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Remarks on the coloring number of graphs †

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Abstract

We give two characterizations of graphs with coloring number $\leq \kappa$ in terms of elementary submodels; one under ZFC and another under SSH and the version of very weak square principle of [8].

These characterizations suggest that the graphs with coloring number $\leq \kappa$ behave very much like the Boolean algebras with κ-Freese-Nation property (see [5], [8]).

1 Introduction

A graph $G = \langle G, K \rangle$ $(K \subseteq [G]^2)$ has *coloring number* $\leq \kappa$ (notation: $col(G) \leq \kappa$) if there is a well ordering $\Subset$ on G such that $K^a_\Subset = \{b \in G \,:\, b \Subset a$ and $\{a, b\} \in$

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† This is an extended version of the note with the same title. Some details and proofs omitted in the version to appear in RIMS Kôkyûroku are added in typewriter font. The most up-to-date version of this note is downloadable as:

`http://kurt.scitec.kobe-u.ac.jp/~fuchino/papers/RIMS10-graph-square-x.pdf`

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$K\}$ has cardinality $< \kappa$ for all $a \in G$ ([3]). The *coloring number* $col(G)$ of G is then defined as the minimum of such κ's. It is easy to see that the chromatic number $\chi(G)$ of G is less or equal to $col(G)$.

The purpose of this note is to show that the graphs with coloring number $\leq \kappa$ behave quite similarly to the Boolean algebras with κ-Freese-Nation property (see e.g. [5], [8]).

In Section 2 we give a characterization of graphs with coloring number $\leq \kappa$ in terms of elementary submodels (Theorem 2.4). As an application of the characterization, we present in Section 3 a short proof of the countability of the coloring number of the plane.

In Section 4, we show that the characterization of Section 2 can be yet sharpened under SSH and the version of the very weak square principle introduced in [8] (Theorem 4.2).

Both Theorems 2.4 and 4.2 find their parallels in the theory of Boolean algebras with κ-Freese-Nation property (see PROPOSITION 3 and THEOREM 10 in [8]).

The following theorem also underlines the analogy between the Boolean algebras with the κ-Freese-Nation property and the graphs with coloring number $\leq \kappa$ in the case of $\kappa = \aleph_0$. Note that Boolean algebras with $\aleph_0$-Freese Nation property are also called *openly generated.*

If $G = \langle G, K\rangle$ is graph then we identify any subset H of G with the graph $G \restriction H = \langle H, K \cap [H]^2\rangle$.

Theorem 1.1 ([6] and [7]). *The following assertions are equivalent over* ZFC*:*

(α) *For any Boolean algebra B if there are club many subalgebras of B of cardinality $\aleph_1$ which are openly generated then B is openly generated.*

(β) *For any graph G if $col(H) \leq \aleph_0$ for every $H \in [G]^{\aleph_1}$ then $col(G) \leq \aleph_0$.* $\square$

Theorem 1.1 in the formulation as above is a sort of bluff since we actually proved that each of the assertions (α) and (β) is equivalent to the set-theoretic principle FRP introduced in [4].

2 A characterization of graphs with coloring number $\leq \kappa$

We use here the following notations. The first one was already used in the introduction:

For a linear ordering $\Subset$ on a graph $G = \langle G, K\rangle$ we denote

(2.1) $K^a_{\Subset} = \{b \in G : b \Subset a \text{ and } \{a,b\} \in K\}$.

If $H \subseteq G$ then we write

(2.2) $K^{H,a}_{\Subset} = \{b \in H : b \Subset a \text{ and } \{a,b\} \in K\}$

For a graph $G = \langle G, K\rangle$, $H \subseteq G$ and $a \in G$, let

(2.3) $K^a_H = \{b \in H : \langle a,b\rangle \in K\}$.

We write $H \sqsubseteq_\kappa G$ if $|\,K^a_H\,| < \kappa$ for all $a \in G \setminus H$.

