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THE EASTON COLLAPSE AND A SATURATED FILTER

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ABSTRACT. Suppose that there is a huge cardinal. We prove that a two-stage iteration of Easton collapses produces a saturated filter on the successor of a regular cardinal.

1. INTRODUCTION

In the pioneering work [10] Kunen established:

Theorem (Kunen). *Suppose that κ is huge with target λ. Then in some forcing extension $\kappa=\omega_1$, $\lambda=\omega_2$ and ω_1 carries an ω_2-saturated filter.*

Kunen's forcing has the form $P*\dot{S}(\kappa,\lambda)$, where P forces that $\kappa=\omega_1$ and $\dot{S}(\kappa,\lambda)$ is the Silver collapse introduced in [15]. The poset P is constructed by recursion so that $P*\dot{S}(\kappa,\lambda)$ can be completely embedded into $j(P)$, where $j:V\rightarrow M$ is the original huge embedding. Kunen's construction has since been modified to get models containing filters that are strongly saturated in various senses. We refer the reader to [5] for a comprehensive survey of the development.

In [7] Foreman, Magidor and Shelah proved the following striking result: If λ is supercompact, then the Levy collapse $C(\omega_1,\lambda)$ forces that ($\lambda=\omega_2$ and) ω_1 carries a saturated filter. The hypothesis was later reduced by Todorčević (see [2]) to λ being Woodin, which follows from Kunen's hypothesis as well. In contrast Foreman and Magidor [6] showed that $C(\omega_2,\lambda)$ forces the nonexistence of a saturated filter on ω_2 under PFA.

Let us assume again κ is huge with target λ. Todorčević's result implies that a saturated filter on ω_1 can be forced to exist by the iteration $C(\omega,\kappa)*\dot{C}(\kappa,\lambda)$ as well. What about ω_2? Namely we ask:

Question. *Does $C(\omega_1,\kappa)*\dot{C}(\kappa,\lambda)$ force that ω_2 carries an ω_3-saturated filter?*

One motivation for the question comes from the following unpublished result of Woodin: $C(\omega_1,\kappa)*\dot{C}(\kappa,\lambda)$ forces that an ω_2-dense filter on ω_2 exists in some inner model. (See [5] for an exposition in the case of ω_1.) Moreover if the answer is positive, then we would get saturated filters on many cardinals by simply iterating Levy collapses. This would in turn help to simplify Foreman's construction [3, 4] of a model in which every regular uncountable cardinal carries a saturated filter.

In this paper we define a poset $E(\mu,\kappa)$ for a pair of regular cardinals $\mu<\kappa$, and call it the Easton collapse. It is the product of standard collapsing posets with Easton support, and forces $\kappa=\mu^+$ if κ is Mahlo. In place of the original question, we answer the corresponding question for the iteration of Easton collapses:

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Theorem. *Suppose that κ is huge with target λ. Let $\mu < \kappa$ be regular. Then $E(\mu,\kappa) * \dot{E}(\kappa,\lambda)$ forces that κ carries a λ-saturated filter.*

In §4 we prove our theorem in somewhat refined form.

2. Preliminaries

We refer the reader to [9] for background material.

Throughout the paper we use μ, κ and λ to denote a regular cardinal. Unless otherwise stated it is understood that $\mu < \kappa < \lambda$.

Let P and Q be posets. We say that a map $\pi : P \to Q$ is a projection if the following hold:

(1) π is order-preserving, i.e. $p' \leq_P p \to \pi(p') \leq_Q \pi(p)$,
(2) $\pi(1_P) = 1_Q$ and
(3) $q \leq_Q \pi(p) \to \exists p^* \leq_P p(\pi(p^*) \leq_Q q)$.

Suppose that $\pi : P \to Q$ is a projection. Then $\operatorname{ran}\pi$ is dense in Q. It is straightforward to check that the map $q \mapsto \sum\{p \in P : \pi(p) \leq q\}$ is a complete embedding of Q into $B(P)$, the completion of P. It is also easy to see that if D is dense open in Q, $\pi^{-1}(D)$ is dense in P. So if $G \subset P$ is generic, π"G generates a generic filter over Q. Let $H \subset Q$ be V-generic. In $V[H]$ let P/H be the set $\pi^{-1}(H)$ ordered by $\leq_P$. It is straightforward to check that the map $p \mapsto (\pi(p), \dot{p})$, where $\dot{p}$ is a Q-name with $\pi(p) \Vdash_Q \dot{p} = p$, is a dense embedding of P into $Q * (P/\dot{H})$. Finally note that the composition of two projections is a projection.

