Remarks on the coloring number of graphs

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Abstract

We give two characterizations of graphs with coloring number \( \leq \kappa \) in terms of elementary submodels; one under ZFC and another under SSH and the version of very weak square principle of [8].

These characterizations suggest that the graphs with coloring number \( \leq \kappa \) behave very much like the Boolean algebras with \( \kappa \)-Freese-Nation property (see [5], [8]).

1 Introduction

A graph \( G = \langle G, K \rangle (K \subseteq [G]^2) \) has coloring number \( \leq \kappa \) (notation: \( \text{col}(G) \leq \kappa \)) if there is a well ordering \( \subseteq \) on \( G \) such that \( K^a_\subseteq = \{ b \in G : b \subseteq a \text{ and } \{ a, b \} \in \subseteq \} \)
$K$ has cardinality $< \kappa$ for all $a \in G$ ([3]). The coloring number $\text{col}(G)$ of $G$ is then defined as the minimum of such $\kappa$'s. It is easy to see that the chromatic number $\chi(G)$ of $G$ is less or equal to $\text{col}(G)$.

The purpose of this note is to show that the graphs with coloring number $\leq \kappa$ behave quite similarly to the Boolean algebras with $\kappa$-Freese-Nation property (see e.g. [5], [8]).

In Section 2 we give a characterization of graphs with coloring number $\leq \kappa$ in terms of elementary submodels (Theorem 2.4). As an application of the characterization, we present in Section 3 a short proof of the countability of the coloring number of the plane.

In Section 4, we show that the characterization of Section 2 can be yet sharpened under SSH and the version of the very weak square principle introduced in [8] (Theorem 4.2).

Both Theorems 2.4 and 4.2 find their parallels in the theory of Boolean algebras with $\kappa$-Freese-Nation property (see Proposition 3 and Theorem 10 in [8]).

The following theorem also underlines the analogy between the Boolean algebras with the $\kappa$-Freese-Nation property and the graphs with coloring number $\leq \kappa$ in the case of $\kappa = \aleph_0$. Note that Boolean algebras with $\aleph_0$-Freese Nation property are also called *openly generated.*

If $G = \langle G, K \rangle$ is graph then we identify any subset $H$ of $G$ with the graph $G \upharpoonright H = \langle H, K \cap [H]^2 \rangle$.

**Theorem 1.1** ([6] and [7]). *The following assertions are equivalent over ZFC:*

(\alpha) For any Boolean algebra $B$ if there are club many subalgebras of $B$ of cardinality $\aleph_1$ which are openly generated then $B$ is openly generated.

(\beta) For any graph $G$ if $\text{col}(H) \leq \aleph_0$ for every $H \in [G]^\aleph_1$ then $\text{col}(G) \leq \aleph_0$.

\[\square\]

Theorem 1.1 in the formulation as above is a sort of bluff since we actually proved that each of the assertions (\alpha) and (\beta) is equivalent to the set-theoretic principle FRP introduced in [4].

**2 A characterization of graphs with coloring number $\leq \kappa$**

We use here the following notations. The first one was already used in the introduction:

For a linear ordering $\Subset$ on a graph $G = \langle G, K \rangle$ we denote
(2.1) \[ K^a_{\Subset} = \{ b \in G : b \Subset a \text{ and } \{a, b\} \in K \} \]

If \( H \subseteq G \) then we write
\[ K^{H,a}_{\Subset} = \{ b \in H : b \Subset a \text{ and } \{a, b\} \in K \} \]

For a graph \( G = \langle G, K \rangle \), \( H \subseteq G \) and \( a \in G \), let
\[ K^a_H = \{ b \in H : \langle a, b \rangle \in K \} \]

We write \( H \subseteq_\kappa G \) if \( |K^a_H| < \kappa \) for all \( a \in G \setminus H \).

A mapping \( f : G \rightarrow [G]^{<\kappa} \) is a \( \kappa \)-coloring mapping on \( G \) if for any \( a, b \in G \) with \( \{a, b\} \in K \), at least one of \( a \in f(b) \) or \( b \in f(a) \) holds.