A mapping $f : G \to [G]^{<\kappa}$ is a *κ-coloring mapping on G* if for any a, $b \in G$ with $\{a,b\} \in K$, at least one of $a \in f(b)$ or $b \in f(a)$ holds.

Lemma 2.1 ([7]). *For any graph G and any infinite cardinal κ, the following are equivalent:*

(a) *$col(G) \leq \kappa$.*

(b) *There is a κ-coloring mapping on G.*

Proof. (a) $\Rightarrow$ (b): Suppose that $col(G) \leq \kappa$ and let $\Subset$ be a well-ordering on G such that $|\,K^a_{\Subset}\,| < \kappa$ for all $a \in G$. Then $f : G \to [G]^{<\kappa}$ defined by $f(a) = K^a_{\Subset}$ for $a \in G$ is a κ-coloring mapping.

(b) $\Rightarrow$ (a): Suppose that $f : G \to [G]^{<\kappa}$ is a κ-coloring mapping on G. Let $\Subset$ be a well-ordering on G such that all initial segments of G of order-type of the form $\kappa \cdot \alpha$ with respect to $\Subset$ are closed with respect to f. Then $\Subset$ is as desired:

Claim 2.1.1. $|\,K^a_{\Subset}\,| < \kappa$ *for all* $a \in G$.

$\vdash$ Suppose that $a \in G$ is the $\kappa \cdot \alpha + \beta$'th element with respect to $\Subset$ where $\beta < \kappa$. Then the first $\kappa \cdot \alpha$ elements of G are closed with respect to f and hence if b is among them and $\{a,b\} \in K$ then we have $b \in f(a)$. Thus

$$K^a_{\Subset} \subseteq \{b \in G : b \text{ is the } \gamma\text{'th element for some } \kappa \cdot \alpha \leq \gamma < \kappa \cdot \alpha + \beta\} \cup f(a).$$

The right side of the inclusion has size $< \kappa$ (note that we need here the infinity of κ). Hence $|\,K^a_{\Subset}\,| < \kappa$. $\dashv$ (Claim 2.1.1)

$\square$ (Lemma 2.1)

Lemma 2.2. *Suppose that $\langle G_\alpha : \alpha < \delta\rangle$ is a filtration of a graph $G = \langle G, K\rangle$ and κ is an infinite cardinal. If $G_\alpha \sqsubseteq_\kappa G$ and $col(G_{\alpha+1}) \leq \kappa$ for all $\alpha < \delta$, then we have $col(G) \leq \kappa$.*

Proof. For $a \in G$ let $o(a) = \min\{\alpha < \delta : a \in G_{\alpha+1}\}$. For $\alpha < \delta$, let $\Subset_{\alpha+1}$ be a well-ordering of $G_{\alpha+1}$ witnessing $col(G_{\alpha+1}) \leq \kappa$. Let $\Subset$ be the ordering on G defined by:

$$(2.4) \quad a \Subset b \quad \Leftrightarrow \quad o(a) < o(b) \text{ or } \Big(o(a) = o(b) \text{ and } a \Subset_{o(a)+1} b \Big).$$

Then $\Subset$ is a well ordering on G. The following claim shows that $\Subset$ witnesses that G has coloring number $< \kappa$.

Claim 2.2.1. *$|\,K^a_{\Subset}\,| < \kappa$ for all $a \in G$.*

$\vdash$ For $a \in G$, we have $K^a_{\Subset} \subseteq K^a_{G_{o(a)}} \cup K^{G_{o(a)+1},a}_{\Subset_{o(a)+1}}$. Since the right side of the inclusion is of cardinality $< \kappa$, it follows that $|\,K^a_{\Subset}\,| < \kappa$. $\dashv$ (Claim 2.2.1)

□ (Lemma 2.2)

Lemma 2.3. *Suppose that H_0 and H_1 are subsets of G with $H_0 \sqsubseteq_\kappa G$ and $H_1 \sqsubseteq_\kappa G$. Then we have $H_0 \cap H_1 \sqsubseteq_\kappa G$.*

Proof. Suppose that $a \in G \setminus (H_0 \cap H_1)$. Then we have $a \in G \setminus H_0$ or $a \in G \setminus H_1$. If $a \in G \setminus H_0$, then $K^a_{H_0 \cap H_1} \subseteq K^a_{H_0}$. And hence $|\,K^a_{H_0 \cap H_1}\,| < \kappa$. If $a \in G \setminus H_1$, then $K^a_{H_0 \cap H_1} \subseteq K^a_{H_1}$. And hence again we have $|\,K^a_{H_0 \cap H_1}\,| < \kappa$.

