Title: Remarks on the coloring number of graphs (Interplay between large cardinals and small cardinals)

Author(s): Fuchino, Sakae

Citation: 数理解析研究所講究録 (2011), 1754: 6-16

Issue Date: 2011-08

URL: http://hdl.handle.net/2433/171198

Type: Departmental Bulletin Paper

Textversion: publisher

Kyoto University
Remarks on the coloring number of graphs

神戸大学大学院・システム情報学研究科 渋野 昌 (Sakaé Fuchino)*
Graduate School of System Informatics
Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
fuchino@diamond.kobe-u.ac.jp

Abstract

We give two characterizations of graphs with coloring number $\leq \kappa$ in terms of elementary submodels; one under ZFC and another under SSH and the version of very weak square principle of [8]. These characterizations suggest that the graphs with coloring number $\leq \kappa$ behave very much like the Boolean algebras with $\kappa$-Freese-Nation property (see [5], [8]).

1 Introduction

A graph $G = (G, K)$ ($K \subseteq [G]^2$) has coloring number $\leq \kappa$ (notation: $\text{col}(G) \leq \kappa$) if there is a well ordering $\subseteq$ on $G$ such that $K^a_{\subseteq} = \{b \in G : b \subseteq a \text{ and } \{a, b\} \in$...
has cardinality \(<\kappa\) for all \(a \in G\) ([3]). The coloring number \(\text{col}(G)\) of \(G\) is then defined as the minimum of such \(\kappa\)'s. It is easy to see that the chromatic number \(\chi(G)\) of \(G\) is less or equal to \(\text{col}(G)\).

The purpose of this note is to show that the graphs with coloring number \(\leq \kappa\) behave quite similarly to the Boolean algebras with \(\kappa\)-Freese-Nation property (see e.g. [5], [8]).

In Section 2 we give a characterization of graphs with coloring number \(\leq \kappa\) in terms of elementary submodels (Theorem 2.4). As an application of the characterization, we present in Section 3 a short proof of the countability of the coloring number of the plane.

In Section 4 we show that the characterization of Section 2 can be yet sharpened under SSH and the version of the very weak square principle introduced in [8] (Theorem 4.2).

Both Theorems 2.4 and 4.2 find their parallels in the theory of Boolean algebras with \(\kappa\)-Freese-Nation property (see Proposition 3 and Theorem 10 in [8]).

The following theorem also underlines the analogy between the Boolean algebras with the \(\kappa\)-Freese-Nation property and the graphs with coloring number \(\leq \kappa\) in the case of \(\kappa = \aleph_0\). Note that Boolean algebras with \(\aleph_0\)-Freese Nation property are also called openly generated.

If \(G = \langle G, K \rangle\) is graph then we identify any subset \(H\) of \(G\) with the graph \(G \upharpoonright H = \langle H, K \cap [H]^2 \rangle\).

**Theorem 1.1** ([6] and [7]). The following assertions are equivalent over ZFC:

\begin{enumerate}
  \item[(\(\alpha\))] For any Boolean algebra \(B\) if there are club many subalgebras of \(B\) of cardinality \(\aleph_1\) which are openly generated then \(B\) is openly generated.
  \item[(\(\beta\))] For any graph \(G\) if \(\text{col}(H) \leq \aleph_0\) for every \(H \in [G]^\aleph_1\) then \(\text{col}(G) \leq \aleph_0\).
\end{enumerate}

Theorem 1.1 in the formulation as above is a sort of bluff since we actually proved that each of the assertions \((\alpha)\) and \((\beta)\) is equivalent to the set-theoretic principle FRP introduced in [4].

**2 A characterization of graphs with coloring number \(\leq \kappa\)**

We use here the following notations. The first one was already used in the introduction:

For a linear ordering \(\subseteq\) on a graph \(G = \langle G, K \rangle\) we denote
(2.1) \( K^a_{\subseteq} = \{ b \in G : b \subseteq a \text{ and } \{a, b\} \in K \} \).

If \( H \subseteq G \) then we write

(2.2) \( K^H,a_{\subseteq} = \{ b \in H : b \subseteq a \text{ and } \{a, b\} \in K \} \).

