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Remarks on Scheepers' theorem on the cardinality of Lindelöf spaces

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Abstract

Scheepers proved the following theorem: If the existence of a measurable cardinal is consistent, then it is consistent that every points-G_δ indestructibly Lindelöf space has cardinality at most $2^{\aleph_0}$. In this paper, We will review the proof of Scheepers' theorem and remark that we can slightly improve the statement of the main theorem in Scheepers' paper and yet simplify its proof.

1 Introduction

We call a topological space in which each singleton is a G_δ set a *points-G_δ space.* A first-countable T_1 space is a points-G_δ space. Arhangel'skiĭ proved that any first-countable T_2 Lindelöf space has cardinality at most $2^{\aleph_0}$, and asked whether the assumption "first-countable T_2 Lindelöf" on the space may be weakened to "points-G_δ T_2 Lindelöf." Note that a points-G_δ space is T_1 but not necessarily T_2.

Arhangel'skiĭ also proved that any points-G_δ Lindelöf space has cardinality less than the least measurable cardinal. Juhász showed that this upper bound is optimal, that is, he proved that there is a points-G_δ (but not T_2) Lindelöf space of arbitrarily large cardinality below the least measurable.

To investigate Arhangel'skiĭ's question, Tall introduced the notion of *indestructibly Lindelöf* spaces.

Definition 1.1. For a topological space (X,τ) and a forcing notion $\mathbb{P}$, $\tau^{\mathbb{P}}$ denotes a $\mathbb{P}$-name representing the topology on X generated by τ in a generic extension by $\mathbb{P}$.

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We say a forcing notion $\mathbb{P}$ *destroys* a Lindelöf space (X, τ) if we have

$$\Vdash_{\mathbb{P}} \text{“}(\check{X}, \tau^{\mathbb{P}}) \text{ is not Lindelöf.”}$$

A Lindelöf space (X, τ) is called an *indestructibly Lindelöf* space if (X, τ) is not destroyed by any $<\omega_1$-closed forcing notion.

Scheepers and Tall proved that indestrutible Lindelöfness is nicely characterized in terms of games of transfinite length.

For a topological space (X, τ), the *game* $\mathsf{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ *on* (X, τ) is played by two players ONE and TWO for ω_1 innings as follows. In the inning $\beta < \omega_1$, ONE chooses an open cover $\mathcal{U}_\beta$ of X and then TWO chooses an open set H_β from $\mathcal{U}_\beta$. TWO wins in this game if $\{H_\beta : \beta < \omega_1\}$ covers X.

For the use in the present paper, we define a slightly modified form of this game. The *game* $\mathsf{G}_1^{<\omega_1}(\mathcal{O}, \mathcal{O})$ *on* (X, τ) is played in the same way as $\mathsf{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$, but TWO wins if there is $\gamma < \omega_1$ such that $\{H_\beta : \beta < \gamma\}$ covers X, otherwise ONE wins.

The equivalence (1) $\Leftrightarrow$ (2) in the following theorem is due to Scheepers and Tall [8, Theorem 1]. The equivalence (2) $\Leftrightarrow$ (3) is easily checked.

Theorem 1.2. [6, Theorem 2.6] *For a space (X, τ) the following are equivalent.*

(1) *(X, τ) is an indestructibly Lindelöf space.*
(2) *(X, τ) is a Lindelöf space and* ONE *does not have a winning strategy in $\mathsf{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ on (X, τ).*
(3) ONE *does not have a winning strategy in $\mathsf{G}_1^{<\omega_1}(\mathcal{O}, \mathcal{O})$ on (X, τ).*

Tall [9] proved that, assuming the consistency of the existence of a supercompact cardinal, it is consistent that $2^{\aleph_0} = \aleph_1$ and any points-G_δ indestructibly Lindelöf space has size at most $\aleph_1$. After that, Scheepers showed that the large cardinal assumption can be weakened to the consistency of the existence of a measurable cardinal.

Theorem 1.3. [7, Corollary 7] *If the existence of a measurable cardinal is consistent, then it is consistent that $2^{\aleph_0} = \aleph_1$ and any points-G_δ indestructibly Lindelöf space has size at most $\aleph_1$.*

We will give a proof of the above theorem in an improved way, which simplifies the original proof.

2 Combinatorial aspects of precipitous ideals

Let $\mathcal{I}$ be a nonprincipal $\aleph_1$-complete ideal on an infinite cardinal κ. We say a subset A of κ is *$\mathcal{I}$-positive* if $A \notin \mathcal{I}$, and let $\mathcal{I}^+$ denote the collection of all $\mathcal{I}$-positive subsets of κ (that is, $\mathcal{I}^+ = \mathcal{P}(\kappa) \setminus \mathcal{I}$).

