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Generalized Mathias Forcing

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Abstract Here we prove some basic facts about Mathias-like forcings using properness and superproperness instead of fusion arguments. We then show how these apply to the standard Mathias forcing and to Mathias-like forcings involving projections on a separable infinite dimensional Hilbert space.

1 Order Properties

Definition 1.1 *For p, q in a preorder $\mathbb{P}$,*

(i) $p|q \Leftrightarrow \exists r \in \mathbb{P}(r \leq p, q)$,

(ii) $p\top q \Leftrightarrow \nexists r \in \mathbb{P}(r \leq p, q)$, *and*

(iii) $p \prec q \Leftrightarrow \forall r \leq p(r|q)$.

$\mathbb{P}$ *is* separative *if* $\prec$ *is identical to* $\leq$.

In the results that follow, N is countable and $\mathbb{P} \in N \prec H(\chi)$, for sufficiently large χ.

Definition 1.2 $p \in \mathbb{P}$ *is* N-generic *if, for every dense* $D \in N \cap \mathscr{P}(\mathbb{P})$, $D \cap N$ *is predense below* p. $\mathbb{P}$ *is* proper *if (for all/some* N*) the set of* N*-generic elements is dense below* $\mathbb{P} \cap N$.

As the downwards closure of any $D \in N \cap \mathscr{P}(\mathbb{P})$ is again in N, it makes no difference above if we quantify over predense, open dense or maximal antichain D. We also have the following characterisation.

Proposition 1.3 $p \in \mathbb{P}$ *is* N*-generic if and only if, whenever* $p \geq q \in C \in N \cap \mathscr{P}(\mathbb{P})$, *we have* $r \in C \cap N$ *with* $q|r$.

Proof: If p is N-generic and $p \geq q \in C \in N \cap \mathscr{P}(\mathbb{P})$ then let $D = C \cup \{d \in \mathbb{P} : \forall c \in C(d\top c)\} \in N \cap \mathscr{P}(\mathbb{P})$. As D is predense we can find $r \in D \cap N$ such that $r|q \in C$ and hence $r \in C \cap N$.

On the other hand say that whenever $p \geq q \in C \in N \cap \mathscr{P}(\mathbb{P})$, we have $r \in C \cap N$ with $q|r$. Given dense $D \in N \cap \mathscr{P}(\mathbb{P})$ and $q \leq p$ we can find $s \leq q$ such that $s \in D$. Thus we have $r \in D \cap N$ such that $s|r$ and hence $q|r$, i.e. p is N-generic. □

We will also need the following stronger concept in 2.12.

Definition 1.4 $p \in \mathbb{P}$ *is* N-supergeneric *if, for every dense* $D \in N \cap \mathscr{P}(\mathbb{P})$, *we have* $q \in D \cap N$ *with* $q \succ p$. $\mathbb{P}$ *is* superproper *if (for all/some* N*) the set of* N*-supergeneric elements is dense below* $\mathbb{P} \cap N$.

Again, it makes no difference above if we quantify over predense, open dense or maximal antichain $D \in N$. We also have the following analog of 1.3.

Proposition 1.5 $p \in \mathbb{P}$ *is* N*-supergeneric if and only if, whenever* $p \geq q \in C \in N \cap \mathscr{P}(\mathbb{P})$, *we have* $r \in C \cap N$ *with* $r \succ p$.

Proof: If p is N-supergeneric and $p \geq q \in C \in N \cap \mathscr{P}(\mathbb{P})$ then let

$$D = C \cup \{d \in \mathbb{P} : \forall c \in C(d \top c)\} \in N \cap \mathscr{P}(\mathbb{P}).$$

As D is predense we can find $r \in D \cap N$ with $r \succ p \geq q$ and hence $r | q \in C$ so $r \in C \cap N$.

On the other hand say that whenever $p \geq q \in C \in N \cap \mathscr{P}(\mathbb{P})$, we have $r \in C \cap N$ with $r \succ p$. Given dense $D \in N \cap \mathscr{P}(\mathbb{P})$ and $q \leq p$ we can find $s \leq q$ such that $s \in D$. Thus we have $r \in D \cap N$ such that $r \succ p$, i.e. p is N-supergeneric. □

In what follows we fix preorders $\mathbb{P}$ and $\mathbb{Q}$.