This shows $H_0 \cap H_1 \sqsubseteq_\kappa G$. □ (Lemma 2.3)

Theorem 2.4. *For any graph $G = \langle G, K \rangle$ and an infinite cardinal κ, the following are equivalent:*

(a) *$col(G) \leq \kappa$.*

(a') *There is a well-ordering $\Subset$ of G of order-type $|\,G\,|$ such that $|\,K^a_{\Subset}\,| < \kappa$ for all $a \in G$.*

(b) *G has a κ-coloring mapping.*

(c) *For a/all sufficiently large regular χ and for all $M \prec \mathcal{H}(\chi)$ such that $\langle G, K \rangle \in M$ and $\kappa + 1 \subseteq M$ we have $G \cap M \sqsubseteq_\kappa G$.*

Proof. (a) $\Rightarrow$ (b) was already proved in Lemma 2.1. (a') $\Rightarrow$ (a) is trivial. The proof of (b) $\Rightarrow$ (a) in Lemma 2.1 actually proves (b) $\Rightarrow$ (a').

For (a) $\Rightarrow$ (c), suppose that $G = \langle G, K \rangle$ has coloring number $\leq \kappa$. Let χ be a sufficiently large regular cardinal and $M \prec \mathcal{H}(\chi)$ be such that $G \in M$ and $\kappa + 1 \subseteq M$. By elementarity and (a) $\Leftrightarrow$ (b), there is $f \in M$ such that f is a κ-coloring mapping on G. Note that by $\kappa + 1 \subseteq M$ and by elementarity, $G \cap M$ is closed with respect to f. For $a \in G \setminus M$ and $b \in K^a_{G \cap M}$, since $a \notin f(b) \subseteq M$, we have $b \in f(a)$. Thus $K^a_{G \cap M} \subseteq f(a)$ and hence $|\,K^a_{G \cap M}\,| < \kappa$. This shows that $G \cap M \sqsubseteq_\kappa G$.

Now we prove (c) $\Rightarrow$ (a) by induction on $|G|$.

If $|G| \leq \kappa$, then (c) $\Rightarrow$ (a) holds since G then has coloring number $\leq \kappa$ anyway — any well-ordering of G of order-type $|G|$ will witness this.

Suppose that $|G| > \kappa$ and we have shown the implication (c) $\Rightarrow$ (a) for all graphs of cardinality $< |G|$. Let $\lambda = |G|$, $\lambda^* = \mathrm{cf}(\lambda)$ and $\langle M_\alpha : \alpha < \lambda^* \rangle$ a continuously increasing chain of elementary submodels of $\mathcal{H}(\chi)$ such that

(2.5) $G \in M_0$; $\kappa + 1 \subseteq M_0$;

(2.6) $|M_\alpha| < \lambda$ for all $\alpha < \lambda^*$; and

(2.7) $G \subseteq \bigcup_{\alpha<\lambda^*} M_\alpha$.