**Lemma 2.1** ([7]). For any graph \( G \) and any infinite cardinal \( \kappa \), the following are equivalent:

(a) \( \text{col}(G) \leq \kappa \).

(b) There is a \( \kappa \)-coloring mapping on \( G \).

**Proof.** (a) \( \Rightarrow \) (b): Suppose that \( \text{col}(G) \leq \kappa \) and let \( \Subset \) be a well-ordering on \( G \) such that \( |K^a_{\Subset}| < \kappa \) for all \( a \in G \). Then \( f : G \rightarrow [G]^{<\kappa} \) defined by \( f(a) = K^a_{\Subset} \) for \( a \in G \) is a \( \kappa \)-coloring mapping.

(b) \( \Rightarrow \) (a): Suppose that \( f : G \rightarrow [G]^{<\kappa} \) is a \( \kappa \)-coloring mapping on \( G \). Let \( \Subset \) be a well-ordering on \( G \) such that all initial segments of \( G \) of order-type of the form \( \kappa \cdot \alpha \) with respect to \( \Subset \) are closed with respect to \( f \). Then \( \Subset \) is as desired:

**Claim 2.1.1.** \( |K^a_{\Subset}| < \kappa \) for all \( a \in G \).

\[ \vdash \text{Suppose that } a \in G \text{ is the } \kappa \cdot \alpha + \beta \text{'th element with respect to } \Subset \text{ where } \beta < \kappa. \text{ Then the first } \kappa \cdot \alpha \text{ elements of } G \text{ are closed with respect to } f \text{ and hence if } b \text{ is among them and } \{a, b\} \in K \text{ then we have } b \in f(a). \text{ Thus} \]

\[ K^a_{\Subset} \subseteq \{ b \in G : b \text{ is the } \gamma \text{'th element for some } \kappa \cdot \alpha \leq \gamma < \kappa \cdot \alpha + \beta \} \]

\[ \cup f(a). \]

The right side of the inclusion has size \( < \kappa \) (note that we need here the infinity of \( \kappa \)). Hence \( |K^a_{\Subset}| < \kappa. \)

\[ \dashv \text{(Claim 2.1.1)} \]

\[ \square \text{(Lemma 2.1)} \]

**Lemma 2.2.** Suppose that \( \langle G_\alpha : \alpha < \delta \rangle \) is a filtration of a graph \( G = \langle G, K \rangle \) and \( \kappa \) is an infinite cardinal. If \( G_\alpha \subseteq_\kappa G \) and \( \text{col}(G_{\alpha+1}) \leq \kappa \) for all \( \alpha < \delta \), then we have \( \text{col}(G) \leq \kappa \).
Proof. For $a \in G$ let $o(a) = \min\{\alpha < \delta : a \in G_{\alpha+1}\}$. For $\alpha < \delta$, let $\preceq_{\alpha+1}$ be a well-ordering of $G_{\alpha+1}$ witnessing $\text{col}(G_{\alpha+1}) \leq \kappa$. Let $\preceq$ be the ordering on $G$ defined by:

$$
(2.4) \quad a \preceq b \iff o(a) < o(b) \text{ or } \left(o(a) = o(b) \text{ and } a \not\preceq_{o(a)+1} b\right).
$$

Then $\preceq$ is a well ordering on $G$. The following claim shows that $\preceq$ witnesses that $G$ has coloring number $< \kappa$.

Claim 2.2.1. $|K_{\preceq}^a| < \kappa$ for all $a \in G$.

$\vdash$ For $a \in G$, we have $K_{\preceq}^a \subseteq K_{G_{o(a)}}^a \cup K_{G_{o(a)+1}}^a$. Since the right side of the inclusion is of cardinality $< \kappa$, it follows that $|K_{\preceq}^a| < \kappa$. $\square$ (Claim 2.2.1)

Lemma 2.3. Suppose that $H_0$ and $H_1$ are subsets of $G$ with $H_0 \subseteq \kappa G$ and $H_1 \subseteq \kappa G$. Then we have $H_0 \cap H_1 \subseteq \kappa G$.