For a graph \( G = \langle G, K \rangle \), \( H \subseteq G \) and \( a \in G \), let

(2.3) \( K^a_H = \{ b \in H : \langle a, b \rangle \in K \} \).

We write \( H \subseteq_\kappa G \) if \( |K^a_H| < \kappa \) for all \( a \in G \setminus H \).

A mapping \( f : G \rightarrow [G]^{<\kappa} \) is a \( \kappa \)-coloring mapping on \( G \) if for any \( a, b \in G \) with \( \{a, b\} \in K \), at least one of \( a \in f(b) \) or \( b \in f(a) \) holds.

Lemma 2.1 ([7]). For any graph \( G \) and any infinite cardinal \( \kappa \), the following are equivalent:

(a) \( \text{col}(G) \leq \kappa \).

(b) There is a \( \kappa \)-coloring mapping on \( G \).

Proof. (a) \( \Rightarrow \) (b): Suppose that \( \text{col}(G) \leq \kappa \) and let \( \subseteq \) be a well-ordering on \( G \) such that \( |K^a_{\subseteq}| < \kappa \) for all \( a \in G \). Then \( f : G \rightarrow [G]^{<\kappa} \) defined by \( f(a) = K^a_{\subseteq} \) for \( a \in G \) is a \( \kappa \)-coloring mapping.

(b) \( \Rightarrow \) (a): Suppose that \( f : G \rightarrow [G]^{<\kappa} \) is a \( \kappa \)-coloring mapping on \( G \). Let \( \subseteq \) be a well-ordering on \( G \) such that all initial segments of \( G \) of order-type of the form \( \kappa \cdot \alpha \) with respect to \( \subseteq \) are closed with respect to \( f \). Then \( \subseteq \) is as desired:

Claim 2.1.1. \( |K^a_{\subseteq}| < \kappa \) for all \( a \in G \).

\( \vdash \) Suppose that \( a \in G \) is the \( \kappa \cdot \alpha + \beta \)'th element with respect to \( \subseteq \) where \( \beta < \kappa \). Then the first \( \kappa \cdot \alpha \) elements of \( G \) are closed with respect to \( f \) and hence if \( b \) is among them and \( \{a, b\} \in K \) then we have \( b \in f(a) \). Thus

\[
K^a_{\subseteq} \subseteq \{ b \in G : b \text{ is the } \gamma \text{'th element for some } \kappa \cdot \alpha \leq \gamma < \kappa \cdot \alpha + \beta \} \
\cup f(a).
\]

The right side of the inclusion has size \( < \kappa \) (note that we need here the infinity of \( \kappa \)). Hence \( |K^a_{\subseteq}| < \kappa \). \( \dashv \) (Claim 2.1.1)

\( \square \) (Lemma 2.1)

Lemma 2.2. Suppose that \( \langle G_\alpha : \alpha < \delta \rangle \) is a filtration of a graph \( G = \langle G, K \rangle \) and \( \kappa \) is an infinite cardinal. If \( G_\alpha \subseteq_\kappa G \) and \( \text{col}(G_{\alpha+1}) \leq \kappa \) for all \( \alpha < \delta \), then we have \( \text{col}(G) \leq \kappa \).
Proof. For a ∈ G let o(a) = min{α < δ : a ∈ Gα+1}. For α < δ, let ∈α+1 be a well-ordering of Gα+1 witnessing col(Gα+1) ≤ κ. Let ∈ be the ordering on G defined by:

\[(2.4) \quad a ∈ b ⇔ o(a) < o(b) \text{ or } \left( o(a) = o(b) \text{ and } a ∈_o(a)+1 b \right)\]

Then ∈ is a well ordering on G. The following claim shows that ∈ witnesses that G has coloring number < κ.

Claim 2.2.1. \( |K^a_∈| < κ \) for all a ∈ G.

\[\dashv \] For a ∈ G, we have \( K^a_∈ \subseteq K^a_∈ \cup K^G_{o(a)+1} \). Since the right side of the inclusion is of cardinality < κ, it follows that \( |K^a_∈| < κ \). \( \dashv \) (Claim 2.2.1)

\[\square\] (Lemma 2.2)

Lemma 2.3. Suppose that \( H_0 \) and \( H_1 \) are subsets of G with \( H_0 ⊆_κ G \) and \( H_1 ⊆_κ G \). Then we have \( H_0 ∩ H_1 ⊆_κ G \).