We say that a cardinal γ is strongly regular if $\gamma^{<\gamma} = \gamma$. A set d of strongly regular cardinals is called Easton if $\sup(d \cap \gamma) < \gamma$ for all regular γ.

Suppose that X be a set of ordinals and P_γ is a poset for $\gamma \in X$. Define

$$\overset{\mathrm{E}}{\prod}_{\gamma \in X} P_\gamma = \{p : \operatorname{dom} p \subset X \text{ is Easton } \wedge \forall \gamma \in \operatorname{dom} p(p(\gamma) \in P_\gamma)\}.$$

$\overset{\mathrm{E}}{\prod}_{\gamma \in X} P_\gamma$ is ordered coordinatewise: $p' \leq p$ iff $\operatorname{dom} p' \supset \operatorname{dom} p$ and $p'(\gamma) \leq_\gamma p(\gamma)$ for all $\gamma \in \operatorname{dom} p$.

Let $Y \subset X$. Then $\overset{\mathrm{E}}{\prod}_{\gamma \in X} P_\gamma$ is canonically isomorphic to $\overset{\mathrm{E}}{\prod}_{\gamma \in Y} P_\gamma \times \overset{\mathrm{E}}{\prod}_{\gamma \in X-Y} P_\gamma$. Suppose in addition $\pi_\gamma : P_\gamma \to Q_\gamma$ is a projection for $\gamma \in Y$. Then it is easy to see that the map $p \mapsto \langle \pi_\gamma(p(\gamma)) : \gamma \in \operatorname{dom} p \cap Y\rangle$ is a projection from $\overset{\mathrm{E}}{\prod}_{\gamma \in X} P_\gamma$ to $\overset{\mathrm{E}}{\prod}_{\gamma \in Y} Q_\gamma$.

We say that P has (κ,κ,μ)-cc if for every $X \in [P]^\kappa$ there is $Y \in [X]^\kappa$ such that every $Z \in [Y]^\mu$ has a common extension. Needless to say, (κ,κ,μ)-cc implies κ-cc. If Q is separative and can be completely embedded into P, then the (κ,κ,μ)-cc of P implies that of Q.

Lemma 1. *Suppose that κ is Mahlo and P_γ is a poset of size $< \kappa$ for $\gamma < \kappa$. Then $\overset{\mathrm{E}}{\prod}_{\mu \leq \gamma < \kappa} P_\gamma$ has (κ,κ,μ)-cc.*

Proof. Let $\{p_\xi : \xi < \kappa\} \subset \overset{\mathrm{E}}{\prod}_{\mu \leq \gamma < \kappa} P_\gamma$. It suffices to find $X \in [\kappa]^\kappa$ and $\delta < \kappa$ such that $\operatorname{dom} p_\xi - \delta$ is mutually disjoint and $p_\xi | \delta$ is constant for $\xi \in X$.

Since $\operatorname{dom} p_\xi$ is Easton, $\sup(\operatorname{dom} p_\xi \cap \xi) < \xi$ for all regular $\xi < \kappa$. Since κ is Mahlo, we get a stationary $S \subset \kappa$ and $\delta < \kappa$ such that $\operatorname{dom} p_\xi \cap \xi \subset \delta$ for all $\xi \in S$.

Since $\operatorname{dom} p_\xi$ is bounded in κ, $C = \{\zeta < \kappa : \forall \xi < \zeta (\operatorname{dom} p_\xi \subset \zeta)\}$ is club. Note that if $\xi < \zeta$ are both from $S \cap C$, we have $\operatorname{dom} p_\xi \cap \operatorname{dom} p_\zeta = \operatorname{dom} p_\xi \cap \zeta \cap \operatorname{dom} p_\zeta \subset \delta$. Since $|\prod_{\mu \le \gamma < \delta} P_\gamma| < \kappa$, there is $X \in [S \cap C]^\kappa$ such that $p_\xi | \delta$ is constant for $\xi \in X$, as desired. □