For $\alpha < \lambda^*$, let $G_\alpha = G \cap M_\alpha$. Then $\langle G_\alpha : \alpha < \lambda^* \rangle$ is a filtration of G by (2.6) and (2.7). $G_\alpha \sqsubseteq_\kappa G$ for all $\alpha < \kappa$ by (2.5) and by the assumption of (c). By Lemma 2.3, G_α also satisfies (c) for $\alpha < \lambda^*$. Since $|G_\alpha| < \lambda$, it follows that $col(G_\alpha) \leq \kappa$ for all $\alpha < \lambda^*$ by the induction hypothesis. Hence we have $col(G) \leq \kappa$ by Lemma 2.2. □ (Theorem 2.4)

3 Coloring number of the plane

The *plane*, or *the unit distance graph of the plane*, is the graph $G^1(\mathbb{R}^2)$ defined by $G^1(\mathbb{R}^2) = \langle \mathbb{R}^2, K^1_{\mathbb{R}^2} \rangle$ where $K^1 = \{\{x,y\} \in [\mathbb{R}^2]^2 : d(x,y) = 1\}$. Applying Theorem 2.4, we can show easily that the coloring number of the plane is equal to $\aleph_0$.

Theorem 3.1. $col(G^1(\mathbb{R}^2)) = \aleph_0$.

Proof. In [2] it is noted that the list-chromatic number $list(G^1(\mathbb{R}^2))$ of $G^1(\mathbb{R}^2)$ is infinite since finite regular graph of arbitrarily large degree d can be embedded in $G^1(\mathbb{R}^2)$ (e.g., throwing down of n-dimensional cube onto the plane) and the list-chromatic number of such finite graph is d (see [1]). Thus we have $\aleph_0 \leq list(G^1(\mathbb{R}^2)) \leq col(G^1(\mathbb{R}^2))$.

To prove the inequality $col(G^1(\mathbb{R}^2)) \leq \aleph_0$, let χ be sufficiently large and $N \prec \mathcal{H}(\chi)$. Note that we have $G^1(\mathbb{R}^2) \in N$ since the plane is definable. Suppose $x \in \mathbb{R}^2 \setminus N$. Let us write simply K for $K^1_{\mathbb{R}^2}$ By Theorem 2.4, it is enough to show that $K^x_{\mathbb{R}^2\cap N}$ is finite. Actually, we can show that $|K^x_{\mathbb{R}^2\cap N}| \leq 1$:

Toward a contradiction, suppose that $|K^x_{\mathbb{R}^2\cap N}| > 1$. Then there are two distinct y, $z \in G \cap N$ such that $d(x,y) = d(x,z) = 1$. But then $X = \{u \in \mathbb{R}^2 : d(u,y) = d(u,z) = 1\}$ is a two element set definable with parameters from N. It follows that $x \in X \subseteq N$. This is a contradiction to the choice of x.

□ (Theorem 3.1)

With the same proof we can also show:

$col(G^{\mathrm{Odd}}(\mathbb{R}^2)) = col(G^{\mathbb{N}}(\mathbb{R}^2)) = col(G^{\mathbb{Q}}(\mathbb{R}^2)) = col(G^{\mathrm{algebraic}}(\mathbb{R}^2)) = \cdots = \aleph_0$.

Theorem 3.1 may be already known. However I could not find any direct mention or proof of the theorem in the literature. Also, in [2] the authors prove $list(G^{\mathrm{Odd}}(\mathbb{R}^2)) \leq \aleph_0$ directly and it seems that idea of the proof cannot be extended to a proof of $col(G^{\mathrm{Odd}}(\mathbb{R}^2)) \leq \aleph_0$.

I first learned a proof of $col(G^1(\mathbb{R}^2)) \leq \aleph_0$ from Hiroshi Sakai in November 2009 who proved the inequality straightforwardly.

Theorem 2.4 is often quite useful to decide the coloring number of infinite graphs. For example, $col(K(\kappa,\kappa)) = \kappa$ and $col(K(\kappa,\lambda)) = \kappa^+$ for any $\aleph_0 \leq \kappa < \lambda$; $col(G^{\mathrm{Odd}}(\mathbb{R}^3)) = \aleph_1$ etc. can be seen immediately by this theorem.

We shall demonstrate the last equality. Recall $G^{\mathrm{Odd}}(\mathbb{R}^3) = \langle \mathbb{R}^3, K^{\mathrm{Odd}}_{\mathbb{R}^3}\rangle$ where $K^{\mathrm{Odd}}_{\mathbb{R}^3} = \{\langle \vec{x},\vec{y}\rangle \in [\mathbb{R}^3]^2 : d(\vec{x},\vec{y})$ is an odd (natural) number$\}$.