Proof. Suppose that a ∈ G \( \setminus (H_0 ∩ H_1) \). Then we have a ∈ G \( \setminus H_0 \) or a ∈ G \( \setminus H_1 \).
If a ∈ G \( \setminus H_0 \), then \( K^a_{H_0 ∩ H_1} \subseteq K^a_{H_0} \). And hence \( |K^a_{H_0 ∩ H_1}| < κ \). If a ∈ G \( \setminus H_1 \), then \( K^a_{H_0 ∩ H_1} \subseteq K^a_{H_1} \). And hence again we have \( |K^a_{H_0 ∩ H_1}| < κ \).

This shows \( H_0 ∩ H_1 ⊆_κ G \). \( \square \) (Lemma 2.3)

Theorem 2.4. For any graph \( G = \langle G, K \rangle \) and an infinite cardinal κ, the following are equivalent:

(a) \( \text{col}(G) ≤ κ \).

(a') There is a well-ordering ∈ of G of order-type \( |G| \) such that \( |K^a_∈| < κ \) for all a ∈ G.

(b) \( G \) has a κ-coloring mapping.

(c) For \( a/\)all sufficiently large regular χ and for all \( M < H(χ) \) such that \( \langle G, K \rangle \in M \) and \( κ + 1 ⊆ M \) we have \( G ∩ M ⊆_κ G \).

Proof. (a) ⇒ (b) was already proved in Lemma 2.1. (a') ⇒ (a) is trivial. The proof of (b) ⇒ (a) in Lemma 2.1 actually proves (b) ⇒ (a').

For (a) ⇒ (c), suppose that \( G = \langle G, K \rangle \) has coloring number ≤ κ. Let χ be a sufficiently large regular cardinal and \( M < H(χ) \) be such that \( G ∈ M \) and \( κ + 1 ⊆ M \). By elementarity and (a) ⇔ (b), there is \( f ∈ M \) such that \( f \) is a κ-coloring mapping on \( G \). Note that by \( κ + 1 ⊆ M \) and by elementarity, \( G ∩ M \) is closed with respect to \( f \). For a ∈ G \( \setminus M \) and b ∈ \( K^G_{a ∩ M} \), since \( a ∉ f(b) ⊆ M \), we have \( b ∈ f(a) \). Thus \( K^a_{G ∩ M} ⊆ f(a) \) and hence \( |K^a_{G ∩ M}| < κ \). This shows that \( G ∩ M ⊆_κ G \).
Now we prove (c) ⇒ (a) by induction on |G|.

If |G| ≤ κ, then (c) ⇒ (a) holds since G then has coloring number ≤ κ anyway — any well-ordering of G of order-type |G| will witness this.

Suppose that |G| > κ and we have shown the implication (c) ⇒ (a) for all graphs of cardinality < |G|. Let λ = |G|, λ* = cf(λ) and \( M_\alpha : \alpha < \lambda^* \) a continuously increasing chain of elementary submodels of \( H(\chi) \) such that

\[
\begin{align*}
    (2.5) & \quad G \in M_0; \kappa + 1 \subseteq M_0; \\
    (2.6) & \quad |M_\alpha| < \lambda \text{ for all } \alpha < \lambda^*; \text{ and} \\
    (2.7) & \quad G \subseteq \bigcup_{\alpha<\lambda^*} M_\alpha.
\end{align*}
\]

For \( \alpha < \lambda^* \), let \( G_\alpha = G \cap M_\alpha. \) Then \( \langle G_\alpha : \alpha < \lambda^* \rangle \) is a filtration of G by (2.6) and (2.7). \( G_\alpha \subseteq G \) for all \( \alpha < \kappa \) by (2.5) and by the assumption of (c).