For $\gamma \ge \mu$ we equip the set ${}^{<\mu}\gamma$ with reverse inclusion. Needless to say, ${}^{<\mu}\gamma$ is μ-closed and forces $|\gamma| = \mu$. Let us sketch a proof of

Lemma 2. *If $\gamma^{<\kappa} = \gamma$, then ${}^{<\mu}\gamma$ is isomorphic to a dense subset of ${}^{<\mu}\kappa \times {}^{<\kappa}\gamma$.*

Proof. Define

$$D = \{(q, r) \in {}^{<\mu}\kappa \times {}^{<\kappa}\gamma : \sup\{\beta + 1 : \beta \in \operatorname{ran} q\} = \operatorname{dom} r\}.$$

It is easy to see that D is dense in ${}^{<\mu}\kappa \times {}^{<\kappa}\gamma$. The following three facts should suffice to construct an isomorphism between ${}^{<\mu}\gamma$ and D by recursion.

First $(\emptyset, \emptyset) \in D$. Second each $(q, r) \in D$ has γ immediate extensions in D. Third if $\langle (q_\alpha, r_\alpha) : \alpha < \delta \rangle$ is a descending sequence in D with $\delta < \mu$, then we have $(\bigcup_{\alpha < \delta} q_\alpha, \bigcup_{\alpha < \delta} r_\alpha) \in D$. □

Corollary 3. *If $\gamma \ge \kappa$ is strongly regular, there is a projection from ${}^{<\mu}\gamma$ to ${}^{<\kappa}\gamma$.*

Let F be a filter on a set. We denote by F^+ the set of F-positive sets ordered by: $X' \le X$ iff $\exists C \in F (X' \cap C \subset X)$. Then F^+ is a separative poset. We say that F is (κ, κ, μ)-saturated if F^+ has (κ, κ, μ)-cc.

3. The Easton collapse

In this section we define the Easton collapse $E(\mu, \kappa)$ and prove its basic properties.

For a set X of ordinals define

$$E(\mu, X) = \overset{\mathrm{E}}{\prod}_{\mu \le \gamma \in X} {}^{<\mu}\gamma.$$

It is easy to see that $E(\mu, X)$ is μ-directed closed and forces $|\gamma| \le \mu$ for all strongly regular $\gamma \in X$. $E(\mu, \kappa)$ is a subset of V_κ, hence has size κ if κ is inaccessible. If κ is Mahlo, then $E(\mu, \kappa)$ has κ-cc by Lemma 1, and hence forces $\kappa = \mu^+$. If $\mu < \kappa \le \nu < \lambda$ are all regular, Corollary 3 provides a projection from $E(\mu, \lambda - \kappa) = \overset{\mathrm{E}}{\prod}_{\kappa \le \gamma < \lambda} {}^{<\mu}\gamma$ to $\overset{\mathrm{E}}{\prod}_{\nu \le \gamma < \lambda} {}^{<\nu}\gamma = E(\nu, \lambda)$.

Here is the main result of this section:

Lemma 4. *Suppose that P has κ-cc and size $\le \kappa$. Then there is a projection $\pi : P \times E(\kappa, \lambda) \to P * \dot{E}(\kappa, \lambda)$ such that $\pi(p, q)$ has the form $(p, \dot{q})$, where*

- $\Vdash_P \operatorname{dom} \dot{q} = \operatorname{dom} q$ *and*
- *each $\dot{q}(\gamma)$ depends only on $q(\gamma)$, i.e. if in addition $\pi(p', q') = (p', \dot{q}')$ and $q(\gamma) = q'(\gamma)$, then $\Vdash_P \dot{q}(\gamma) = \dot{q}'(\gamma)$.*

Proof. Since P has κ-cc and size $\le \kappa$, forcing with P does not change the class of (strongly) regular cardinals $\ge \kappa$. If $\gamma \ge \kappa$ is regular and $\Vdash \dot{\alpha} < \gamma$, then there is $\beta < \gamma$ with $\Vdash \dot{\alpha} < \beta$. If $\gamma \ge \kappa$ is strongly regular, there exist exactly γ representatives from the P-names $\dot{\alpha}$ such that $\Vdash \dot{\alpha} < \gamma$. Thus we can take P-names $\dot{\tau}(\xi)$ so that for every strongly regular $\gamma \ge \kappa$

- if $\xi < \gamma$, then $\Vdash \dot{\tau}(\xi) < \gamma$ and

- if $\Vdash \dot\alpha < \gamma$, then there is $\xi < \gamma$ with $\Vdash \dot\alpha = \dot\tau(\xi)$.

