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# EXISTENCE THEOREMS FOR SADDLE POINTS OF SET-VALUED MAPS VIA NONLINEAR SCALARIZATION METHODS*
# (非線形スカラー化手法を用いた集合値写像の鞍点の存在定理)

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Abstract
In the paper, we introduce five types of concepts for saddle points of set-valued maps and show existence theorems for these saddle points by using nonlinear scalarizing functions for sets introduced by Kuwano, Tanaka, and Yamada in 2009.

## 1 Introduction

Let $X$ and $Y$ be two real topological vector spaces, $F$ a map on $X\times Y$. In real-valued case, $(x_0,y_0)\in X\times Y$ is a saddle point of $F$ if

$$F(x_0,y)\leq F(x_0,y_0)\leq F(x,y_0)$$

for any $x\in X$ and $y\in Y$. In vector-valued case, a saddle point $(x_0,y_0)\in X\times Y$ with respect to partial ordering $\leq_C$ induced by a convex cone $C$ is defined by

$$F(x,y_0)\not\leq_C F(x_0,y_0)\not\leq_C F(x_0,y)$$

for any $x\in X$ and $y\in Y$, and it is called $C$-saddle point of $F$. Many researchers have been investigated existence theorems for saddle points and $C$-saddle points. In [7] and [8], we consider five types of generalizations for $C$-saddle points and investigate sufficient conditions for the existence of these saddle points by using nonlinear scalarization methods for sets proposed in [4].

The aim of the paper is to introduce three types of existence theorems for cone saddle points of set-valued maps.

The organization of the paper is as follows. In Section 2, we review mathematical methodology proposed in [3] on comparison between two sets in an ordered vector space and some basic concepts of set-valued optimization. In Section 3, we consider two types of nonlinear scalarizing functions for sets proposed by the unified approach in [4], and

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investigate their properties. In Section 4, we introduce five types of concepts for cone saddle points of set-valued maps, and three types of existence theorems for these saddle points proved in [7, 8].

## 2 Mathematical Preliminaries

Throughout the paper, $X$ and $Y$ are two real topological vector spaces and $C$ is a proper closed convex cone in $Y$ (that is, $C \neq Y$, $C + C = C$ and $\lambda C \subset C$ for all $\lambda \geq 0$) with nonempty topological interior. We define a partial ordering $\leq_C$ on $Y$ as follows:

$$x \leq_C y \quad \text{if} \quad y - x \in C \quad \text{for} \quad x, y \in Y.$$

Let $F$ be a set-valued map from $S \subset X$ into $2^Y$ where $S := \{x \in X | F(x) \neq \emptyset\}$ and assume that $S$ is a convex set. For $A \in 2^Y \setminus \{\emptyset\}$, we denote the topological interior of $A$ by int$A$. Also, we denote the algebraic sum, algebraic difference of $A$ and $C$ by $A + C := \bigcup_{a \in A}(a + C)$, $A - C := \bigcup_{a \in A}(a - C)$, respectively. In addition, we denote the composite function of two functions $f$ and $g$ by $g \circ f$. When $x \leq_C y$ for $x, y \in Y$, we define the order interval between $x$ and $y$ by $[x, y] := \{z \in Y | x \leq_C z \,\text{and}\, z \leq_C y\}$.

At first, we review some basic concepts of set-relation.

**Definition 2.1.** (See Ref. [3].) For any $A, B \in 2^Y \setminus \{\emptyset\}$ and convex cone $C$ in $Y$, we write

$A \leq_C^{(1)} B$ by $A \subset \bigcap_{b \in B}(b - C)$, equivalently $B \subset \bigcap_{a \in A}(a + C)$,
$A \leq_C^{(2)} B$ by $A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset$,
$A \leq_C^{(3)} B$ by $B \subset (A + C)$,
$A \leq_C^{(4)} B$ by $\left(\bigcap_{a \in A}(a + C)\right) \cap B \neq \emptyset$,
$A \leq_C^{(5)} B$ by $A \subset (B - C)$,
$A \leq_C^{(6)} B$ by $A \cap (B - C) \neq \emptyset$, equivalently $(A + C) \cap B \neq \emptyset$.