Proof. Suppose that $a \in G \setminus (H_0 \cap H_1)$. Then we have $a \in G \setminus H_0$ or $a \in G \setminus H_1$. If $a \in G \setminus H_0$, then $K_{H_0 \cap H_1}^a \subseteq K_{H_0}^a$. And hence $|K_{H_0 \cap H_1}^a| < \kappa$. If $a \in G \setminus H_1$, then $K_{H_0 \cap H_1}^a \subseteq K_{H_1}^a$. And hence again we have $|K_{H_0 \cap H_1}^a| < \kappa$.

This shows $H_0 \cap H_1 \subseteq \kappa G$. $\square$ (Lemma 2.3)

Theorem 2.4. For any graph $G = \langle G, K \rangle$ and an infinite cardinal $\kappa$, the following are equivalent:

(a) $\text{col}(G) \leq \kappa$.

(a') There is a well-ordering $\preceq$ of $G$ of order-type $|G|$ such that $|K_{\preceq}^a| < \kappa$ for all $a \in G$.

(b) $G$ has a $\kappa$-coloring mapping.

(c) For $\forall a$ all sufficiently large regular $\chi$ and for all $M \prec H(\chi)$ such that $\langle G, K \rangle \in M$ and $\kappa+1 \subseteq M$ we have $G \cap M \subseteq \kappa G$.

Proof. (a) $\Rightarrow$ (b) was already proved in Lemma 2.1. (a') $\Rightarrow$ (a) is trivial. The proof of (b) $\Rightarrow$ (a) in Lemma 2.1 actually proves (b) $\Rightarrow$ (a').

For (a) $\Rightarrow$ (c), suppose that $G = \langle G, K \rangle$ has coloring number $\leq \kappa$. Let $\chi$ be a sufficiently large regular cardinal and $M \prec H(\chi)$ be such that $G \in M$ and $\kappa+1 \subseteq M$. By elementarity and (a) $\Leftrightarrow$ (b), there is $f \in M$ such that $f$ is a $\kappa$-coloring mapping on $G$. Note that by $\kappa+1 \subseteq M$ and by elementarity, $G \cap M$ is closed with respect to $f$. For $a \in G \setminus M$ and $b \in K_{G \cap M}^a$, since $a \not\preceq f(b) \subseteq M$, we have $b \in f(a)$. Thus $K_{G \cap M}^a \subseteq f(a)$ and hence $|K_{G \cap M}^a| < \kappa$. This shows that $G \cap M \subseteq \kappa G$.
Now we prove (c) \(\Rightarrow\) (a) by induction on \(|G|\).

If \(|G| \leq \kappa\), then (c) \(\Rightarrow\) (a) holds since \(G\) then has coloring number \(\leq \kappa\) anyway — any well-ordering of \(G\) of order-type \(|G|\) will witness this.

Suppose that \(|G| > \kappa\) and we have shown the implication (c) \(\Rightarrow\) (a) for all graphs of cardinality \(< |G|\). Let \(\lambda = |G|,\ \lambda^* = \text{cf}(\lambda)\) and \(<M_{\alpha} : \alpha < \lambda^*>\) a continuously increasing chain of elementary submodels of \(\mathcal{H}(\chi)\) such that

\[
\begin{align*}
(2.5) & \quad G \in M_0; \ \kappa + 1 \subseteq M_0; \\
(2.6) & \quad |M_{\alpha}| < \lambda \text{ for all } \alpha < \lambda^*; \text{ and} \\
(2.7) & \quad G \subseteq \bigcup_{\alpha<\lambda^*} M_{\alpha}.
\end{align*}
\]

For \(\alpha < \lambda^*\), let \(G_{\alpha} = G \cap M_{\alpha}\). Then \(<G_{\alpha} : \alpha < \lambda^*>\) is a filtration of \(G\) by (2.6) and (2.7). \(G_{\alpha} \subseteq G\) for all \(\alpha < \kappa\) by (2.5) and by the assumption of (c).