For $(p,q) \in P \times E(\kappa,\lambda)$ define

$$\pi(p,q) = (p,\dot q),$$

where $\dot q$ is a P-name such that

- $\Vdash \operatorname{dom}\dot q = \operatorname{dom} q$ and
- $\Vdash \dot q(\gamma) = \langle \dot\tau(q(\gamma)(\eta)) : \eta \in \operatorname{dom} q(\gamma)\rangle$ for every $\gamma \in \operatorname{dom} q$.

Since P has κ-cc, $\operatorname{dom} q$ remains an Easton subset of $\lambda - \kappa$ after forcing with P. Moreover $\Vdash \dot q(\gamma)(\eta) < \gamma$ by $q(\gamma)(\eta) < \gamma$ and the choice of $\dot\tau(\xi)$. Thus $\pi(p,q) \in P * \dot E(\kappa,\lambda)$.

Claim. *π is a projection.*

Proof. It is easy to see that π is order-preserving and $\pi(1_P,\emptyset) = (1_P,\emptyset)$.

Now assume $(p,q) \in P \times E(\kappa,\lambda)$ and $(p',\dot q') \le \pi(p,q)$ in $P * \dot E(\kappa,\lambda)$. Let $(p,\dot q) = \pi(p,q)$. Define

$$p^* = p'.$$

Then $p^* \le p$ by $(p',\dot q') \le (p,\dot q)$. It remains to find $q^* \le q$ in $E(\kappa,\lambda)$ such that $\pi(p^*,q^*) \le (p',\dot q')$ in $P * \dot E(\kappa,\lambda)$. Define

$$d^* = \{\gamma : \exists r \in P(r \Vdash \gamma \in \operatorname{dom}\dot q')\}.$$

Since P has κ-cc and $\Vdash \dot q' \in \dot E(\kappa,\lambda)$, d^* is an Easton subset of $\lambda - \kappa$. Moreover $\operatorname{dom} q \subset d^*$ because

$$p' \Vdash \operatorname{dom} q = \operatorname{dom}\dot q \subset \operatorname{dom}\dot q' \subset d^*.$$

The left equality follows from the definition of $\dot q$, the middle inclusion from $(p',\dot q') \le (p,\dot q)$, and the right inclusion from the definition of d^*.

Fix $\gamma \in d^*$. Since P has κ-cc and $\Vdash \dot q' \in \dot E(\kappa,\lambda)$, there is $\delta^*_\gamma < \kappa$ such that $\Vdash \gamma \in \operatorname{dom}\dot q' \to \operatorname{dom}\dot q'(\gamma) \subset \delta^*_\gamma$. If $\gamma \in \operatorname{dom} q$, then $\operatorname{dom} q(\gamma) \subset \delta^*_\gamma$ because

$$p' \Vdash \operatorname{dom} q(\gamma) = \operatorname{dom}\dot q(\gamma) \subset \operatorname{dom}\dot q'(\gamma) \subset \delta^*_\gamma.$$

The left equality follows from the definition of $\dot q$, the middle inclusion from $(p',\dot q') \le (p,\dot q)$, and the right inclusion from $p' \Vdash \gamma \in \operatorname{dom}\dot q'$ and the choice of δ^*_γ.

Now define q^* with $\operatorname{dom} q^* = d^*$ and $\operatorname{dom} q^*(\gamma) = \delta^*_\gamma$ for every $\gamma \in d^*$ so that

- $q^*(\gamma)(\eta) = q(\gamma)(\eta)$ if $\gamma \in \operatorname{dom} q$ and $\eta \in \operatorname{dom} q(\gamma)$, or else
- $q^*(\gamma)(\eta)$ is the minimal ξ such that
  $\Vdash \gamma \in \operatorname{dom}\dot q' \wedge \eta \in \operatorname{dom}\dot q'(\gamma) \to \dot q'(\gamma)(\eta) = \dot\tau(\xi)$.