**Proposition 2.1.** (See [3].) *For any $A, B \in 2^Y \setminus \{\emptyset\}$, the following statements hold:*

| | |
|---|---|
| $A \leq_C^{(1)} B$ *implies* $A \leq_C^{(2)} B$, | $A \leq_C^{(1)} B$ *implies* $A \leq_C^{(4)} B$, |
| $A \leq_C^{(2)} B$ *implies* $A \leq_C^{(3)} B$, | $A \leq_C^{(4)} B$ *implies* $A \leq_C^{(5)} B$, |
| $A \leq_C^{(3)} B$ *implies* $A \leq_C^{(6)} B$, | $A \leq_C^{(5)} B$ *implies* $A \leq_C^{(6)} B$. |

**Proposition 2.2.** (See [4].) *For any $A, B \in 2^Y \setminus \{\emptyset\}$, the following statements hold:*

(i) *For each $j = 1, \ldots, 6$,*
$A \leq_C^{(j)} B$ *implies* $(A + y) \leq_C^{(j)} (B + y)$ *for* $y \in Y$, *and*
$A \leq_C^{(j)} B$ *implies* $\alpha A \leq_C^{(j)} \alpha B$ *for* $\alpha \geq 0$.

(ii) *For each $j = 1, \ldots, 5$, $\leq_C^{(j)}$ is transitive.*

(iii) *For each $j = 3, 5, 6$, $\leq_C^{(j)}$ is reflexive.*

From (b) and (c) of Proposition 2.2, $\leq_C^{(6)}$ is difficult to say as order. Hence, we consider mainly the cases of $j = 1, \ldots, 5$ in the paper.

By using the set-relations defined in Definition 2.1, we consider the following five kinds of set-valued optimization problems with $j = 1, \dots, 5$:

$$(j\text{-SVOP}) \begin{cases} j\text{-Optimize} & F(x) \\ \text{Subject to} & x \in S. \end{cases}$$

Then, we introduce some concepts of solutions for ($j$-SVOP). Let $x_0 \in S$. For each $j = 1, \dots, 5$, $x_0$ is a *minimal solution of* ($j$-SVOP) if for any $x \in S \setminus \{x_0\}$,

$$F(x) \leq_C^{(j)} F(x_0) \quad \text{implies} \quad F(x_0) \leq_C^{(j)} F(x); \tag{2.1}$$

and $x_0$ is a *maximal solution of* ($j$-SVOP) if for any $x \in S \setminus \{x_0\}$,

$$F(x_0) \leq_C^{(j)} F(x) \quad \text{implies} \quad F(x) \leq_C^{(j)} F(x_0). \tag{2.2}$$

If $C$ is replaced by int$C$, then $x_0$ is a *weak minimal solution* (resp., *weak maximal solution*) of ($j$-SVOP). We denote the family of sets satisfying (2.1) (resp., (2.2)) by $\mathrm{Min}_{(j)}F(S)$ (resp., $\mathrm{Max}_{(j)}F(S)$) and the case of weak minimal (resp., weak maximal) by $\mathrm{WMin}_{(j)}F(S)$ (resp., $\mathrm{WMax}_{(j)}F(S)$) where $F(S) = \{F(x) | x \in S\}$. It is clear that if $x_0$ is a minimal (resp., maximal) solution of ($j$-SVOP) then $x_0$ is a weak minimal (resp., weak maximal) solution of ($j$-SVOP).

Let us recall some definitions of $C$-notions (see [2].) A subset $A$ of $Y$ is said to be *$C$-convex* (resp., *$C$-closed*) if $A + C$ is convex (resp., closed). Moreover, we say that $F$ is *$C$-notion on $S$* if $F(x)$ has the property $C$-notion for every $x \in S$.

Next, we introduce several definitions of $C$-convexity and $C$-continuity for set-valued maps. These notions are used in Sections 3 and 4.

**Definition 2.2.** (See [4].) For each $j = 1, \dots, 5$,

(i) $F$ is called a *type ($j$) naturally quasi $C$-convex function* if for each $x, y \in S$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$F(\lambda x + (1-\lambda)y) \leq_C^{(j)} \mu F(x) + (1-\mu)F(y).$$

(ii) $F$ is called a *type ($j$) naturally quasi $C$-concave function* if for each $x, y \in S$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$\mu F(x) + (1-\mu)F(y) \leq_C^{(j)} F(\lambda x + (1-\lambda)y).$$