Theorem A.3.1. $col(G^{\mathrm{Odd}}(\mathbb{R}^3)) = \aleph_1$.

Proof. For notational simplicity, let $G = G^{\mathrm{Odd}}(\mathbb{R}^3) = \langle G, K\rangle$ with $G = \mathbb{R}^3$ and $K = K^{\mathrm{Odd}}_{\mathbb{R}^3}$. Suppose that χ is sufficiently large. By Theorem 2.4, it is enough to show that $G \cap M \sqsubseteq_{\aleph_1} G$ for all $M \prec \mathcal{H}(\chi)$ but $G \cap M \not\sqsubseteq_{\aleph_0} G$ for some $M \prec \mathcal{H}(\chi)$.

Suppose that $M \prec \mathcal{H}(\chi)$. If $\mathbb{R} \subseteq M$ then $G \subseteq M$ and we have $G \cap M \sqsubseteq_{\aleph_i} G$ vacuously.

Otherwise, letting $C = \{\langle x,y,0\rangle \in \mathbb{R}^3 : d(\langle x,y,0\rangle,\vec{0}) = 1\}$, we have $C \not\subseteq M$. Let $\vec{x} \in C \setminus M$. Then, for any odd $n \in \omega$, $\sqrt{n^2-1} \in M$ and $d(\vec{x},\langle 0,0,\sqrt{n^2-1}\rangle) = n$. Thus $\langle 0,0,\sqrt{n^2-1}\rangle \in K^{\vec{x}}_{G\cap M}$. This shows that $G \cap M \not\sqsubseteq_{\aleph_0} G$.

To show $G \cap M \sqsubseteq_{\aleph_1} G$, assume for contradiction that there is $\vec{x} \in G \setminus M$ such that $K^{\vec{x}}_{G\cap M}$ is uncountable. Then there is an odd $n \in \omega$ such that $X = \{\vec{y} \in G \cap M : d(\vec{x},\vec{y}) = n\}$ is uncountable. Let y_0, y_1, y_3 be three distinct elements of X. $Y = \{\vec{z} \in G : d(\vec{z},\vec{y_0}) = d(\vec{z},\vec{y_1}) = d(\vec{z},\vec{y_2}) = n\}$ is a two-elements set definable with parameters form M. It follows that $\vec{x} \in Y \subseteq M$. This is a contradiction to the choice of $\vec{x}$. □ (Theorem A.3.1)

4 Coloring number under very weak square

The following version of the very weak square was introduced in [8].

For a regular cardinal κ and $\mu > \kappa$, let $\square^{***}_{\kappa,\mu}$ be the following assertion: there exists a sequence $\langle C_\alpha : \alpha < \mu^+ \rangle$ and a club set $D \subseteq \mu^+$ such that, for all $\alpha \in D$ with $\mathrm{cf}(\alpha) \geq \kappa$, we have

(4.1) $C_\alpha \subseteq \alpha$, C_α is unbounded in α; and

(4.2) $[\alpha]^{<\kappa} \cap \{C_{\alpha'} : \alpha' < \alpha\}$ dominates $[C_\alpha]^{<\kappa}$ (with respect to $\subseteq$).

For a (sufficiently large regular) cardinal χ and $M \prec \mathcal{H}(\chi)$, M is *κ-internally cofinal* if $[M]^{<\kappa} \cap M$ is cofinal in $[M]^{<\kappa}$ with respect to $\subseteq$. For $\mathcal{D} \subseteq [\mathcal{H}(\chi)]^{<\kappa}$, M is *$\mathcal{D}$-internally cofinal* if $\mathcal{D} \cap M$ is cofinal in $[M]^{<\kappa}$ with respect to $\subseteq$.