Definition 1.6 $i : \mathbb{P} \to \mathbb{Q}$ *is a* dense embedding *if it is*

(i) order preserving, *i.e.* $\forall p, q \in \mathbb{P}(p \leq q \Rightarrow i(p) \leq i(q))$,

(ii) incompatibility preserving, *i.e.* $\forall p, q \in \mathbb{P}(p \top q \Rightarrow i(p) \top i(q))$ *and*

(iii) has dense image, *i.e.* $\forall q \in \mathbb{Q} \exists p \in \mathbb{P}(i(p) \leq q)$.

Despite the term 'embedding', dense embeddings are not necessarily one-to-one.

Proposition 1.7 *If* $i : \mathbb{P} \to \mathbb{Q}$ *is order and incompatibility preserving then, for all* $p, q \in \mathbb{P}$,

$$i(p) \prec i(q) \Rightarrow p \prec q.$$

Proof: For $r \in \mathbb{P}$, $r \leq p \Rightarrow i(r) \leq i(p) \Rightarrow i(r) | i(q) \Rightarrow r | q$ by 1.6(i), 1.1(iii) and 1.6(ii) respectively. □

Proposition 1.8 *If* $\mathbb{P}$ *is* σ*-closed it is superproper.*

Proof: Let (D_n) enumerate the dense sets of $\mathbb{P}$ in N. Given $d_{-1} \in \mathbb{P} \cap N$, recursively choose $d_n \in D_n \cap N$ such that $d_n \leq d_{n-1}$, for each $n \in \omega$. As $\mathbb{P}$ is σ-closed we can find $d \in \mathbb{P}$ with $d \leq d_n$, for all $n \in \omega$. This d is N-supergeneric. □

Proposition 1.9 *If* $i : \mathbb{P} \to \mathbb{Q}$ *is a dense embedding in* N *and* $i(p)$ *is* N*-supergeneric, so too is* p.

Proof: Take dense $D \in N \cap \mathscr{P}(\mathbb{P})$. As $i[D]$ is dense, $\exists q \in D(i(q) \succ i(p))$. By 1.7, $q \succ p$. □

Corollary 1.10 *If* $i : \mathbb{P} \to \mathbb{Q}$ *is a dense embedding and* $\mathbb{Q}$ *is superproper, so is* $\mathbb{P}$.

2 Mathias-Like Preorders

Definition 2.1 *Say we have preorders* $\mathbb{P}$ *and* $X \times \mathbb{P}$ *such that, for all* $x, y \in X$ *and* $p, q \in \mathbb{P}$,

(i) $(x, p) \leq (y, q) \Rightarrow p \leq q$, *and*

(ii) $p \leq q \Rightarrow (x, p) \leq (x, q)$.

Then we say $X \times \mathbb{P}$ *is* Mathias-like.

Proposition 2.2 *If* $X \times \mathbb{P}$ *is Mathias-like it is densely embeddable in* $\mathbb{P} * (X \times \dot{G})$.

Proof: We prove that the map $(n,p) \mapsto (p,(n,p))$ is a dense embedding. It is immediately seen to be incompatibility preserving and, by (i), it is an order isomorphism too. Now say we have $(p,(\dot{x},\dot{q})) \in \mathbb{P} * (X \times \dot{G})$. Take $r \leq p$ such that $r \Vdash (\dot{x},\dot{q}) = (x,q)$, for some $x \in X$ and $q \in \mathbb{P}$. As $r \leq p \Vdash \dot{q} \in \dot{G}$, we have $r \Vdash q \in \dot{G}$ so any $\mathbb{P}$-generic $G \ni r$ also contains q and hence we can find $s \leq r, q$. But then, by (ii), $(s,(x,s)) \leq (p,(\dot{x},\dot{q}))$. □

Lemma 2.3 *If $\omega \times \mathbb{P}$ is Mathias-like and $p \in \mathbb{P}$ is N-generic then so is (n,p), for all $n \in \omega$.*

Proof: Take $(m,q) \in C \in N \cap \mathscr{P}(\omega \times \mathbb{P})$ such that $(m,q) \leq (n,p)$. By (i), $q \leq p$ and also $q \in \{r : (m,r) \in C\} \in N$ so $\exists r \in N$ such that $(m,r) \in C$ and $q|r$. By (ii), $(m,q)|(m,r) \in N$. □

Corollary 2.4 *If $\mathbb{P}$ is proper then so is $\omega \times \mathbb{P}$.*

Proof: By (ii) and the previous lemma. □

We now apply these results to certain Mathias-like forcings. Generally, the idea is to get a forcing with nice properties which also generically adds some element of a preorder which is either below or incompatible with every element of that preorder in the ground model.

