mapping from E to E. Then f has a fixed point.*

Recently, T. C. Lim [11] proved a common fixed point theorem for the family of decreasing commutative mapping, which is a generalization of Theorem 1.

A partially ordered set E is called a partially ordered vector space if E is a vector space and $x+z \leq y+z$ and $\alpha x \leq \alpha y$ hold whenever $x, y, z \in E$, $x \leq y$, and $\alpha \in R$. If a partially ordered vector space E is a lattice, that is, any two elements have a supremum and an infimum, then E is called a *Riesz space*.

Let X be a vector space and E a partially ordered vector space. A mapping f from X to E is said to be *concave* if $f(tx+(1-t)y) \geq tf(x)+(1-t)f(y)$ for any $x, y \in X$ and $t \in [0,1]$. A mapping f from X to E is called *sublinear* if the following are satisfied.
(S1) For any $x, y \in X$, $p(x+y) \leq p(x)+p(y)$.
(S2) For any $x \in X$ and $\alpha \geq 0$, $p(\alpha x) = \alpha p(x)$.

3. The Hahn-Banach Theorem

Lemma 2. *Let p be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E, K a nonempty convex subset of X and q a concave mapping from K to E such that $q \leq p$ on K. For any $x \in X$, let*

$$f(x) = \inf\{p(x+ty) - tq(y) \mid t \in [0,\infty) \text{ and } y \in K\}.$$

Then f is sublinear such that $f \leq p$. Moreover if g is a linear mapping from X to E, then $g \leq f$ is equivalent to $g \leq p$ on X and $q \leq g$ on K.

Proof. For any $x \in X$, $\{p(x+ty) - tq(y) \mid t \in [0,\infty) \text{ and } y \in K\}$ is bounded from below. Indeed, since

$$\begin{aligned} p(x+ty) - tq(y) &\geq p(ty) - p(-x) - tq(y) \\ &\geq -p(-x), \end{aligned}$$

it is established. Since E is Dedekind complete, f is well-defined and we have $f(x) \geq -p(-x)$. By definition of f, we have $f(x) \leq p(x)$ and $f(\alpha x) = \alpha f(x)$ for any $\alpha \geq 0$. Thus (S2) is established. Let $x_1, x_2 \in X$. For any $y_1, y_2 \in K$ and $s, t > 0$, we have

$$\begin{aligned} &p(x_1+sy_1) - sq(y_1) + p(x_2+ty_2) - tq(y_2) \\ &\geq p(x_1+x_2+(s+t)w) - (s+t)q(w) \\ &\geq f(x_1+x_2), \end{aligned}$$

where $w = \frac{1}{s+t}(sy_1+ty_2) \in K$. For $p(x_1+sy_1) - sq(y_1)$, take infimum with respect to $s > 0$ and $y \in K$, we have

$$f(x_1) + p(x_2+ty_2) - tq(y_2) \geq f(x_1+x_2)$$

and for $p(x_2+ty_2) - tq(y_2)$, also take infimum with respect to $t > 0$ and $y \in K$, we have

$$f(x_1) + f(x_2) \geq f(x_1+x_2).$$

This shows that $f(x_1) + f(x_2) \geq f(x_1+x_2)$. Thus (S1) is established. Suppose that g is a linear mapping from X to E. If $g \leq f$, then we have $g \leq p$. Moreover for any $y \in K$, since

$$-g(y) = g(-y) \leq f(-y) \leq p(-y+y) - q(y) = -q(y),$$

we have $g \geq q$ on K. To prove the converse, suppose that $g \leq p$ on X and $q \leq g$ on K. For any $x \in X$, $y \in K$ and $t > 0$, we have

$$g(x) = g(x+ty) - tg(y) \leq p(x+ty) - tq(y).$$

This implies that $g \leq f$. □

The above Lemma 2 is proved in case where the range space is a Dedekind complete Riesz space, see [14, Lemma 1.5.1].

By Theorem 1 and Lemma 2, we obtain the following.