Note that $q^*(\gamma)(\eta) < \gamma$ by $q \in E(\kappa,\lambda)$ in the first case, and by $\Vdash \dot q \in \dot E(\kappa,\lambda)$ and the choice of $\dot\tau(\xi)$ in the second case. Thus $q^* \in E(\kappa,\lambda)$ and $q^* \le q$.

Let $(p^*,\dot q^*) = \pi(p^*,q^*)$. Since $p^* = p'$, it remains to prove that $p' \Vdash \dot q^* \le \dot q'$. First recall that

$$\Vdash \operatorname{dom}\dot q^* = \operatorname{dom} q^* = d^* \supset \operatorname{dom}\dot q'.$$

It remains to prove that for every $\gamma \in d^*$ and $\eta \in \delta^*_\gamma$

$$p' \Vdash \gamma \in \operatorname{dom}\dot q' \wedge \eta \in \operatorname{dom}\dot q'(\gamma) \to \dot q^*(\gamma)(\eta) = \dot q'(\gamma)(\eta).$$

If $\gamma \in \operatorname{dom} q$ and $\eta \in \operatorname{dom} q(\gamma)$, the claim follows from

$$p' \Vdash \dot q^*(\gamma)(\eta) = \dot\tau(q^*(\gamma)(\eta)) = \dot\tau(q(\gamma)(\eta)) = \dot q'(\gamma)(\eta).$$

The left equality follows from the definition of $\dot{q}^*$, the middle from that of q^*, and the right from $(p', \dot{q}') \leq (p, \dot{q})$.

In the remaining case the claim follows from

$$\Vdash \gamma \in \operatorname{dom} \dot{q}' \wedge \eta \in \operatorname{dom} \dot{q}'(\gamma) \to \dot{q}^*(\gamma)(\eta) = \dot{r}(q^*(\gamma)(\eta)) = \dot{q}'(\gamma)(\eta).$$

The left equality follows from the definition of $\dot{q}^*$, and the right from that of q^*. □

This completes the proof. □

Remark. Lemma 4 should hold for suitable modifications of the collapses of Levy and Silver. See [13] or [14] for the corresponding lemma for the modified Silver collapse and the resulting model in which a saturated filter exists and Chang's conjecture holds.

In [11] Laver introduced a poset $L(\kappa, \lambda)$, here called the Laver collapse. It is the product of collapsing posets with Easton support and bounded height. Using Kunen's method Laver constructed a forcing of the form $P * \dot{L}(\kappa, \lambda)$, which produces an $(\omega_2, \omega_2, \omega)$-saturated filter on ω_1. Although Lemma 4 should hold for a suitable modification of the Laver collapse as well, we need to work with the Easton collapse because a projection, say from $L(\mu, \lambda - \kappa)$ to $L(\kappa, \lambda)$ is not available to us. For the same reason we cannot substitute the collapses of Levy or of Silver for the Easton collapse.

For a P-name $\dot{Q}$ for a poset let $T(P, \dot{Q})$ denote the term forcing. It is known that the identity map from $P \times T(P, \dot{Q})$ to $P * \dot{Q}$ is a projection. See [5] for details. In [1] Cummings observed that $T(P, {}^{<\dot{\kappa}}\gamma)$ is equivalent to ${}^{<\kappa}\gamma$ if P has κ-cc and size $\leq \kappa$, and $\gamma^{<\kappa} = \gamma$. The proof of Lemma 4 shows in effect that $T(P, \dot{E}(\kappa, \lambda))$ is equivalent to $E(\kappa, \lambda)$. To see that the filter in our model is λ-saturated only, it suffices to prove this fact or even Lemma 4 without additional clauses.