Definition 2.5 $\mathbb{M} = [\omega]^{<\omega} \times [\omega]^{\omega}$ *with* $(a,A) \leq (b,B) \Leftrightarrow b \subseteq a \wedge A \subseteq B \wedge a \backslash b \subseteq B$. *For* $S \subseteq [\omega]^{\omega}$, $\mathbb{M}(S) = [\omega]^{<\omega} \times S$ *with this same order.*

This $\mathbb{M}$ is none other than the standard Mathias forcing. Many authors place the further restriction on elements $(a,A) \in \mathbb{M}$ that $\max(a) \leq \min(A)$. For us this would be an unnecessary complication and, in any case, the restricted version is immediately seen to be dense in the version used here so any forcing properties proved about the former will also apply to the latter. (Alternatively, you could note that the above results still hold for a suborder $\mathbb{S} \subseteq \omega \times \mathbb{P}$ so long as $p \leq q \wedge (n,q) \in \mathbb{S} \Rightarrow (n,p) \in \mathbb{S}$).

Definition 2.6 *Given* $G \subseteq \mathbb{M}$, $A(G) = \bigcup_{(a,A)\in G} a$.

Proposition 2.7 *For all* $B \in [\omega]^{\omega}$, $\mathbb{1} \Vdash_{\mathbb{M}} A(\dot{G}) \subseteq^* B \vee |A(\dot{G}) \cap B| < \infty$.

Proposition 2.8 $\mathbb{M}$ *is Mathias-like (w.r.t.* $\mathbb{P} = [\omega]^{\omega}$ *ordered by* $\subseteq$*).*

Proposition 2.9 $[\omega]^{\omega}$ *ordered by* $\subseteq$ *is superproper.*

Proof: $([\omega]^{\omega}, \subseteq^*)$ is σ-closed and hence superproper by 1.8. As the identity on $([\omega]^{\omega}, \subseteq)$ is a dense embedding into $([\omega]^{\omega}, \subseteq^*)$, $([\omega]^{\omega}, \subseteq)$ is superproper too by 1.10. □

Corollary 2.10 $\mathbb{M}$ *is proper and densely embeddable in* $[\omega]^{\omega} * \mathbb{M}(\dot{G})$.

Proof: By 2.9, 2.4 and 2.2. □

Note that for $[\omega]^{\omega}$ ordered by $\subseteq$, $A \prec B \Leftrightarrow A \subseteq^* B \Leftrightarrow |A \backslash B| < \infty$.

Definition 2.11 $X \times \mathbb{P}$ *has the* pure decision property *if, for every sentence ϕ of the forcing language and every $(x,p) \in X \times \mathbb{P}$, there exists $q \leq p$ such that (x,q) decides ϕ (i.e. $(x,q) \Vdash \phi$ or $(x,q) \Vdash \neg\phi$).*

Theorem 2.12 $\mathbb{M}$ *has the pure decision property.*

Proof: Take a sentence ϕ of the forcing language and let $D = \{p \in \mathbb{M} : p \Vdash \phi\}$. Take countable $N \prec H(\chi)$ with $D \in N$ and N-supergeneric $A \in [\omega]^\omega$. For each $a \in [\omega]^{<\omega}$, let

$$D_a = \{B : (a, B) \in D\} \in N$$

and note that if there exists $B \subseteq A$ with $B \in D_a$ then there exists $B_a \in D_a(\cap N)$ with $A \subseteq^* B_a$. We claim this can be done for all $a \in [\omega]^{<\omega}$. To see this, take any $a \in [\omega]^{<\omega}$ such that B_a is undefined (i.e. for which there exists no $B \subseteq A$ with $B \in D_a$). We first claim that, for all but finitely many $n \in A$, $B_{a\cup\{n\}}$ is undefined. If not then, for $\psi = \phi$ or $\neg\phi$, there exists infintely many $n \in A$ such that $(a \cup \{n\}, B_{a\cup\{n\}}) \Vdash \psi$. Recursively pick distinct $(m_n) \subseteq A$ such that $m_n \in \bigcap_{k<n} B_{a\cup\{m_k\}}$ and $B_{a\cup\{m_n\}} \Vdash \psi$. Let $M = \{m_n : n \in \omega\}$ and note that any $(b, B) \leq (a, M)$ with $b \neq a$ will satisfy $(b, B) \leq (a \cup \{m_n\}, B_{a\cup\{m_n\}})$, where n minimal for $m_n \in b$. But this implies that $(a, M) \Vdash \psi$ even though $M \subseteq A$, contradicting the fact that B_a was undefined.