By Lemma 2.3, \( G_\alpha \) also satisfies (c) for \( \alpha < \lambda^* \). Since \( |G_\alpha| < \lambda \), it follows that \( \text{col}(G_\alpha) \leq \kappa \) for all \( \alpha < \lambda^* \) by the induction hypothesis. Hence we have \( \text{col}(G) \leq \kappa \) by Lemma 2.2. \( \square \) (Theorem 2.4)

3 Coloring number of the plane

The \textit{plane}, or the \textit{unit distance graph of the plane}, is the graph \( G^1(\mathbb{R}^2) \) defined by \( G^1(\mathbb{R}^2) = \langle \mathbb{R}^2, K^1_{\mathbb{R}} \rangle \) where \( K^1 = \{ \{x, y\} : d(x, y) = 1 \} \). Applying Theorem 2.4, we can show easily that the coloring number of the plane is equal to \( \aleph_0 \).

**Theorem 3.1.** \( \text{col}(G^1(\mathbb{R}^2)) = \aleph_0. \)

**Proof.** In [2] it is noted that the list-chromatic number \( \text{list}(G^1(\mathbb{R}^2)) \) of \( G^1(\mathbb{R}^2) \) is infinite since finite regular graph of arbitrarily large degree \( d \) can be embedded in \( G^1(\mathbb{R}^2) \) (e.g., throwing down of \( n \)-dimensional cube onto the plane) and the list-chromatic number of such finite graph is \( d \) (see [1]). Thus we have \( \aleph_0 \leq \text{list}(G^1(\mathbb{R}^2)) \leq \text{col}(G^1(\mathbb{R}^2)). \)

To prove the inequality \( \text{col}(G^1(\mathbb{R}^2)) \leq \aleph_0, \) let \( \chi \) be sufficiently large and \( N < H(\chi) \). Note that we have \( G^1(\mathbb{R}^2) \in N \) since the plane is definable. Suppose \( x \in \mathbb{R}^2 \setminus N. \) Let us write simply \( K \) for \( K^1_{\mathbb{R}}. \) By Theorem 2.4, it is enough to show that \( K^1_{\mathbb{R} \cap N} \) is finite. Actually, we can show that \( |K^1_{\mathbb{R} \cap N}| \leq 1. \)

Toward a contradiction, suppose that \( |K^1_{\mathbb{R} \cap N}| > 1. \) Then there are two distinct \( y, z \in G \cap N \) such that \( d(x, y) = d(x, z) = 1. \) But then \( X = \{ u \in \mathbb{R}^2 : d(u, y) = d(u, z) = 1 \} \) is a two element set definable with parameters from \( N. \) It follows that \( x \in X \subseteq N. \) This is a contradiction to the choice of \( x. \) \( \square \) (Theorem 3.1)
With the same proof we can also show:
\[
\text{col}(G^{\text{odd}}(\mathbb{R}^2)) = \text{col}(G(\mathbb{R}^2)) = \text{col}(G^{Q}(\mathbb{R}^2)) = \text{col}(G^{\text{algebraic}}(\mathbb{R}^2)) = \cdots = \aleph_0.
\]

Theorem 3.1 may be already known. However, I could not find any direct mention or proof of the theorem in the literature. Also, in [2] the authors prove \(\text{list}(G^{\text{odd}}(\mathbb{R}^2)) \leq \aleph_0\) directly and it seems that idea of the proof cannot be extended to a proof of \(\text{col}(G^{\text{odd}}(\mathbb{R}^2)) \leq \aleph_0\).

I first learned a proof of \(\text{col}(G^1(\mathbb{R}^2)) \leq \aleph_0\) from Hiroshi Sakai in November 2009 who proved the inequality straightforwardly.

Theorem 2.4 is often quite useful to decide the coloring number of infinite graphs. For example, \(\text{col}(K(\kappa, \kappa)) = \kappa\) and \(\text{col}(K(\kappa, \lambda)) = \kappa^+\) for any \(\aleph_0 \leq \kappa < \lambda\); \(\text{col}(G^{\text{odd}}(\mathbb{R}^3)) = \aleph_1\) etc. can be seen immediately by this theorem.