4. The main theorem

This section is devoted to the proof of

Theorem. *Suppose that κ is almost huge with target λ and λ is Mahlo. Let $\mu < \nu$ be both regular with $\mu < \kappa \leq \nu < \lambda$. Then $E(\mu, \kappa) * \dot{E}(\nu, \lambda)$ forces that $\mathcal{P}_\kappa \nu$ carries a (λ, λ, μ)-saturated normal filter.*

Proof. Let $j : V \to M$ witness that κ is almost huge with target λ, i.e. $\kappa = \operatorname{crit}(j)$, $\lambda = j(\kappa)$ and ${}^{<\lambda}M \subset M$. Then we have $j(E(\mu, \kappa)) = E(\mu, \lambda)$, which is canonically isomorphic to $E(\mu, \kappa) \times E(\mu, \lambda - \kappa)$. As stated in §3, there is a projection from $E(\mu, \lambda - \kappa)$ to $E(\nu, \lambda)$. Since $E(\mu, \kappa)$ has κ-cc and size κ, there is a projection from $E(\mu, \kappa) \times E(\nu, \lambda)$ to $E(\mu, \kappa) * \dot{E}(\nu, \lambda)$ as in Lemma 4. Thus we get a projection $\pi : E(\mu, \lambda) \to E(\mu, \kappa) * \dot{E}(\nu, \lambda)$ such that $\pi(p)$ has the form $(p|\kappa, \dot{q})$, where $E(\mu, \kappa) \Vdash \operatorname{dom} \dot{q} = \operatorname{dom} p - \nu$ and each $\dot{q}(\gamma)$ depends only on $p(\gamma)$.

Now let $\bar{G} \subset E(\mu, \lambda)$ be V-generic. Then π"$\bar{G}$ generates a V-generic filter over $E(\mu, \kappa) * \dot{E}(\nu, \lambda)$, say $G * H$. We claim that $V[G][H]$ is the desired model. Since j"$G = G \subset \bar{G}$, we can lift $j : V \to M$ to $j : V[G] \to M[\bar{G}]$ in $V[\bar{G}]$. Since λ is Mahlo in V, $E(\mu, \lambda)$ has λ-cc in V. Hence we have ${}^{<\lambda}M[\bar{G}] \subset M[\bar{G}]$ in $V[\bar{G}]$ by ${}^{<\lambda}M \subset M$ in V.

Work in $V[G]$. Since $E(\mu,\kappa)$ has size κ in V, λ remains Mahlo and hence $E(\nu,\lambda)$ has λ-cc. Thus a nice $E(\nu,\lambda)$-name for a subset of $\mathcal{P}_\kappa\nu$ can be viewed as an $E(\nu,\xi)$-name for some $\xi<\lambda$. So we can list the set of all such names with cofinal repetition as $\{\dot{X}_\xi : \xi<\lambda\}$.

Now work in $V[\bar{G}]$. Since ${}^{<\lambda}M[\bar{G}] \subset M[\bar{G}]$, $E(j(\nu), j(\xi))^{M[\bar{G}]}$ is λ-directed closed for $\xi<\lambda$. So we can define for $\xi<\lambda$

$$r_\xi = \text{the greatest lower bound of } j``(H\cap E(\nu,\xi)^{V[G]}) \text{ in } E(j(\nu),j(\xi))^{M[\bar{G}]}.$$

Note that $\xi<\zeta<\lambda$ implies $r_\zeta|j(\xi)=r_\xi$. Thus we can define a descending sequence $\langle r^*_\xi : \xi<\lambda\rangle$ in $E(j(\nu),j(\lambda))^{M[\bar{G}]}$ by recursion so that

- $r^*_\xi \le r_\xi$ in $E(j(\nu),j(\xi))^{M[\bar{G}]}$ and
- if $\dot{X}_\xi$ is a $E(\nu,\xi)^{V[G]}$-name, then r^*_ξ decides $j``\nu \in j(\dot{X}_\xi)$ in $M[\bar{G}]$.

Define

$$U = \{(\dot{X}_\xi)_H : \xi<\lambda \wedge M[\bar{G}] \models r^*_\xi \Vdash j``\nu \in j(\dot{X}_\xi)\}.$$

Standard arguments show that U is a $V[G][H]$-normal ultrafilter on $\mathcal{P}_\kappa\nu^{V[G][H]}$.