Lemma 3. *Let f be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E. Then there exists a linear mapping g from X to E such that $g \leq f$.*

Proof. Put $f^*(x) = -f(-x)$ for any $x \in X$. Let $y \in X$ and

$$Y = \{h \in E^X \mid h \text{ is sublinear, } f^* \leq h \leq f\}.$$

Then Y is an ordered set by its canonical order. Since E is Dedekind complete, E^X is also so. Consider an arbitrary chain $Z \subset Y$. Since E^X is Dedekind complete and Z is bounded from below, there exists a $g = \inf Z$ in E^X. Then g is sublinear. Since Y is bounded from below, it holds that $g \in Y$. Thus Y is complete. Let $K = \{y\}$. We define a decreasing operator S by

$$Sh(x) = \inf\{h(x+ty) - h(ty) \mid t \in [0, \infty), \, y \in K\}$$

for any $h \in Y$. By Lemma 2, Sh is sublinear and S is a mapping from Y to Y. Thus by Theorem 1, we have a fixed point $g \in Y$. Then for any $x \in X$, we have $g(x) \leq g(x+y) - g(y)$ and

$$g(x) + g(y) \leq g(x+y) \leq g(x) + g(y).$$

Thus g is linear and $g(y) = f(y)$. □

By Lemmas 2 and 3, we can prove the Hahn-Banach theorem and the Mazur-Orlicz theorem in case where the range space is a Dedekind complete partially ordered vector space.

Theorem 4. *Let p be a sublinear mapping from a vector space X to a Dedekind complete ordered vector space E, Y a vector subspace of X and q a linear mapping from Y to E such that $q \leq p$ on Y. Then q can be extended to a linear mapping g defined on the whole space X such that $g \leq p$.*

Proof. By Lemma 2, there exists a sublinear mapping f such that $f \leq p$. By Lemma 3, there exists a linear mapping g such that $g \leq f$. Then putting $K = Y$ in Lemma 2, we have $g \leq p$ on X and $q \leq g$ on Y. Since q is linear, we have $q = g$ on Y. Thus the assertion holds. □

We obtain the Mazur-Orlicz theorem in a Dedekind complete partially ordered vector space.

Theorem 5. *Let p be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E. Let $\{x_j \mid j \in I\}$ be a family of elements of X and $\{y_j \mid j \in I\}$ a family of elements of E. Then the following* (1) and (2) *are equivalent.*

(1) *There exists a linear mapping f from X to E such that $f(x) \leq p(x)$ for any $x \in X$ and $y_j \leq f(x_j)$ for any $j \in I$.*

(2) *For any $n \in N$, $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $j_1, j_2, \ldots, j_n \in I$, we have*

$$\sum_{i=1}^{n} \alpha_i y_{j_i} \leq p\left(\sum_{i=1}^{n} \alpha_i x_{j_i}\right).$$

Proof. The assertion from (1) to (2) is clear. For any $x \in X$, by (2), we have

$$-p(-x) \leq p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) - \sum_{i=1}^{n} \alpha_i y_{j_i}.$$

Put $p_0(x) = \inf\{p(x + \sum_{i=1}^{n} \alpha_i x_{j_i}) - \sum_{i=1}^{n} \alpha_i y_{j_i} \mid n \in N, \, \alpha_i \geq 0 \text{ and } j_i \in I\}$. Since E is Dedekind complete, p_0 is well-defined and p_0 is sublinear. Thus by Lemma 3, there exists a linear mapping f from X to E such that $f(x) \leq p_0(x)$ for any $x \in X$. Since $p_0(-x_j) \leq -y_j$, we have

$$y_j \leq -p_0(-x_j) \leq f(x_j).$$

Since $p_0(x) \leq p(x)$, we have $f(x) \leq p(x)$. Thus the assertion holds. □

4. The separation theorem

Let X be a vector space, E a Dedekind complete partially ordered vector space. Let A be a nonempty subset of X. ${}^{l}A$ denotes the linear span of A and ${}^{i}A$ denotes the relative algebraic interior of A, that is, ${}^{i}A = \{x \in X \mid \text{For any } x' \in X \text{ there exists } \varepsilon > 0 \text{ such that } x+\lambda(x'-x) \in A \text{ for any } \lambda \in [0, \varepsilon)\}$. Let f be a linear mapping from X to E, g a linear mapping from E to E and $u_0 \in E$. Then $H = \{(x, y) \in X \times E \mid f(x) + g(y) = u_0\}$ is empty or an affine manifold in $X \times E$. Let A, B be nonempty subsets of $X \times E$. It is said that an affine manifold H separates A and B if