We define the *precipitous ideal game* $\mathsf{PIG}(\mathcal{I})$ as follows. The game is played by two players ONE and TWO for ω innings. In each inning $n < \omega$, ONE first chooses $O_n \in \mathcal{I}^+$ and then TWO chooses $T_n \in \mathcal{I}^+$, obeying the rule that for each n, $O_n \supseteq T_n \supseteq O_{n+1}$. TWO wins in this game if $\bigcap\{T_n : n < \omega\}$ is nonempty, otherwise ONE wins.

We refer the reader to [4, Section 22] for the definition and basic properties of a precipitous ideal. In the present paper we do not require a precipitous ideal on κ to be κ-complete unless explicitly stated.*1 Galvin, Jech and Magidor [2] investigated game-theoretic properties of precipitous ideals. They proved that the precipitousness of the ideal $\mathcal{I}$ is characterized using the game $\mathsf{PIG}(\mathcal{I})$ in the following way.

Theorem 2.1. ([5, Theorem 1], [2]) *For an ideal $\mathcal{I}$ on a cardinal κ, $\mathcal{I}$ is precipitous if and only if* ONE *does not have a winning strategy in* $\mathsf{PIG}(\mathcal{I})$.

The game $\mathsf{PIG}(\mathcal{I})$ looks like a well-known *descending chain game* (also called a *Banach–Mazur game*) but is slightly different. We define a descending chain game $\mathsf{DG}^{\omega}(P, \leq)$ on a partially ordered set $(P, \leq)$ as follows. The game is played by two players ONE and TWO for ω innings. In each inning $n < \omega$, ONE first chooses $o_n \in P$ and then TWO chooses $t_n \in P$, forming a descending sequence $o_0 \geq t_0 \geq o_1 \geq \cdots$. TWO wins in this game if the set $\{t_n : n < \omega\}$ has a lower bound in P, otherwise ONE wins. Note that the game $\mathsf{DG}^{\omega}(\mathcal{I}^+, \subseteq)$ is played in the same way as $\mathsf{PIG}(\mathcal{I})$ but TWO attempts to make the set $\bigcap\{T_n : n < \omega\}$ $\mathcal{I}$-positive, not just nonempty.

It is easy to see that, for a measure ultrafilter $\mathcal{U}$ on a measurable cardinal κ, the dual ideal $\mathcal{I} = \mathcal{U}^*$ of $\mathcal{U}$ has the property that TWO has a winning strategy in the game $\mathsf{DG}^{\omega}(\mathcal{I}^+, \subseteq)$, and hence $\mathcal{I}$ is precipitous. It is well-known that, after collapsing κ to $\aleph_1$ by Lévy collapse, $\overline{\mathcal{I}}$, the ideal on $\aleph_1$ which is generated by $\mathcal{I}$, is precipitous. But then TWO cannot have a winning strategy in $\mathsf{DG}^{\omega}((\overline{\mathcal{I}})^+, \subseteq)$ by the following theorem.

Theorem 2.2. [2, Theorem 1] *If $\kappa \leq 2^{\aleph_0}$, then* TWO *does not have a winning strategy in the game $\mathsf{DG}^{\omega}(\mathcal{J}^+, \subseteq)$ for any ideal $\mathcal{J}$ on κ.*

On the other hand, when we collapse a measurable cardinal κ to $\aleph_2$ or greater, then the ideal $\overline{\mathcal{I}}$ generated by the dual ideal $\mathcal{I}$ of a measure ultrafilter on κ retains the property that TWO has a winning strategy in the game $\mathsf{DG}^{\omega}((\overline{\mathcal{I}})^+, \subseteq)$ [2, Theorem 4].

Now we introduce a descending chain game of transfinite length, which was originally introduced by Foreman [1]. For a partially ordered set $(P, \leq)$, we define the game $\mathsf{DG}^{<\omega_1}_{\mathrm{TWO}}(P, \leq)$ as follows. The game is played by two players ONE and TWO for ω_1 innings. For each inning β for $\beta = 0$ or a successor ordinal β, ONE first

*1 In Scheepers' paper [7], a precipitous ideal in the sense of the present paper was called a *weakly precipitous ideal*, and the term "a precipitous ideal" was used only for a κ-complete precipitous ideal on κ.

chooses $o_\beta \in P$ and then TWO chooses $t_\beta \in P$, forming a descending sequence $o_0 \geq t_0 \geq o_1 \geq \cdots \geq t_\beta \geq o_{\beta+1} \geq t_{\beta+1} \geq \cdots$. For each inning β for a limit ordinal β, ONE does nothing and TWO chooses $t_\beta \in P$ so that $t_\xi \geq t_\beta$ for all $\xi < \beta$, if possible. TWO wins in this game if the play is sustained all over ω_1 innings, otherwise (that is, at some limit inning $\beta < \omega_1$ it happens that $\{t_\xi : \xi < \beta\}$ does not have a lower bound in P and the play gets stuck) ONE wins.