By Lemma 2.3, \(G_{\alpha}\) also satisfies (c) for \(\alpha < \lambda^*\). Since \(|G_{\alpha}| < \lambda\), it follows that \(\text{col}(G_{\alpha}) \leq \kappa\) for all \(\alpha < \lambda^*\) by the induction hypothesis. Hence we have \(\text{col}(G) \leq \kappa\) by Lemma 2.2. \(\square\) (Theorem 2.4)

### 3 Coloring number of the plane

The plane, or the unit distance graph of the plane, is the graph \(G^1(\mathbb{R}^2)\) defined by \(G^1(\mathbb{R}^2) = \langle \mathbb{R}^2, K^1_{\mathbb{R}^2}\rangle\) where \(K^1 = \{\{x, y\} \in [\mathbb{R}^2]^2 : d(x, y) = 1\}\). Applying Theorem 2.4, we can show easily that the coloring number of the plane is equal to \(\aleph_0\).

**Theorem 3.1.** \(\text{col}(G^1(\mathbb{R}^2)) = \aleph_0\).

**Proof.** In [2] it is noted that the list-chromatic number \(\text{list}(G^1(\mathbb{R}^2))\) of \(G^1(\mathbb{R}^2)\) is infinite since finite regular graph of arbitrarily large degree \(d\) can be embedded in \(G^1(\mathbb{R}^2)\) (e.g., throwing down of \(n\)-dimensional cube onto the plane) and the list-chromatic number of such finite graph is \(d\) (see [1]). Thus we have \(\aleph_0 \leq \text{list}(G^1(\mathbb{R}^2)) \leq \text{col}(G^1(\mathbb{R}^2))\).

To prove the inequality \(\text{col}(G^1(\mathbb{R}^2)) \leq \aleph_0\), let \(\chi\) be sufficiently large and \(N < \mathcal{H}(\chi)\). Note that we have \(G^1(\mathbb{R}^2) \in N\) since the plane is definable. Suppose \(x \in \mathbb{R}^2 \setminus N\). Let us write simply \(K\) for \(K^1_{\mathbb{R}^2}\). By Theorem 2.4, it is enough to show that \(K^2_{\mathbb{R}^2 \cap N}\) is finite. Actually, we can show that \(|K^2_{\mathbb{R}^2 \cap N}| \leq 1\).

Toward a contradiction, suppose that \(|K^2_{\mathbb{R}^2 \cap N}| > 1\). Then there are two distinct \(y, z \in G \cap N\) such that \(d(x, y) = d(x, z) = 1\). But then \(X = \{u \in \mathbb{R}^2 : d(u, y) = d(u, z) = 1\}\) is a two element set definable with parameters from \(N\). It follows that \(x \in X \subseteq N\). This is a contradiction to the choice of \(x\). \(\square\) (Theorem 3.1)
With the same proof we can also show:

\[
\text{col}(G^{\text{Odd}}(\mathbb{R}^2)) = \text{col}(G^N(\mathbb{R}^2)) = \text{col}(G^Q(\mathbb{R}^2)) = \text{col}(G^{\text{algebraic}}(\mathbb{R}^2)) = \cdots = \aleph_0.
\]

Theorem 3.1 may be already known. However I could not find any direct mention or proof of the theorem in the literature. Also, in [2] the authors prove \(\text{list}(G^{\text{Odd}}(\mathbb{R}^2)) \leq \aleph_0\) directly and it seems that idea of the proof cannot be extended to a proof of \(\text{col}(G^{\text{Odd}}(\mathbb{R}^2)) \leq \aleph_0\).

I first learned a proof of \(\text{col}(G^1(\mathbb{R}^2)) \leq \aleph_0\) from Hiroshi Sakai in November 2009 who proved the inequality straightforwardly.

Theorem 2.4 is often quite useful to decide the coloring number of infinite graphs. For example, \(\text{col}(K(\kappa, \kappa)) = \kappa\) and \(\text{col}(K(\kappa, \lambda)) = \kappa^+\) for any \(\aleph_0 \leq \kappa < \lambda; \text{col}(G^{\text{Odd}}(\mathbb{R}^3)) = \aleph_1\) etc. can be seen immediately by this theorem.

We shall demonstrate the last equality. Recall \(G^{\text{Odd}}(\mathbb{R}^3) = (\mathbb{R}^3, K^{\text{Odd}}_{\mathbb{R}^3})\) where \(K^{\text{Odd}}_{\mathbb{R}^3} = \{(\vec{x}, \vec{y}) \in [\mathbb{R}^3]^2 : d(\vec{x}, \vec{y}) \text{ is an odd (natural) number}\}\).