Suppose now that κ is a regular cardinal and $\mu > \kappa$ is such that $\mathrm{cf}(\mu) < \kappa$. Let $\mu^* = \mathrm{cf}(\mu)$. For a sufficiently large χ and $x \in \mathcal{H}(\chi)$, let us call a sequence $\langle M_{\alpha,\beta} : \alpha < \mu^+, \beta < \mu^* \rangle$ a *(κ, μ)-dominating matrix (of elementary submodels of $\mathcal{H}(\chi)$) over x* if the following conditions (4.3) – (4.6) hold:

(4.3) $M_{\alpha,\beta} \prec \mathcal{H}(\chi)$, $x \in M_{\alpha,\beta}$, $\kappa + 1 \subseteq M_{\alpha,\beta}$ and $|M_{\alpha,\beta}| < \mu$ for all $\alpha < \mu^+$ and $\beta < \mu^*$;

(4.4) $\langle M_{\alpha,\beta} : \beta < \mu^* \rangle$ is an increasing sequence for each fixed $\alpha < \mu^+$;

(4.5) if $\alpha < \mu^+$ is such that $\mathrm{cf}(\alpha) \geq \kappa$, then there is $\beta^* < \mu^*$ such that, for every $\beta^* \leq \beta < \mu^*$, $M_{\alpha,\beta}$ is κ-internally cofinal.

For $\alpha < \mu^+$, let $M_\alpha = \bigcup_{\beta<\mu^*} M_{\alpha,\beta}$. By (4.3) and (4.4), we have $M_\alpha \prec \mathcal{H}(\chi)$.

(4.6) $\langle M_\alpha : \alpha < \mu^+ \rangle$ is continuously increasing and $\mu^+ \subseteq \bigcup_{\alpha<\mu^+} M_\alpha$.

Theorem 4.1 (Theorem 7 in [8]). *Suppose that κ is a regular cardinal and $\mu > \kappa$ is such that $\mathrm{cf}(\mu) < \kappa$. If we have $\mathrm{cf}([\lambda]^{<\kappa}, \subseteq) = \lambda$ for cofinally many $\lambda < \mu$ and $\square^{***}_{\kappa,\mu}$ holds, then, for any sufficiently large χ and $x \in \mathcal{H}(\chi)$, there is a (κ, μ)-dominating matrix over x.*

Theorem 4.2. *Assume* SSH *and $\square^{***}_{\kappa,\mu}$ for a regular uncountable κ and all singular cardinal μ with $\mathrm{cf}(\mu) < \kappa < \mu$.*

Then, for any graph $G = \langle G, K \rangle$ the following are equivalent:

(a) *$col(G) \leq \kappa$.*

(d) *For a/all sufficiently large regular χ and κ-internally cofinal $M \prec \mathcal{H}(\chi)$ with $G \in M$ we have $G \cap M \sqsubseteq_\kappa G$.*

(e) *For a/all sufficiently large regular χ there is $\mathcal{D} \subseteq [\mathcal{H}(\chi)]^{<\kappa}$ such that $\mathcal{D}$ is cofinal in $[\mathcal{H}(\chi)]^{<\kappa}$ and, for any $\mathcal{D}$-internally cofinal $M \prec \mathcal{H}(\chi)$, we have $G \cap M \sqsubseteq_\kappa G$.*

Proof. (a) $\Rightarrow$ (d) follows from Theorem 2.4, (d) $\Rightarrow$ (e) is trivial (just put $\mathcal{D} = [\mathcal{H}(\chi)]^{<\kappa}$).

For (e) $\Rightarrow$ (a), we proceed with induction on $|G|$. If $|G| \leq \kappa$ then the implication (e) $\Rightarrow$ (a) is trivial since $col(G) \leq \kappa$ holds always for any graph of size $\leq \kappa$. Suppose now that $|G| > \kappa$ and we have shown the implication (e) $\Rightarrow$ (a) for all graphs of cardinality $< |G|$.

Assume that G satisfies (e) with χ and $\mathcal{D}$. Let χ^* be sufficiently large above χ such that we have in particular $\mathcal{H}(\chi) \in \mathcal{H}(\chi^*)$.