Now, for each $a \in [\omega]^{<\omega}$ such that B_a is not defined, let $C_a \subseteq A$ with $|A \backslash C_a| < \infty$ be such that $B_{a\cup\{n\}}$ is not defined for all $n \in A \cap C_a$. Given $a \in [\omega]^{<\omega}$ such that B_a is not defined, recursively pick distinct (m_n) such that $m_n \in \bigcap_{b\subseteq\{m_k:k<n\}} C_{a\cup b}$. Let $M = \{m_n : n \in \omega\}$ and note that, for any $(c, B) \leq (a, M)$, we have $c = a \cup b \cup \{m_n\}$ with $b \subseteq \{m_k : k < n\}$, for n maximal for $m_n \in c$, which, as $m_n \in C_{a\cup b}$, means B_c is not defined and hence $B \notin D_c$. But this contradicts the fact that something below (a, M) must decide ϕ. □

Now let H be a separable infinite dimensional Hilbert space H. Order $\mathcal{P}_\infty(H)$, the collection of infinite rank (self-adjoint) projections on H, by $P \leq^* Q \Leftrightarrow PQ - P$ is compact. Order

$$[\mathcal{P}_\infty(H)]^{<\omega}_{\min} = \{(\mathcal{P}, P) : P \in \mathcal{P} \in [\mathcal{P}_\infty(H)]^{<\omega} \wedge \forall Q \in \mathcal{P}(P \leq^* Q)\}$$

by $(\mathcal{P}, P) \leq (\mathcal{Q}, Q) \Leftrightarrow \mathcal{P} \supseteq \mathcal{Q}$.

Proposition 2.13 *$[\mathcal{P}_\infty(H)]^{<\omega}_{\min}$ is superproper.*

Proof: For $(\mathcal{P}, P), (\mathcal{Q}, Q) \in [\mathcal{P}_\infty(H)]^{<\omega}_{\min}$, $\mathcal{P} \supseteq \mathcal{Q} \Rightarrow P \leq^* Q$, while if $R \leq^* P, Q$ then we have $(\mathcal{P} \cup \mathcal{Q} \cup \{R\}, R) \leq (\mathcal{P}, P), (\mathcal{Q}, Q))$. So the map $(\mathcal{P}, P) \mapsto P$ is a dense embedding. As $(\mathcal{P}_\infty(H), \leq^*)$ is σ-closed and hence superproper by 1.8, $[\mathcal{P}_\infty(H)]^{<\omega}_{\min}$ is too by 1.10. □

Now take any dense $(v_n) \subseteq H$, let $\mathcal{V}^{<\infty}_{(v_n)} = \{\mathrm{span}_{n\in F}(v_n) : F \in [\omega]^{<\omega}\}$ and define a preorder on $\mathbb{M}^* = \mathcal{V}^{<\infty}_{(v_n)} \times \omega \times [\mathcal{P}_\infty(H)]^{<\omega}_{\min}$ by

$$(V, n, \mathcal{P}, P) \leq (W, m, \mathcal{Q}, Q) \Leftrightarrow V \supseteq W \wedge \mathcal{P} \supseteq \mathcal{Q} \wedge \forall R \in \mathcal{Q}(||R|_{V\cap W^\perp}|| + 1/n \leq 1/m),$$

taking $1/0 = \infty \geq 1/n$, for all $n \in \omega$, i.e. ignoring the last condition when $m = 0$. Note that if $||R|_{V\cap W^\perp}|| + 1/n \leq 1/m$ and $||R|_{W\cap X^\perp}|| + 1/m \leq 1/l$ then

$$||R|_{V\cap X^\perp}|| + 1/n \leq ||R|_{V\cap W^\perp}|| + ||R|_{W\cap X^\perp}|| + 1/n \leq ||R|_{W\cap X^\perp}|| + 1/m \leq 1/l,$$

so $\leq$ is indeed transitive. For $S \subseteq [\mathcal{P}_\infty(H)]^{<\omega}_{\min}$, $\mathbb{M}^*(S) = \mathcal{V}^{<\infty}_{(v_n)} \times \omega \times S$ with this order.