**Theorem A.3.1.** \(\text{col}(G^{\text{Odd}}(\mathbb{R}^3)) = \aleph_1\).

**Proof.** For notational simplicity, let \(G = G^{\text{Odd}}(\mathbb{R}^3) = (G, K)\) with \(G = \mathbb{R}^3\) and \(K = K^{\text{Odd}}_{\mathbb{R}^3}\). Suppose that \(\chi\) is sufficiently large. By Theorem 2.4, it is enough to show that \(G \cap M \subseteq_{\aleph_1} G\) for all \(M \prec \mathcal{H}(\chi)\) but \(G \cap M \not\subseteq_{\aleph_0} G\) for some \(M \prec \mathcal{H}(\chi)\).

Suppose that \(M \prec \mathcal{H}(\chi)\). If \(\mathbb{R} \subseteq M\) then \(G \subseteq M\) and we have \(G \cap M \subseteq_{\aleph_1} G\) vacuously.

Otherwise, let \(\mathcal{C} = \{(x, y, 0) \in \mathbb{R}^3 : d((x, y, 0), 0) = 1\}\), we have \(C \not\subseteq M\) for \(\bar{x} \in C \setminus M\). Then, for any odd \(n \in \omega\), \(\sqrt{n^2 - 1} \in M\) and \(d(\bar{x}, (0, 0, \sqrt{n^2 - 1})) = n\). Thus \((0, 0, \sqrt{n^2 - 1}) \in K^{\bar{x}}_{G \cap M}\). This shows that \(G \cap M \not\subseteq_{\aleph_0} G\).

To show \(G \cap M \subseteq_{\aleph_1} G\), assume for contradiction that there is \(\bar{x} \in G \setminus M\) such that \(K^{\bar{x}}_{G \cap M}\) is uncountable. Then there is an odd \(n \in \omega\) such that \(X = \{\vec{y} \in G \cap M : d(\bar{x}, \vec{y}) = n\}\) is uncountable. Let \(y_0, y_1, y_3\) be three distinct elements of \(X\). \(Y = \{\vec{z} \in G : d(\bar{z}, \vec{y}_0) = d(\bar{z}, \vec{y}_1) = d(\bar{z}, \vec{y}_3) = n\}\) is a two-elements set definable with parameters from \(M\). It follows that \(\bar{z} \in Y \subseteq M\). This is a contradiction to the choice of \(\bar{x}\). \(\square\) (Theorem A.3.1)

### 4 Coloring number under very weak square

The following version of the very weak square was introduced in [8].
For a regular cardinal $\kappa$ and $\mu > \kappa$, let $\square^{**}_{\kappa,\mu}$ be the following assertion: there exists a sequence $(C_\alpha : \alpha < \mu^+)$ and a club set $D \subseteq \mu^+$ such that, for all $\alpha \in D$ with $\text{cf}(\alpha) \geq \kappa$, we have

\begin{align*}
(4.1) \quad C_\alpha \subseteq \alpha, \ C_\alpha \text{ is unbounded in } \alpha; \text{ and }
(4.2) \quad [\alpha]^{<\kappa} \cap \{C_{\alpha'} : \alpha' < \alpha\} \text{ dominates } [C_\alpha]^{<\kappa} \text{ (with respect to } \subseteq).}
\end{align*}

For a (sufficiently large regular) cardinal $\chi$ and $M \prec \mathcal{H}(\chi)$, $M$ is $\kappa$-internally cofinal if $[M]^{<\kappa} \subseteq M$ is cofinal in $[M]^{<\kappa}$ with respect to $\subseteq$. For $D \subseteq [\mathcal{H}(\chi)]^{<\kappa}$, $M$ is $D$-internally cofinal if $D \cap M$ is cofinal in $[M]^{<\kappa}$ with respect to $\subseteq$.