$$H_- = \{(x, y) \in X \times E \mid f(x) + g(y) \leq u_0\} \supset A$$

and

$$H_+ = \{(x, y) \in X \times E \mid f(x) + g(y) \geq u_0\} \supset B$$

hold. Let A be a nonempty subsets of $X \times E$. The operator P_X defined by $P_X(x, y) = x$ for any $(x, y) \in X \times E$ is called the projection of $X \times E$ onto X. Then P_X is a linear mapping from $X \times E$ onto X. We define

$$P_X(A) = \{x \in X \mid \text{ there exists } y \in E \text{ such that } (x, y) \in X \times E\}.$$

Then we have $P_X(A + B) = P_X(A) + P_X(A)$ for $A \neq \emptyset$ and $B \neq \emptyset$. The set

$$C(A) = \{\lambda z \in X \times E \mid \lambda \geq 0,\ z \in A\}$$

is called the cone of A. Then if A is convex, then $C(A)$ is convex. A subset $A \in X \times E$ is cone if $\lambda > 0$ implies $\lambda A \subset A$. We obtain the following separation theorem in a Cartesian product of the vector space and the Dedekind complete partially ordered vector space.

Theorem 6. *Let A and B be subsets of $X \times E$ such that $C(A - B)$ is convex cone, $P_X(A - B)$ satisfies the following* (i) *and* (ii):
(i) $0 \in {}^{i}P_X(A - B))$ *and* ${}^{l}P_X(A - B) = X$.
(ii) *If $(x, y_1) \in A$ and $(x, y_2) \in B$, then $y_1 \geq y_2$ holds.*
Then there exists a linear mapping f from X to E and a $y_0 \in E$ such that the affine manifold $H = \{(x, y) \in X \times E \mid f(x) - y = y_0\}$ separates A and B.

Proof. By assumption (i) and the definition of ${}^{i}P_X(A - B)$, for any $x \in X$ and $\lambda > 0$, there exist $y_1, y_2 \in E$ such that $(\lambda x, y_1 - y_2) \in A - B$. Then there exists $x_1, x_2 \in X$ such that $(\lambda x, y_1 - y_2) = (x_1 - x_2, y_1 - y_2) = (x_1, y_1) - (x_2, y_2) \in A - B$. For any $x \in X$ define $E_x = \{y \in E \mid (x, y) \in C(A - B)\}$. Since $\lambda^{-1}(y_1 - y_2) \in E_x$ for any $\lambda \in (0, \varepsilon)$, we have $E_x \neq \emptyset$ for all $x \in X$. Moreover, let $y \in E_0$ and $y \neq 0$, then there exists $\lambda > 0$ such that $(x_1, y_1) \in A$, $(x_1, y_1) \in B$ and $(0, y) = \lambda\{(x_1, y_1) - (x_2, y_2)\}$ and $x_1 = x_2$. By assumption (ii), we have $y = \lambda(y_1 - y_2) \geq 0$. Thus $y \in E_+$. Since $C(A - B)$ is convex cone, we have $E_x + E_{x'} \subset E_{x+x'}$ for any $x, x' \in X$. For any $x \in X$, there exists $y' \in E$ with $-y' \in E_{-x}$ by the definition of E_x. Then $y - y' \in E_x + E_{-x} \subset E_0 \subset E_+$ for any $y \in E_x$. Thus $y' \leq y$ for any $y \in E_x$. Put $p(x) = \inf\{y \mid y \in E_x\}$, then $p(x)$ is sublinear. Since E is Dedekind complete, by Lemma 3, there exists a linear mapping f from X to E such that $f(x) \leq p(x)$ for all $x \in E$. Then for any $(x_1, y_1) \in A$, $(x_2, y_2) \in B$, take $x = x_1 - x_2$, we have

$$f(x_1 - x_2) \leq p(x_1 - x_2) \leq y_1 - y_2$$

Therefore,

$$f(x_1) - y_1 \leq f(x_2) - y_2.$$

Since E is Dedekind complete, there exists a $y_0 \in E$ such that

$$f(x_1) - y_1 \leq y_0 \leq f(x_2) - y_2$$

for any $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$, and $y_0 \in E$ satisfies that $\sup\{f(x_1) - y_1 \mid (x_1, y_1) \in A\} \leq y_0 \leq \inf\{f(x_2) - y_2 \mid (x_2, y_2) \in B\}$. □

We also obtain the following.