Finally we work in $V[G][H]$. Since $E(\mu,\lambda)$ projects down to $E(\mu,\kappa)*\dot{E}(\nu,\lambda)$ in V, there is a $E(\mu,\lambda)^V/(G*H)$-name $\dot{U}$ such that

$$E(\mu,\lambda)^V/(G*H) \Vdash \dot{U} \text{ is a } V[G][H]\text{-normal ultrafilter on } \mathcal{P}_\kappa\nu^{V[G][H]}.$$

Define

$$F = \{X \subset \mathcal{P}_\kappa\nu : E(\mu,\lambda)^V/(G*H) \Vdash X \in \dot{U}\}.$$

Standard arguments show that F is a normal filter on $\mathcal{P}_\kappa\nu$. We claim that F is (λ,λ,μ)-saturated. Standard arguments show that

$$X \mapsto \sum\{p \in E(\mu,\lambda)^V/(G*H) : p \Vdash X \in \dot{U}\}$$

defines a complete embedding of F^+ into $B(E(\mu,\lambda)^V/(G*H))$. So it suffices to prove that $E(\mu,\lambda)^V/(G*H)$ has (λ,λ,μ)-cc. Let $\{p_\xi : \xi<\lambda\} \subset E(\mu,\lambda)^V/(G*H)$. Since $E(\mu,\kappa)$ has κ-cc and forces $\dot{E}(\nu,\lambda)$ to be κ-closed in V, it suffices to find $S\in[\lambda]^\lambda$ such that if $x\in[S]^\mu$ and $\langle p_\xi : \xi\in x\rangle \in V$, $\{p_\xi : \xi\in x\}$ has a common extension in $E(\mu,\lambda)^V/(G*H)$.

Let R be the set of regular cardinals $<\lambda$ in V. Since λ is Mahlo and $E(\mu,\kappa)*\dot{E}(\nu,\lambda)$ has λ-cc in V, R is stationary. As in the proof of Lemma 1 we get a stationary $S\subset R$ such that $\{\operatorname{dom} p_\xi : \xi\in S\}$ forms a Δ-system, say with root d. Moreover we may assume that $p_\xi|d$ is constant and $\operatorname{dom} p_\xi \cap \kappa \subset d$ for $\xi\in S$.

Suppose $x\in[S]^\mu$ and $\langle p_\xi : \xi\in x\rangle\in V$. Define $p=\bigcup_{\xi\in x} p_\xi$. We claim that p is a lower bound of $\{p_\xi : \xi\in x\}$ in $E(\mu,\lambda)^V/(G*H)$. Since $p_\xi|d$ is constant on S, p is a lower bound of $\{p_\xi : \xi\in x\}$ in $E(\mu,\lambda)^V$.

It remains to prove that $\pi(p)\in G*H$. Let $(p|\kappa,\dot{q})=\pi(p)$ and $(p_\xi|\kappa,\dot{q}_\xi)=\pi(p_\xi)$ for $\xi\in S$. Since $p_\xi|\kappa$ is constant on S, we have $p|\kappa=p_\xi|\kappa$ for every $\xi\in S$. Hence $p|\kappa\in G$ by $(p_\xi|\kappa,\dot{q}_\xi)=\pi(p_\xi)\in G*H$. To see that $\dot{q}_G\in H$, note first that $(\dot{q}_\xi)_G\in H$ by $(p_\xi|\kappa,\dot{q}_\xi)\in G*H$. Since $\operatorname{dom}(\dot{q}_\xi)_G=\operatorname{dom} p_\xi-\nu$, $\{\operatorname{dom}(\dot{q}_\xi)_G : \xi\in S\}$ forms a Δ-system with root $d-\nu$. Moreover $(\dot{q}_\xi)_G|(d-\nu)$ is constant on S. Thus $\dot{q}_G=\bigcup_{\xi\in x}(\dot{q}_\xi)_G$ is the greatest lower bound of $\{(\dot{q}_\xi)_G : \xi\in x\}$ in $E(\nu,\lambda)^{V[G]}$. Therefore $\dot{q}_G\in H$, as desired. $\square$

Remark. For the moment let us assume that κ is huge with target λ. As remarked in §3, our strategy requires forcing with Easton collapses rather than with Laver collapses. This requires in turn invoking an argument of Magidor [8] that involves local master conditions, even under the stronger hypothesis as above. In fact we can dispense with the argument in the case $\nu > \kappa$. Moreover the proof in this case, if modified as in [12], shows that $[\lambda]^{\kappa}$ carries a (λ, λ, μ)-saturated κ-complete filter in the extension.

In [11] Laver observed that a strong form of Chang's conjecture holds in his model. We do not know whether our model in the case $\nu = \kappa$ satisfies the conjecture.

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