Suppose now that $\kappa$ is a regular cardinal and $\mu > \kappa$ is such that $\text{cf}(\mu) < \kappa$. Let $\mu^* = \text{cf}(\mu)$. For a sufficiently large $\chi$ and $x \in \mathcal{H}(\chi)$, let us call a sequence $\langle M_{\alpha,\beta} : \alpha < \mu^+, \beta < \mu^* \rangle$ a $(\kappa, \mu)$-dominating matrix (of elementary submodels of $\mathcal{H}(\chi)$) over $x$ if the following conditions (4.3) – (4.6) hold:

\begin{align*}
(4.3) \quad M_{\alpha,\beta} \prec \mathcal{H}(\chi), \ x \in M_{\alpha,\beta}, \ \kappa + 1 \subseteq M_{\alpha,\beta} \text{ and } |M_{\alpha,\beta}| < \mu \text{ for all } \alpha < \mu^+ \text{ and } \beta < \mu^*; \\
(4.4) \quad \langle M_{\alpha,\beta} : \beta < \mu^* \rangle \text{ is an increasing sequence for each fixed } \alpha < \mu^+; \\
(4.5) \quad \text{if } \alpha < \mu^+ \text{ is such that } \text{cf}(\alpha) \geq \kappa, \text{ then there is } \beta^* < \mu^* \text{ such that, for every } \beta^* \leq \beta < \mu^*, \ M_{\alpha,\beta} \text{ is } \kappa \text{-internally cofinal.}
\end{align*}

For $\alpha < \mu^+$, let $M_\alpha = \bigcup_{\beta < \mu^*} M_{\alpha,\beta}$. By (4.3) and (4.4), we have $M_\alpha \prec \mathcal{H}(\chi)$.

\begin{align*}
(4.6) \quad \langle M_\alpha : \alpha < \mu^+ \rangle \text{ is continuously increasing and } \mu^+ \subseteq \bigcup_{\alpha < \mu^+} M_\alpha.
\end{align*}

**Theorem 4.1** (THEOREM 7 in [8]). Suppose that $\kappa$ is a regular cardinal and $\mu > \kappa$ is such that $\text{cf}(\mu) < \kappa$. If we have $\text{cf}([\lambda]^{<\kappa}, \subseteq) = \lambda$ for cofinally many $\lambda < \mu$ and $\square^{**}_{\kappa,\mu}$ holds, then, for any sufficiently large $\chi$ and $x \in \mathcal{H}(\chi)$, there is a $(\kappa, \mu)$-dominating matrix over $x$.

**Theorem 4.2.** Assume SSH and $\square^{**}_{\kappa,\mu}$ for a regular uncountable $\kappa$ and all singular cardinal $\mu$ with $\text{cf}(\mu) < \kappa < \mu$.

Then, for any graph $G = (G, K)$ the following are equivalent:

\begin{itemize}
  \item[(a)] $\text{col}(G) \leq \kappa$.
  \item[(d)] For all sufficiently large regular $\chi$ and $\kappa$-internally cofinal $M \prec \mathcal{H}(\chi)$ with $G \in M$ we have $G \cap M \subseteq_\kappa G$.
  \item[(e)] For all sufficiently large regular $\chi$ there is $D \subseteq [\mathcal{H}(\chi)]^{<\kappa}$ such that $D$ is cofinal in $[\mathcal{H}(\chi)]^{<\kappa}$ and, for any $D$-internally cofinal $M \prec \mathcal{H}(\chi)$, we have $G \cap M \subseteq_\kappa G$.
\end{itemize}
Proof. (a) ⇒ (d) follows from Theorem 2.4, (d) ⇒ (e) is trivial (just put 
\[ D = [\mathcal{H}(\chi)]^{<\kappa}. \]

For (e) ⇒ (a), we proceed with induction on \(|G|\). If \(|G| \leq \kappa\) then the
implication (e) ⇒ (a) is trivial since \(\text{col}(G) \leq \kappa\) holds always for any graph of
size \(\leq \kappa\). Suppose now that \(|G| > \kappa\) and we have shown the implication (e) ⇒
(a) for all graphs of cardinality < \(|G|\).

Assume that \(G\) satisfies (e) with \(\chi\) and \(\mathcal{D}\). Let \(\chi^*\) be sufficiently large above
\(\chi\) such that we have in particular \(\mathcal{H}(\chi) \in \mathcal{H}(\chi^*)\).