Definition 2.14 *Given $G \subseteq \mathbb{M}^*$, $V(G) = \overline{\bigcup_{(V,n,\mathcal{P},P)} V}$.*

Proposition 2.15 *For all $P \in \mathcal{P}_\infty(H)$, $\mathbb{1} \Vdash_{\mathbb{M}^*} P_{V(\dot{G})} \leq^* P \vee P_{V(\dot{G})} \top^* P$.*

Proposition 2.16 *$\mathbb{M}^*$ is Mathias-like.*

Corollary 2.17 *$\mathbb{M}^*$ is proper and densely embeddable in $[\mathcal{P}_\infty(H)]^{<\omega}_{\min} * \mathbb{M}^*(\dot{G})$.*

Proof: By 2.13, 2.4 and 2.2. □

This is nice in the sense that $[\mathcal{P}_\infty(H)]^{<\omega}_{\min}$ forcing equivalent to $\mathcal{P}_\infty(H)$. However, if we are not worried about that, we can generically add a closed subspace like $V(\dot{G})$ by generically adding a block subspace. Specifically, take $l^2 \subseteq \mathbb{F}^\omega$ as our Hilbert space and let

$$V^{\mathbb{A}}_{\text{block}} = \{v \in l^2 : \text{dom}(v) < \infty \wedge \forall n \in \omega(v(n) \in \mathbb{A})\},$$

where $\mathbb{A}$ is a countable dense subset of $\mathbb{F}$ (the algebraic numbers for example). Let

$$[V^{\mathbb{A}}_{\text{block}}]_{\text{dom}} = \{\mathcal{V} \subseteq V^{\mathbb{A}}_{\text{block}} : \forall v \neq w \in \mathcal{V}(\text{dom}(v) \cap \text{dom}(w))\}.$$

We define an order on $\mathbb{M}^{\text{block}} = [V^{\mathbb{A}}_{\text{block}}]^{<\omega}_{\text{dom}} \times [V^{\mathbb{A}}_{\text{block}}]^{\omega}_{\text{dom}}$ by

$$(\mathcal{F}, \mathcal{V}) \leq (\mathcal{G}, \mathcal{W}) \Leftrightarrow \mathcal{G} \subseteq \mathcal{F} \wedge \text{every } v \in \mathcal{V} \text{ is a finite linear combination of elements of } \mathcal{W}.$$

Definition 2.18 *Given* $G \subseteq \mathbb{M}^{\text{block}}$, $V(G) = \overline{\text{span}}\{v : \exists(\mathcal{F}, \mathcal{V}) \in G(v \in \mathcal{F})\} = \overline{\text{span}}(\bigcup_{(\mathcal{F},\mathcal{V})\in G} \mathcal{F})$.

Proposition 2.19 *For all* $P \in \mathcal{P}_\infty(H)$, $\mathbb{1} \Vdash_{\mathbb{M}^{\text{block}}} P_{V(\dot{G})} \leq^* P \vee P_{V(\dot{G})} \top^* P$.

Proof: Like the proof that projections onto block subspaces are $\leq^*$-dense in $\mathcal{P}_\infty(H)$.

Proposition 2.20 $\mathbb{M}^{\text{block}}$ *is Mathias-like.*

By proving an analog of 2.13 we also get the following.

Corollary 2.21 $\mathbb{M}^{\text{block}}$ *is proper and densely embeddable in* $[V^{\mathbb{A}}_{\text{block}}]^{\omega}_{\text{dom}} * \mathbb{M}^{\text{block}}(\dot{G})$.

It is also possible to define variations on $\mathbb{M}^*$ and $\mathbb{M}^{\text{block}}$ which also generically add subspaces like $V(\dot{G})$. The precise relation between these variants is not clear, although adding a generic for $\mathcal{V}_\infty(H)$ (the collection of infinite dimensional closed subspaces of H with the inclusion order $\subseteq$) will add one for $\mathcal{P}_\infty(H)$ so it seems possible that adding a generic for $\mathbb{M}^{\text{block}}$ might also add one for $\mathbb{M}^*$. It is also not clear what other properties of $\mathbb{M}$ generalize to these variants, in particular if any of them have the pure decision property.