Corollary 7. *Let A and B be subsets of $X \times E$ such that $C(A-B)$ is convex cone, $P_X(A-B)$ satisfies the following* (i) *and* (ii):
(i) $0 \in {}^iP_X(A-B))$.
(ii) *If $(x, y_1) \in A$ and $(x, y_2) \in B$, then $y_1 \geq y_2$ holds.*
Then there exists a linear mapping f from X to E and a $y_0 \in E$ such that the affine manifold $H = \{(x, y) \in X \times E \mid f(x) - y = y_0\}$ separates A and B.

Proof. Since $X_1 = {}^lP_X(A-B) = {}^{li}P_X(A-B)$ is a subspace of X, A, B, $A-B$ and $C(A-B)$ are subsets of X_1. By Theorem 6, there exists a linear mapping f_1 from X_1 to E such that

$$f_1(x_1 - x_2) \leq y_1 - y_2$$

for any $(x_1, y_1) \in A$, $(x_2, y_2) \in B$. Let X_2 be an algebraical complementary space of X_1. Then an arbitrary $z \in X$ has a unique representation $z = x + y$ with $x \in X_1$ and $y \in X_2$. We define a linear mapping f from X to E by $f(z) = f_1(x)$ for all $z \in X$. Then f satisfies the assertion of Corollary. □

Let C be a non-empty subset of X and f a linear mapping from X to E. For a mapping T from C to F, we define its algebraical conjugate mapping T_c by

$$D(T_c) = \{f \mid \sup\{f(x) - T(x) \,|x \in C\} \in E\}\,, \, T_c(f) = \sup\{f(x) - T(x) \mid x \in C\}\,,$$

where $f \in D(T_c)$. As an application of theorem 6, we have the following Fenchel duality theorem; see [5, 6].

Theorem 8. *Let X be a vector space, E a Dedekind complete partially ordered vector space. Let $D(U)$ and $D(V)$ be convex subsets of E with ${}^iD(U) \cap {}^iD(V) \neq \emptyset$, $U \in Map(D(U), E)$ and $V \in Map(D(V), E)$ be convex mappings, $P_0 = D(U) \cap D(V)$ and $\inf\{\, U(x) + V(x) \mid x \in P_0\} \in E$. Then there exists an $f_0 \in D_0 = D(U_c) \cap D(V_c)$ such that*

$$\inf\{U(x) + V(x) \mid x \in P_0\} = \sup\{-U_c - V_c \mid f \in D_0\} = -U_c(f_0) - V_c(-f_0).$$

Proof. First, we prove the inequality

$$\inf\{U(x) + V(x) \mid x \in P_0\} \leq -U_c(f_0) - V_c(-f_0).$$

Put $y_{00} = \inf\{U(x) + V(x) \mid x \in P_0\}$, then $y_{00} \leq U(x) + V(x)$. Put $V'(x) = y_{00} - V(x)$, then $V'(x) \leq U(x)$ for any $x \in P_0$. Then by assumption, ${}^iP_X(\mathrm{epi}(U)) \cap {}^iP_X(\mathrm{hypo}(V')) = {}^iD(U) \cap {}^iD(V') \neq \emptyset$. Since ${}^iA + {}^iB \subset {}^i(A+B)$ for arbitrary subsets A and B of X if $A \neq \emptyset$ and $B \neq \emptyset$, ${}^iP_X(\mathrm{epi}(U)) \cap {}^iP_X(\mathrm{hypo}(V')) \neq \emptyset$ implies $0 \in {}^iP_X(\mathrm{epi}(U) - \mathrm{hypo}(V'))$. If $(x, y_1) \in \mathrm{epi}(U)$ and $(x, y_2) \in \mathrm{hypo}(V')$ then $y_1 \geq U(x) \geq V'(x) \geq y_2$ for any $x \in P_0$. By Corollary 7, there exist linear mapping f from X to E and $y_0 \in E$ such that $f(x_1) - y_1 \leq y_0 \leq f(x_2) - y_2$ for any $(x_1, y_1) \in \mathrm{epi}(U)$ and $(x_2, y_2) \in \mathrm{hypo}(V')$. Then $\sup\{f(x_1) - y_1 \mid (x_1, y_1) \in \mathrm{epi}(U)\} \leq y_0 \leq \inf\{f(x_2) - y_2 \mid (x_2, y_2) \in \mathrm{hypo}(V')\}$. Take $y_1 = U(x)$, $x_1 = x$ and $y_2 = -V(x) + y_0$, $x_2 = x$ where $x \in {}^iP_X(\mathrm{epi}(U)) \cap {}^iP_X(\mathrm{hypo}(V'))$ then we have