Claim 4.2.1. If \(M\) is a \(\kappa\)-internal cofinal elementary submodel of \(\mathcal{H}(\chi^*)\) such
that

\[
(4.7) \quad G, \chi, \mathcal{D} \in M \text{ and } \kappa + 1 \subseteq M,
\]

then we have \(G \cap M \subseteq \kappa G\).

\(\vdash\) Suppose not. Then there is \(a \in G \setminus M\) such that \(|K_{G \cap M}^{a}| \geq \kappa\). Let
\(N = \mathcal{H}(\chi) \cap M\). By elementarity we have \(N < \mathcal{H}(\chi)\). Let \(\langle N_{\alpha} : \alpha < \kappa \rangle\) be an
increasing sequence such that, for all \(\alpha < \kappa\), we have

\[
(4.8) \quad N_{\alpha} \in \mathcal{D} \cap M; \\
(4.9) \quad N_{\alpha} \in N_{\alpha+1}; \\
(4.10) \quad \text{there is } N_{\alpha}^* \in [N]^{<\kappa} \cap M \text{ such that } N_{\alpha}^* < N \text{ and } N_{\alpha} \subseteq N_{\alpha}^* \subseteq N_{\alpha+1}; \text{ and} \\
(4.11) \quad K_{G \cap M}^{a} \cap (N_{\alpha+1} \setminus N_{\alpha}) \neq \emptyset.
\]

The construction is possible by elementarity of \(M\) and since \(\mathcal{D}\) is cofinal in
\([\mathcal{H}(\chi)]^{<\kappa}\).

Let \(N^* = \bigcup_{\alpha < \kappa} N_{\alpha}\). By (4.10) we have \(N^* < N < \mathcal{H}(\chi)\). By (4.8) and (4.9)
\(N^*\) is \(\mathcal{D}\)-internally cofinal. On the other hand, we have \(|K_{G \cap N^*}^{a}| \geq \kappa\) by (4.11).
This is a contradiction to the assumption of (e). \(\vdash\) (Claim 4.2.1)

Claim 4.2.2. If \(H \subseteq \kappa G\) then for every \(\mathcal{D}\)-internally cofinal \(M < \mathcal{H}(\chi)\) we
have \(H \cap M \subseteq \kappa H\). In particular, \(H\) also satisfies the condition (e).

Proof. Suppose that \(M < \mathcal{H}(\chi)\) is \(\mathcal{D}\)-internally approachable. For \(a \in H \setminus (H \cap M)\), since \(a \in G \setminus (G \cap M)\), we have \(K_{H \cap M}^{a} \subseteq K_{G \cap M}^{a}\). The right side of
the inclusion is of cardinality < \(\kappa\) by the assumption of (e) on \(G\). This shows
that \(H \cap M \subseteq \kappa H\). \(\vdash\) (Claim 4.2.2)

Now we finish the induction step for the proof of (e) ⇒ (a) in two cases. Let
\(\nu = |G|\).

Case I. \(\nu\) is a limit cardinal or \(\nu = \delta^+\) with \(\text{cf}(\delta) \geq \kappa\).

Let \(\nu^* = \text{cf}(\nu)\). Note that, in this case, we have that
(4.12) the cardinals $\lambda < \nu$ such that $cf([\lambda]^{<\kappa}) = \lambda$ are cofinal among cardinals below $\nu$

by SSH.

Let $\langle M_\alpha : \alpha < \nu^* \rangle$ be an increasing sequence of elementary submodels of $\mathcal{H}(\chi^*)$ of cardinality $< \nu$ satisfying (4.7) and $G \subseteq \bigcup_{\alpha < \nu^*} M_\alpha$. We can find such a sequence by (4.12).

Let $G_\alpha = \left\{ \begin{array}{ll} G \cap M_\alpha & \text{if } \alpha = 0 \text{ or } \alpha \text{ is a successor ordinal;} \\ G \cap \left( \bigcup_{\beta < \alpha} M_\beta \right) & \text{otherwise} \end{array} \right.$

for $\alpha < \nu^*$. Then $\langle G_\alpha : \alpha < \nu^* \rangle$ is a filtration of $G$.