Claim 4.2.1. *If M is a κ-internal cofinal elementary submodel of $\mathcal{H}(\chi^*)$ such that*

(4.7) $\quad G,\ \chi,\ \mathcal{D} \in M$ *and* $\kappa + 1 \subseteq M$,

then we have $G \cap M \sqsubseteq_\kappa G$.

$\vdash$ Suppose not. Then there is $a \in G \setminus M$ such that $|K^a_{G\cap M}| \geq \kappa$. Let $N = \mathcal{H}(\chi) \cap M$. By elementarity we have $N \prec \mathcal{H}(\chi)$. Let $\langle N_\alpha : \alpha < \kappa \rangle$ be an increasing sequence such that, for all $\alpha < \kappa$, we have

(4.8) $\quad N_\alpha \in \mathcal{D} \cap M$;

(4.9) $\quad N_\alpha \in N_{\alpha+1}$;

(4.10) there is $N^*_\alpha \in [N]^{<\kappa} \cap M$ such that $N^*_\alpha \prec N$ and $N_\alpha \subseteq N^*_\alpha \subseteq N_{\alpha+1}$; and

(4.11) $\quad K^a_{G\cap M} \cap (N_{\alpha+1} \setminus N_\alpha) \neq \emptyset$.

The construction is possible by elementarity of M and since $\mathcal{D}$ is cofinal in $[\mathcal{H}(\chi)]^{<\kappa}$.

Let $N^* = \bigcup_{\alpha<\kappa} N_\alpha$. By (4.10) we have $N^* \prec N \prec \mathcal{H}(\chi)$. By (4.8) and (4.9) N^* is $\mathcal{D}$-internally cofinal. On the other hand, we have $|K^a_{G\cap N^*}| \geq \kappa$ by (4.11). This is a contradiction to the assumption of (e). $\dashv$ (Claim 4.2.1)

Claim 4.2.2. *If $H \sqsubseteq_\kappa G$ then for every $\mathcal{D}$-internally cofinal $M \prec \mathcal{H}(\chi)$ we have $H \cap M \sqsubseteq_\kappa H$. In particular, H also satisfies the condition* (e).

Proof. Suppose that $M \prec \mathcal{H}(\chi)$ is $\mathcal{D}$-internally approachable. For $a \in H \setminus (H \cap M)$, since $a \in G \setminus (G \cap M)$, we have $K^a_{H\cap M} \subseteq K^a_{G\cap M}$. The right side of the inclusion is of cardinality $< \kappa$ by the assumption of (e) on G. This shows that $H \cap M \sqsubseteq_\kappa H$. $\dashv$ (Claim 4.2.2)

Now we finish the induction step for the proof of (e) $\Rightarrow$ (a) in two cases. Let $\nu = |G|$.

Case I. ν is a limit cardinal or $\nu = \delta^+$ with $\mathrm{cf}(\delta) \geq \kappa$.

Let $\nu^* = cf(\nu)$. Note that, in this case, we have that

(4.12) the cardinals $\lambda < \nu$ such that $cf([\lambda]^{<\kappa}) = \lambda$ are cofinal among cardinals below ν

by SSH.

Let $\langle M_\alpha : \alpha < \nu^* \rangle$ be an increasing sequence of elementary submodels of $\mathcal{H}(\chi^*)$ of cardinality $< \nu$ satisfying (4.7) and $G \subseteq \bigcup_{\alpha<\nu^*} M_\alpha$. We can find such a sequence by (4.12).

Let

$$G_\alpha = \begin{cases} G \cap M_\alpha & \text{if } \alpha = 0 \text{ or } \alpha \text{ is a successor ordinal;} \\ G \cap \left(\bigcup_{\beta<\alpha} M_\beta\right) & \text{otherwise} \end{cases}$$

for $\alpha < \nu^*$. Then $\langle G_\alpha : \alpha < \nu^* \rangle$ is a filtration of G.