We shall demonstrate the last equality. Recall \(G^{\text{odd}}(\mathbb{R}^3) = \langle \mathbb{R}^3, K^{\text{odd}}_{\mathbb{R}^3}\rangle\) where \(K^{\text{odd}}_{\mathbb{R}^3} = \{(\vec{x}, \vec{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 : d(\vec{x}, \vec{y}) \text{ is an odd (natural) number}\}\).

**Theorem A.3.1.** \(\text{col}(G^{\text{odd}}(\mathbb{R}^3)) = \aleph_1\).

**Proof.** For notational simplicity, let \(G = G^{\text{odd}}(\mathbb{R}^3) = \langle G, K \rangle\) with \(G = \mathbb{R}^3\) and \(K = K^{\text{odd}}_{\mathbb{R}^3}\). Suppose that \(\chi\) is sufficiently large. By Theorem 2.4, it is enough to show that \(G \cap M \subseteq_{\aleph_1} G\) for all \(M \prec \mathcal{H}(\chi)\) but \(G \cap M \not\subseteq_{\aleph_0} G\) for some \(M \prec \mathcal{H}(\chi)\).

Suppose that \(M \prec \mathcal{H}(\chi)\). If \(\mathbb{R} \subseteq M\) then \(G \subseteq M\) and we have \(G \cap M \subseteq_{\aleph_1} G\) vacuously.

Otherwise, letting \(C = \{(x, y, 0) \in \mathbb{R}^3 : d((x, y, 0), \vec{0}) = 1\}\), we have \(C \subseteq M\). Let \(\vec{x} \in C \setminus M\). Then, for any odd \(n \in \omega\), \(\sqrt{n^2 - 1} \in M\) and \(d(\vec{x}, \langle 0, 0, \sqrt{n^2 - 1} \rangle) = n\). Thus \(\langle 0, 0, \sqrt{n^2 - 1} \rangle \in K^{\vec{x}}_{G \cap M}\). This shows that \(G \cap M \not\subseteq_{\aleph_0} G\).

To show \(G \cap M \subseteq_{\aleph_1} G\), assume for contradiction that there is \(\vec{x} \in G \setminus M\) such that \(K^{\vec{x}}_{G \cap M}\) is uncountable. Then there is an odd \(n \in \omega\) such that \(X = \{\vec{y} \in G \cap M : d(\vec{x}, \vec{y}) = n\}\) is uncountable. Let \(y_0, y_1, y_3\) be three distinct elements of \(X\). \(Y = \{\vec{x} \in G : d(\vec{x}, \vec{y}_0) = d(\vec{x}, \vec{y}_1) = d(\vec{x}, \vec{y}_3) = n\}\) is a two-elements set definable with parameters form \(M\). It follows that \(\vec{x} \in Y \subseteq M\). This is a contradiction to the choice of \(\vec{x}\).

\(\Box\) (Theorem A.3.1)

### 4 Coloring number under very weak square

The following version of the very weak square was introduced in [8].
For a regular cardinal $\kappa$ and $\mu > \kappa$, let $\square_{\kappa,\mu}^{**}$ be the following assertion: there exists a sequence $(C_\alpha : \alpha < \mu^+)$ and a club set $D \subseteq \mu^+$ such that, for all $\alpha \in D$ with $\text{cf}(\alpha) \geq \kappa$, we have

(4.1) $C_\alpha \subseteq \alpha, C_\alpha$ is unbounded in $\alpha$; and

(4.2) $[\alpha]^{<\kappa} \cap \{C_{\alpha'} : \alpha' < \alpha\}$ dominates $[C_\alpha]^{<\kappa}$ (with respect to $\subseteq$).

For a (sufficiently large regular) cardinal $\chi$ and $M < \mathcal{H}(\chi)$, $M$ is $\kappa$-internally cofinal if $[M]^{<\kappa} \cap M$ is cofinal in $[M]^{<\kappa}$ with respect to $\subseteq$. For $D \subseteq [\mathcal{H}(\chi)]^{<\kappa}$, $M$ is $D$-internally cofinal if $D \cap M$ is cofinal in $[M]^{<\kappa}$ with respect to $\subseteq$.