We will use the following theorem, which was stated in the remark after Theorem 10 in Veličković's paper [10] (see also [3, Corollary 3.2]).

Theorem 2.3. *For a partially ordered set $(P, \leq)$,* TWO *has a winning strategy in* $\mathsf{DG}^\omega(P, \leq)$ *if and only if* TWO *has a winning strategy in* $\mathsf{DG}^{<\omega_1}_{\mathrm{TWO}}(P, \leq)$.

In particular, for an ideal $\mathcal{I}$ on κ, TWO has a winning strategy in $\mathsf{DG}^\omega(\mathcal{I}^+, \subseteq)$ if and only if TWO has a winning strategy in $\mathsf{DG}^{<\omega_1}_{\mathrm{TWO}}(\mathcal{I}^+, \subseteq)$.

3 The main theorem

This section is devoted to the proof of the following theorem, which is a slightly improved form of a theorem due to Scheepers [7, Theorem 4] yet the proof is simplified.*2

Theorem 3.1. *Assume that there is a nonprincipal $\aleph_1$-complete ideal $\mathcal{I}$ on a cardinal κ such that* TWO *has a winning strategy in the game $\mathsf{DG}^\omega(\mathcal{I}^+, \subseteq)$. Then each points-$G_\delta$ indestructibly Lindelöf space has cardinality less than κ.*

We will heavily use the following combinatorial lemma, which corresponds to [7, Lemma 2] but is stated in a little stronger form.

Lemma 3.2. *Let $\mathcal{I}$ be a nonprincipal $\aleph_1$-complete ideal on a cardinal κ. Let X be a points-G_δ topological space with $\kappa \subseteq X$. Then, for $x \in X$, $B \subseteq \kappa$ with $B \in \mathcal{I}^+$ and a sequence $\langle U_k : k < \omega\rangle$ of open neighborhoods of x with $\bigcap_{k<\omega} U_k = \{x\}$, there is an $n < \omega$ such that $B \setminus U_n \in \mathcal{I}^+$.*

This lemma is easily derived from the following observation.

Lemma 3.3. *Let $\mathcal{I}$ be an $\aleph_1$-complete ideal on a cardinal κ. Suppose that $T \in \mathcal{I}^+$ and $\langle S_k : k < \omega\rangle$ is a sequence of subsets of κ such that $\bigcap_{k<\omega} S_k \in \mathcal{I}$. Then there is an $n < \omega$ such that $T \setminus S_n \in \mathcal{I}^+$.*

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*2 The original form of Scheepers' theorem requires the assumption that TWO has a winning *tactic* in $\mathsf{DG}^\omega(\mathcal{I}^+, \subseteq)$, where a tactic means a strategy which suggests a move depending only on the opponent's last move. In the original proof the tactic was used to clear limit innings in the game $\mathsf{G}_1^{<\omega_1}(\mathcal{O}, \mathcal{O})$. Actually, the use of Theorem 2.3 eliminates the need of a tactic and also significantly simplifies the proof.

Proof. Suppose not. Choose T and $\langle S_n : n < \omega\rangle$ so that $T \in \mathcal{I}^+$, $\bigcap_{k<\omega} S_k \in \mathcal{I}$ and $T \setminus S_k \in \mathcal{I}$ for all $k < \omega$. Then we see $T \subseteq \left(\bigcup_{k<\omega}(T \setminus S_k)\right) \cup \bigcap_{k<\omega} S_k$. Since $\mathcal{I}$ is $\aleph_1$-complete, the set of the right-hand side belongs to $\mathcal{I}$, which contradicts $T \in \mathcal{I}^+$. □

Proof of Lemma 3.2. Apply Lemma 3.3 to $T = B$ and $S_k = U_k \cap \kappa$ for each k. Note that $\bigcap_{k<\omega} S_k$ is $\{x\}$ or $\emptyset$, and in either case belongs to $\mathcal{I}$ because $\mathcal{I}$ is nonprincipal. □

Now we are ready to prove the main theorem.

Proof of Theorem 3.1. By the assumption and Theorem 2.3, we choose a nonprincipal $\aleph_1$-complete ideal $\mathcal{I}$ on κ so that there is a winning strategy σ for TWO in the game $\mathsf{DG}^{<\omega_1}_{\mathrm{TWO}}(\mathcal{I}^+, \subseteq)$. We regard σ as a function which maps a sequence $\langle C_\beta : \beta \leq \gamma\rangle$ of $\mathcal{I}$-positive sets to an $\mathcal{I}$-positive set. Since ONE is supposed to "pass" in each limit inning by the rule, we do not care about C_δ for limit $\delta \leq \gamma$.