Claim 4.2.3. *$G_\alpha \sqsubseteq_\kappa G$ for all $\alpha < \nu^*$.*

Proof. If $\alpha < \nu^*$ is 0 or a successor ordinal, this follows from Claim 4.2.1.

If $\alpha < \nu^*$ is a limit and $\mathrm{cf}(\alpha) < \kappa$, Then G_α is a union of less than κ many G_β's where $\beta < \alpha$ may be chosen to be a successor ordinal and hence $G_\beta \sqsubseteq_\kappa G$. It follows that we have $G_\alpha \sqsubseteq_\kappa G$ also in this case.

If $\mathrm{cf}(\alpha) \geq \kappa$, then $\bigcup_{\beta<\alpha} M_\beta$ is κ-internally cofinal and hence we have $G_\beta \sqsubseteq_\kappa G$ again by Claim 4.2.1. $\dashv$ (Claim 4.2.3)

Now by Claim 4.2.2 and by the induction hypothesis, all of G_α, $\alpha < \nu^*$ are of coloring number $\leq \kappa$. By Lemma 2.2, it follows that G also has coloring number $\leq \kappa$.

Case II. $\nu = \mu^+$ with $\mathrm{cf}(\mu) < \kappa$. Let $\mu^* = \mathrm{cf}(\mu)$.

By Theorem 4.1, there is a (κ, μ)-dominating matrix $\langle M_{\alpha,\beta} : \alpha < \nu, \beta < \mu^* \rangle$ of submodels of $\mathcal{H}(\chi^*)$ over $x = \langle G, \mathcal{H}(\chi) \rangle$.

For $\alpha < \nu$ and $\beta < \mu^*$, let $G_{\alpha,\beta} = G \cap M_{\alpha,\beta}$ and $G_\alpha = \bigcup_{\beta<\mu^*} G_{\alpha,\beta} = G \cap \left(\bigcup_{\beta<\mu^*} M_{\alpha,\beta}\right)$. By (4.6), the sequence $\langle G_\alpha : \alpha < \nu \rangle$ is continuously increasing and $\bigcup_{\alpha<\nu} G_\alpha = G$. By (4.3), we have $|G_\alpha| \leq \mu < \nu$. Thus $\langle G_\alpha : \alpha < \nu \rangle$ is a filtration of G.

Let

(4.13) $C = \{\alpha < \nu : \mathrm{cf}(\alpha) \geq \kappa \text{ or } \{\alpha' < \alpha : \mathrm{cf}(\alpha') \geq \kappa\} \text{ is cofinal in } \alpha\}$.

C is a club subset of ν.

Claim 4.2.4. *$G_\alpha \sqsubseteq_\kappa G$ for all $\alpha \in C$.*

$\vdash$ Suppose $\alpha \in C$. If $\mathrm{cf}(\alpha) \geq \kappa$, $M_{\alpha,\beta}$ is κ-internally cofinal for all sufficiently large $\beta < \mu^*$ by (4.5). Hence by Claim 4.2.1, we have $G_{\alpha,\beta} \sqsubseteq_\kappa G$ for all such β. Since $\mu^* < \kappa$, it follows that $G_\alpha \sqsubseteq_\kappa G$.

If $\mathrm{cf}(\alpha) < \kappa$, then let $X \subseteq \alpha$ be a cofinal subset of α with $|X| < \kappa$ such that all $\alpha' \in X$ have cofinality $\geq \kappa$. Since $G_\alpha = \bigcup_{\alpha' \in X} G_{\alpha'}$ and $G_{\alpha'} \sqsubseteq_\kappa G$ for all $\alpha' \in X$ by the first part of the proof, it follows that $G_\alpha \sqsubseteq_\kappa G$. $\dashv$ (Claim 4.2.4)

By Claim 4.2.2 and by the induction hypothesis, we have $col(G_\alpha) \leq \kappa$ for all $\alpha \in C$. Hence by Lemma 2.2 we can conclude that $col(G) \leq \kappa$.

$\Box$ (Theorem 4.2)

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