Suppose now that $\kappa$ is a regular cardinal and $\mu > \kappa$ is such that $\text{cf}(\mu) < \kappa$. Let $\mu^* = \text{cf}(\mu)$. For a sufficiently large $\chi$ and $x \in \mathcal{H}(\chi)$, let us call a sequence $(M_{\alpha,\beta} : \alpha < \mu^+, \beta < \mu^*)$ a $(\kappa, \mu)$-dominating matrix (of elementary submodels of $\mathcal{H}(\chi)$) over $x$ if the following conditions (4.3) – (4.6) hold:

(4.3) $M_{\alpha,\beta} < \mathcal{H}(\chi)$, $x \in M_{\alpha,\beta}$, $\kappa + 1 \subseteq M_{\alpha,\beta}$ and $|M_{\alpha,\beta}| < \mu$ for all $\alpha < \mu^+$ and $\beta < \mu^*$;

(4.4) $(M_{\alpha,\beta} : \beta < \mu^*)$ is an increasing sequence for each fixed $\alpha < \mu^+$;

(4.5) if $\alpha < \mu^+$ is such that $\text{cf}(\alpha) \geq \kappa$, then there is $\beta^* < \mu^*$ such that, for every $\beta^* \leq \beta < \mu^*$, $M_{\alpha,\beta}$ is $\kappa$-internally cofinal.

For $\alpha < \mu^+$, let $M_\alpha = \bigcup_{\beta < \mu^*} M_{\alpha,\beta}$. By (4.3) and (4.4), we have $M_\alpha < \mathcal{H}(\chi)$.

(4.6) $(M_\alpha : \alpha < \mu^+)$ is continuously increasing and $\mu^+ \subseteq \bigcup_{\alpha < \mu^+} M_\alpha$.

**Theorem 4.1** (Theorem 7 in [8]). Suppose that $\kappa$ is a regular cardinal and $\mu > \kappa$ is such that $\text{cf}(\mu) < \kappa$. If we have $\text{cf}([\lambda]^{<\kappa}, \subseteq) = \lambda$ for cofinally many $\lambda < \mu$ and $\square_{\kappa,\mu}^{**}$ holds, then, for any sufficiently large $\chi$ and $x \in \mathcal{H}(\chi)$, there is a $(\kappa, \mu)$-dominating matrix over $x$.

**Theorem 4.2.** Assume SSH and $\square_{\kappa,\mu}^{**}$ for a regular uncountable $\kappa$ and all singular cardinal $\mu$ with $\text{cf}(\mu) < \kappa < \mu$.

Then, for any graph $G = (G, K)$ the following are equivalent:

(a) $\text{col}(G) \leq \kappa$.

(d) For a (or all) sufficiently large regular $\chi$ and $\kappa$-internally cofinal $M < \mathcal{H}(\chi)$ with $G \in M$ we have $G \cap M \subseteq \kappa G$.

(e) For a (or all) sufficiently large regular $\chi$ there is $D \subseteq [\mathcal{H}(\chi)]^{<\kappa}$ such that $D$ is cofinal in $[\mathcal{H}(\chi)]^{<\kappa}$ and, for any $D$-internally cofinal $M < \mathcal{H}(\chi)$, we have $G \cap M \subseteq \kappa G$. 

Proof. (a) \(\Rightarrow\) (d) follows from Theorem 2.4, (d) \(\Rightarrow\) (e) is trivial (just put \(D = [\mathcal{H}(\chi)]^{<\kappa}\)).

For (e) \(\Rightarrow\) (a), we proceed with induction on \(|G|\). If \(|G| \leq \kappa\) then the implication (e) \(\Rightarrow\) (a) is trivial since \(\text{col}(G) \leq \kappa\) holds always for any graph of size \(\leq \kappa\). Suppose now that \(|G| > \kappa\) and we have shown the implication (e) \(\Rightarrow\) (a) for all graphs of cardinality \(<|G|\).

Assume that \(G\) satisfies (e) with \(\chi\) and \(\mathcal{D}\). Let \(\chi^*\) be sufficiently large above \(\chi\) such that we have in particular \(\mathcal{H}(\chi) \in \mathcal{H}(\chi^*)\).