$$f(x) - U(x) \leq y_0 \text{ for any } x \in D(U)$$

and

$$y_0 + y_{00} \leq f(x) + V(x) \text{ for any } x \in D(V).$$

Then there exist

$$U_c(f) = \sup\{f(x_1) - U(x) \mid x \in U\} \leq y_0,$$

$$V_c(-f) = \sup\{(-f)(x_1) - V(x) \mid x \in V\} \leq -(y_0 + y_{00}).$$

Therefore $-U_c(f) - V_c(-f) \geq -y_0 + y_{00} + y_0 = y_{00}$.

On the other hand, since $D_0 = D(U_c) \cap (-D(V_c)) \neq \emptyset$, $U_c(f) + U(x) \geq f(x)$ and $V_c(-f) + V(x) \geq (-f)(x)$, we have $U(x) + V(x) \leq -U_c(f) - V_c(-f)$ for any $x \in P_0$ and $L \in D_0$. Then $y_{00} \geq -U_c(f) - V_c(-f)$. Therefore we have $y_{00} = -U_c(f) - V_c(-f)$. □

References

[1] C. D. Aliprantis and O. Burkinshaw, *Locally Solid Riesz Spaces with Applications to Economics*, second edition, Amer. Math. Soc., Providence, 2003.

[2] N. Bourbaki, *Topologie Générale*, Hermann, Paris, 1940.

[3] R. Cristescu, *Topological vector spaces*, Noordhoff International Publishing, Leyden, 1977.

[4] B. A. Davey and H. A. Priestley, *Introduction to lattices and order*, second edition, Cambridge University Press, New York, 2002.

[5] K.H. Elster and R. Nehse, *Konjugierete operatoren und subdifferentiale*, Math. Operationsforsch.u.Statist. **6** (1975), 641–657.

[6] K.H. Elster and R. Nehse, *Necessary and sufficient conditions for the Order-Completeness of partially ordered vector space*, Math. Nachr. **81** (1978), 301–311.

[7] N. Hirano, H. Komiya, and W. Takahashi, *A generalization of the Hahn-Banach theorem*, J. Math. Anal. Appl. **88** (1982), 333–340.

[8] S. Kakutani, *Two fixed-point theorems concerning bicompact convex sets*, Proc. Imp. Acad. Tokyo **14** (1938), 242–245.

[9] W. A. Kirk, *Fixed point theory: A brief survey*, Universidas de Los Andes, Mérida, 1990.

[10] H. Kneser, *Eine direkte Ableitung des Zornschen Lemmas aus dem Auswahlaxiom*, Math. Z. **53** (1950), 110–113.

[11] T. C. Lim, *On minimal (maximal) common fixed points of a commuting family of decreasing (increasing) maps*, Differential and Difference Equations and Applications (2006), 683–684.

[12] W. A. J. Luxemburg and A. C. Zannen, *Riesz spaces I*, North Holland, Amsterdam, 1971.

[13] R. Nehse, *Some general separation theorems*, Math. Nachr. **84** (1978), 319–327.

[14] P. M. Nieberg, *Banach Lattices*, Springer-Verlag, Berlin, Heidelberg, New York, 1991.

[15] W. Takahashi, *Nonlinear Functional Analysis: Fixed Points Theory and its Applications,* Yokohama Publishers, 2000.

[16] A. C. Zannen, *Riesz spaces II*, North Holland, Amsterdam, 1984.

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