**Claim 4.2.3.** $G_\alpha \subseteq \kappa G$ for all $\alpha < \nu^*$.

**Proof.** If $\alpha < \nu^*$ is 0 or a successor ordinal, this follows from Claim 4.2.1.

If $\alpha < \nu^*$ is a limit and $cf(\alpha) < \kappa$, then $G_\alpha$ is a union of less than $\kappa$ many $G_\beta$'s where $\beta < \alpha$ may be chosen to be a successor ordinal and hence $G_\beta \subseteq \kappa G$.

It follows that we have $G_\alpha \subseteq \kappa G$ also in this case.

If $cf(\alpha) \geq \kappa$, then $\bigcup_{\beta < \alpha} M_\beta$ is $\kappa$-internally cofinal and hence we have $G_\beta \subseteq \kappa G$ again by Claim 4.2.1. \(\dashv\) (Claim 4.2.3)

Now by Claim 4.2.2 and by the induction hypothesis, all of $G_\alpha$, $\alpha < \nu^*$ are of coloring number $\leq \kappa$. By Lemma 2.2, it follows that $G$ also has coloring number $\leq \kappa$.

**Case II.** $\nu = \mu^+$ with $cf(\mu) < \kappa$. Let $\mu^* = cf(\mu)$.

By Theorem 4.1, there is a $(\kappa, \mu)$-dominating matrix $\langle M_{\alpha,\beta} : \alpha < \nu, \beta < \mu^* \rangle$ of submodels of $\mathcal{H}(\chi^*)$ over $x = \langle G, \mathcal{H}(\chi) \rangle$.

For $\alpha < \nu$ and $\beta < \mu^*$, let $G_{\alpha,\beta} = G \cap M_{\alpha,\beta}$ and $G_\alpha = \bigcup_{\beta < \mu^*} G_{\alpha,\beta} = G \cap \left( \bigcup_{\beta < \mu^*} M_{\alpha,\beta} \right)$. By (4.6), the sequence $\langle G_\alpha : \alpha < \nu \rangle$ is continuously increasing and $\bigcup_{\alpha < \nu} G_\alpha = G$. By (4.3), we have $|G_\alpha| \leq \mu < \nu$. Thus $\langle G_\alpha : \alpha < \nu \rangle$ is a filtration of $G$.

Let $\mathcal{C} = \{ \alpha < \nu : cf(\alpha) \geq \kappa \} \cup \{ \alpha' < \alpha : cf(\alpha') \geq \kappa \}$ is cofinal in $\alpha$.

$\mathcal{C}$ is a club subset of $\nu$.

**Claim 4.2.4.** $G_\alpha \subseteq \kappa G$ for all $\alpha \in \mathcal{C}$.
Suppose $\alpha \in C$. If $\text{cf}(\alpha) \geq \kappa$, $M_{\alpha,\beta}$ is $\kappa$-internally cofinal for all sufficiently large $\beta < \mu^*$ by (4.5). Hence by Claim 4.2.1, we have $G_{\alpha,\beta} \subseteq \kappa G$ for all such $\beta$. Since $\mu^* < \kappa$, it follows that $G_{\alpha} \subseteq \kappa G$.

If $\text{cf}(\alpha) < \kappa$, then let $X \subseteq \alpha$ be a cofinal subset of $\alpha$ with $|X| < \kappa$ such that all $\alpha' \in X$ have cofinality $\geq \kappa$. Since $G_{\alpha} = \bigcup_{\alpha' \in X} G_{\alpha'}$ and $G_{\alpha'} \subseteq \kappa G$ for all $\alpha' \in X$ by the first part of the proof, it follows that $G_{\alpha} \subseteq \kappa G$. ⊣ (Claim 4.2.4)

By Claim 4.2.2 and by the induction hypothesis, we have $\text{col}(G_{\alpha}) \leq \kappa$ for all $\alpha \in C$. Hence by Lemma 2.2 we can conclude that $\text{col}(G) \leq \kappa$.

□ (Theorem 4.2)

References