Claim 4.2.1. If \(M\) is a \(\kappa\)-internal cofinal elementary submodel of \(\mathcal{H}(\chi^*)\) such that

\[
G, \chi, \mathcal{D} \in M \text{ and } \kappa + 1 \subseteq M,
\]

then we have \(G \cap M \subseteq_{\kappa} G\).

\(\vdash\) Suppose not. Then there is \(a \in G \setminus M\) such that \(|K_{G \cap M}^a| \geq \kappa\). Let \(N = \mathcal{H}(\chi) \cap M\). By elementarity we have \(N < \mathcal{H}(\chi)\). Let \(\langle N_\alpha : \alpha < \kappa\rangle\) be an increasing sequence such that, for all \(\alpha < \kappa\), we have

\[
\begin{align*}
(4.7) & \quad N_\alpha \in \mathcal{D} \cap M; \\
(4.8) & \quad N_\alpha \in D \cap M; \\
(4.9) & \quad N_\alpha \in N_{\alpha+1}; \\
(4.10) & \quad \text{there is } N_\alpha^* \in [N]^{<\kappa} \cap M \text{ such that } N_\alpha^* < N \text{ and } N_\alpha \subseteq N_\alpha^* \subseteq N_{\alpha+1}; \text{ and} \\
(4.11) & \quad K_{G \cap M}^a \cap (N_{\alpha+1} \setminus N_\alpha) \neq \emptyset.
\end{align*}
\]

The construction is possible by elementarity of \(M\) and since \(\mathcal{D}\) is cofinal in \([\mathcal{H}(\chi)]^{<\kappa}\).

Let \(N^* = \bigcup_{\alpha < \kappa} N_\alpha\). By (4.10) we have \(N^* < N < \mathcal{H}(\chi)\). By (4.8) and (4.9) \(N^*\) is \(\mathcal{D}\)-internally cofinal. On the other hand, we have \(|K_{G \cap N^*}^a| \geq \kappa\) by (4.11). This is a contradiction to the assumption of (e).

\(\neg\) (Claim 4.2.1)

Claim 4.2.2. If \(H \subseteq_{\kappa} G\) then for every \(\mathcal{D}\)-internally cofinal \(M < \mathcal{H}(\chi)\) we have \(H \cap M \subseteq_{\kappa} H\). In particular, \(H\) also satisfies the condition (e).

Proof. Suppose that \(M < \mathcal{H}(\chi)\) is \(\mathcal{D}\)-internally approachable. For \(a \in H \setminus (H \cap M)\), since \(a \in G \setminus (G \cap M)\), we have \(K_{H \cap M}^a \subseteq K_{G \cap M}^a\). The right side of the inclusion is of cardinality \(< \kappa\) by the assumption of (e) on \(G\). This shows that \(H \cap M \subseteq_{\kappa} H\).

\(\neg\) (Claim 4.2.2)

Now we finish the induction step for the proof of (e) \(\Rightarrow\) (a) in two cases. Let \(\nu = |G|\).

Case I. \(\nu\) is a limit cardinal or \(\nu = \delta^+\) with \(\text{cf}(\delta) \geq \kappa\).

Let \(\nu^* = \text{cf}(\nu)\). Note that, in this case, we have that
(4.12) the cardinals \( \lambda < \nu \) such that \( cf([\lambda]^{<\kappa}) = \lambda \) are cofinal among cardinals below \( \nu \) by SSH.

Let \( \langle M_\alpha : \alpha < \nu^* \rangle \) be an increasing sequence of elementary submodels of \( \mathcal{H}(\chi^*) \) of cardinality \( < \nu \) satisfying (4.7) and \( G \subseteq \bigcup_{\alpha<\nu^*} M_\alpha \). We can find such a sequence by (4.12).

Let
\[
G_\alpha = \begin{cases} 
G \cap M_\alpha & \text{if } \alpha = 0 \text{ or } \alpha \text{ is a successor ordinal;} \\
G \cap \left( \bigcup_{\beta<\alpha} M_\beta \right) & \text{otherwise}
\end{cases}
\]
for \( \alpha < \nu^* \). Then \( \langle G_\alpha : \alpha < \nu^* \rangle \) is a filtration of \( G \).