Let X be a topological space with $|X| \geq \kappa$. We shall construct a winning strategy Σ for ONE in the game $\mathsf{G}^{<\omega_1}_1(\mathcal{O}, \mathcal{O})$ on the space X. By Theorem 1.2, this will show that X cannot be an indestructibly Lindelöf space.

We may assume that $\kappa \subseteq X$. For each $x \in X$ fix a sequence $\langle U_{x,k} : k < \omega\rangle$ of open neighborhoods of x such that $\bigcap_{k<\omega} U_{x,k} = \{x\}$. We will design Σ so that Σ always suggests a cover of the form $\{U_{x,f(x)} : x \in X\}$ for some function f from X to ω, and we assume that TWO responds with a point $x \in X$ instead of the corresponding open set $U_{x,f(x)}$ from the cover.

In the beginning, let $C_0 = \kappa$ and $B_0 = \sigma(\langle C_0\rangle)$. For each $x \in X$, apply Lemma 3.2 to x, B_0 and $\langle U_{x,k} : k < \omega\rangle$ to get $f_0(x) < \omega$ such that $C_{x,0} = B_0 \setminus U_{x,f_0(x)} \in \mathcal{I}^+$. Let $\Sigma(\langle\rangle) = \{U_{x,f_0(x)} : x \in X\}$. Suppose that TWO responds with $x_0 \in X$. Let $C_1 = C_{x_0,0}$. We put C_1 into another game $\mathsf{DG}^{<\omega_1}_{\mathrm{TWO}}(\mathcal{I}^+, \subseteq)$ as ONE's move in the inning 1, and let TWO respond with $B_1 = \sigma(\langle C_0, C_1\rangle)$. Then we bring B_1 back to the game $\mathsf{G}^{<\omega_1}_1(\mathcal{O}, \mathcal{O})$, and define $\Sigma(\langle x_0\rangle)$ in the same way, using B_1 instead of B_0.

We just repeat this procedure to construct Σ inductively. Suppose that the game $\mathsf{G}^{<\omega_1}_1(\mathcal{O}, \mathcal{O})$ has been played for initial γ innings. We have two sequences, namely, a sequence $\langle x_\beta : \beta < \gamma\rangle$ of points of X which describes TWO's initial moves in the game $\mathsf{G}^{<\omega_1}_1(\mathcal{O}, \mathcal{O})$ against Σ, and a sequence $\langle C_\beta : \beta \leq \gamma\rangle$ of ONE's initial moves in the game $\mathsf{DG}^{<\omega_1}_{\mathrm{TWO}}(\mathcal{I}^+, \subseteq)$. Let $B_\gamma = \sigma(\langle C_\beta : \beta \leq \gamma\rangle)$. For each $x \in X$, apply Lemma 3.2 to x, B_γ and $\langle U_{x,k} : k < \omega\rangle$ to get $f_\gamma(x) < \omega$ such that $C_{x,\gamma} = B_\gamma \setminus U_{x,f_\gamma(x)} \in \mathcal{I}^+$. Let $\Sigma(\langle x_\beta : \beta < \gamma\rangle) = \{U_{x,f_\gamma(x)} : x \in X\}$. If TWO responds with $x_\gamma \in X$, then let $C_{\gamma+1} = C_{x_\gamma,\gamma}$ and go into the next step.

Since σ is a winning strategy for TWO in $\mathsf{DG}^{<\omega_1}_{\mathrm{TWO}}(\mathcal{I}^+, \subseteq)$, this induction goes through ω_1 innings without getting stuck, and for each $\gamma < \omega_1$ the set B_γ is not covered by the collection $\{U_{x_\beta,f_\beta(x_\beta)} : \beta < \gamma\}$ of open sets chosen by TWO in the initial γ innings. This means that, as long as ONE follows the strategy Σ, TWO cannot complete an open cover of X in any intermediate stage, and hence Σ is a winning strategy for ONE in the game $\mathsf{G}_1^{<\omega_1}(\mathcal{O},\mathcal{O})$. $\square$

Using Theorem 3.1, we can prove Theorem 1.3 in the following way. Let κ be a measurable cardinal and $\mathcal{I}$ the dual ideal of a measure ultrafilter on κ. Collapse κ to $\aleph_2$ by Lévy collapse, and consider the ideal $\overline{\mathcal{I}}$ on $\aleph_2$ which is generated by $\mathcal{I}$. By the observation in Section 2, TWO has a winning strategy in the game $\mathsf{DG}^\omega((\overline{\mathcal{I}})^+,\subseteq)$ and $2^{\aleph_0}=\aleph_1$, and by Theorem 3.1, any points-G_δ indestructibly Lindelöf space has cardinality at most $\aleph_1$. See [7, Section 4] for further consequences of Theorem 3.1.

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