**Claim 4.2.3.** \( G_\alpha \subseteq_\kappa G \) for all \( \alpha < \nu^* \).

**Proof.** If \( \alpha < \nu^* \) is 0 or a successor ordinal, this follows from Claim 4.2.1.

If \( \alpha < \nu^* \) is a limit and \( cf(\alpha) < \kappa \), then \( G_\alpha \) is a union of less than \( \kappa \) many \( G_\beta \)'s where \( \beta < \alpha \) may be chosen to be a successor ordinal and hence \( G_\beta \subseteq_\kappa G \). It follows that we have \( G_\alpha \subseteq_\kappa G \) also in this case.

If \( cf(\alpha) \geq \kappa \), then \( \bigcup_{\beta<\alpha} M_\beta \) is \( \kappa \)-internally cofinal and hence we have \( G_\beta \subseteq_\kappa G \) again by Claim 4.2.1. \( \dashv \) (Claim 4.2.3)

Now by Claim 4.2.2 and by the induction hypothesis, all of \( G_\alpha, \alpha < \nu^* \) are of coloring number \( \leq \kappa \). By Lemma 2.2, it follows that \( G \) also has coloring number \( \leq \kappa \).

**Case II.** \( \nu = \mu^+ \) with \( cf(\mu) < \kappa \). Let \( \mu^* = cf(\mu) \).

By Theorem 4.1, there is a \( (\kappa, \mu) \)-dominating matrix \( \langle M_{\alpha,\beta} : \alpha < \nu, \beta < \mu^* \rangle \) of submodels of \( \mathcal{H}(\chi^*) \) over \( x = \langle G, \mathcal{H}(\chi) \rangle \).

For \( \alpha < \nu \) and \( \beta < \mu^* \), let \( G_{\alpha,\beta} = G \cap M_{\alpha,\beta} \) and \( G_\alpha = \bigcup_{\beta<\mu^*} G_{\alpha,\beta} = G \cap \left( \bigcup_{\beta<\mu^*} M_{\alpha,\beta} \right) \). By (4.6), the sequence \( \langle G_\alpha : \alpha < \nu \rangle \) is continuously increasing and \( \bigcup_{\alpha<\nu} G_\alpha = G \). By (4.3), we have \( |G_\alpha| \leq \mu < \nu \). Thus \( \langle G_\alpha : \alpha < \nu \rangle \) is a filtration of \( G \).

Let
\[
(4.13) \quad C = \{ \alpha < \nu : cf(\alpha) \geq \kappa \text{ or } \{ \alpha' < \alpha : cf(\alpha') \geq \kappa \} \text{ is cofinal in } \alpha \}.
\]

\( C \) is a club subset of \( \nu \).

**Claim 4.2.4.** \( G_\alpha \subseteq_\kappa G \) for all \( \alpha \in C \).
Suppose $\alpha \in C$. If $\text{cf}(\alpha) \geq \kappa$, $M_{\alpha, \beta}$ is $\kappa$-internally cofinal for all sufficiently large $\beta < \mu^*$ by (4.5). Hence by Claim 4.2.1, we have $G_{\alpha, \beta} \subseteq \kappa G$ for all such $\beta$. Since $\mu^* < \kappa$, it follows that $G_{\alpha} \subseteq \kappa G$.

If $\text{cf}(\alpha) < \kappa$, then let $X \subseteq \alpha$ be a cofinal subset of $\alpha$ with $|X| < \kappa$ such that all $\alpha' \in X$ have cofinality $\geq \kappa$. Since $G_{\alpha} = \bigcup_{\alpha' \in X} G_{\alpha'}$ and $G_{\alpha'} \subseteq \kappa G$ for all $\alpha' \in X$ by the first part of the proof, it follows that $G_{\alpha} \subseteq \kappa G$. $\dashv$ (Claim 4.2.4)

By Claim 4.2.2 and by the induction hypothesis, we have $\text{col}(G_{\alpha}) \leq \kappa$ for all $\alpha \in C$. Hence by Lemma 2.2 we can conclude that $\text{col}(G) \leq \kappa$. $\square$ (Theorem 4.2)

References