**Definition 2.3.** (See [8].) For each $j = 1, \dots, 5$,

(i) $F$ is called a *type ($j$) $C$-convexlike function* if for every $x, y \in S$ and $\lambda \in (0,1)$, there exists $z \in S$ such that

$$F(z) \leq_C^{(j)} \lambda F(x) + (1-\lambda)F(y).$$

(ii) $F$ is called a *type ($j$) $C$-concavelike function* if for every $x, y \in S$ and $\lambda \in (0,1)$, there exists $z \in S$ such that

$$\lambda F(x) + (1-\lambda)F(y) \leq_C^{(j)} F(z).$$

**Definition 2.4.** (See [2].) Let $x \in S$. Then,

(i) $F$ is called *$C$-lower continuous at $x$* if for every open set $V$ with $F(x) \cap V \neq \emptyset$, there exists an open neighborhood $U$ of $x$ such that $F(y) \cap (V + C) \neq \emptyset$ for all $y \in U$. We shall say that $F$ is *$C$-lower continuous on $S$* if it is $C$-lower continuous at every point $x \in S$,

(ii) $F$ is called *$C$-upper continuous at $x$* if for every open set $V$ with $F(x) \subset V$, there exists an open neighborhood $U$ of $x$ such that $F(y) \subset V + C$ for all $y \in U$. We shall say that $F$ is *$C$-upper continuous on $S$* if it is $C$-upper continuous at every point $x \in S$.

## 3 Unified Types of Scalarizing Functions for Sets

In [4], we propose the following nonlinear scalarizing functions for sets: Let $V, V' \in 2^Y \setminus \{\emptyset\}$, and direction $k \in \mathrm{int}C$. For each $j = 1, \ldots, 5$, we define $I^{(j)}_{k,V'} : 2^Y \setminus \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$I^{(j)}_{k,V'}(V) := \inf \left\{ t \in \mathbb{R} \;\middle|\; V \leq^{(j)}_C (tk + V') \right\}.$$

In this section, we introduce some properties of these functions and several sufficient conditions for the existence of solutions of ($j$-SVOP).

**Proposition 3.1.** (See [6].) *Let $A, B \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(i) *If $A \leq^{(1)}_C B$, $A$ is $(-C)$-closed and $B$ is $C$-closed then*

$$I^{(1)}_{k,V'}(A) < I^{(1)}_{k,V'}(B).$$

(ii) *For each $j = 2, 3$, if $A \leq^{(j)}_{\mathrm{int}C} B$ and $B$ is $C$-closed then*

$$I^{(j)}_{k,V'}(A) < I^{(j)}_{k,V'}(B).$$

(iii) *For each $j = 4, 5$, if $A \leq^{(j)}_{\mathrm{int}C} B$ and $A$ is $(-C)$-closed then*

$$I^{(j)}_{k,V'}(A) < I^{(j)}_{k,V'}(B).$$

Next, we introduce certain inherited properties on cone-convexity and cone-continuity of set-valued maps proved in [4, 5, 8, 10].

**Lemma 3.1.** (See [4, 5].) *Let $k \in \mathrm{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(i) *For each $j = 1, 2, 3$, if $F$ is type ($j$) naturally quasi $C$-convex, then $I^{(j)}_{k,V'} \circ F$ is quasi convex. Moreover, if $F$ is type ($j$) naturally quasi $C$-concave, then $I^{(j)}_{k,V'} \circ F$ is quasi concave.*

(ii) *For each $j = 4, 5$, if $F$ is type ($j$) naturally quasi $C$-convex and $V'$ is $(-C)$-convex, then $I^{(j)}_{k,V'} \circ F$ is quasi convex. Moreover, if $F$ is type ($j$) naturally quasi $C$-concave and $V'$ is $(-C)$-convex, then $I^{(j)}_{k,V'} \circ F$ is quasi concave.*

**Lemma 3.2.** (See [8].) *Let $k \in \mathrm{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(i) *For each $j = 1, 2, 3$, if $F$ is type $(j)$ $C$-convexlike and $V'$ is $C$-convex, then $I^{(j)}_{k,V'} \circ F$ is convexlike.*
(ii) *For each $j = 4, 5$, if $F$ is type $(j)$ $C$-convexlike and $V'$ is $(-C)$-convex, then $I^{(j)}_{k,V'} \circ F$ is convexlike.*

**Lemma 3.3.** (See [8].) *Let $k \in \mathrm{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(i) *For each $j = 1, 2, 3$, if $F$ is type $(j)$ $C$-concavelike and $V'$ is $C$-convex, then $I^{(j)}_{k,V'} \circ F$ is concavelike.*
(ii) *For each $j = 4, 5$, if $F$ is type $(j)$ $C$-concavelike and $V'$ is $(-C)$-convex, then $I^{(j)}_{k,V'} \circ F$ is concavelike.*

**Lemma 3.4.** (See [10].) *Let $k \in \mathrm{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(i) *For each $j = 1, 4, 5$, if $F$ is $C$-lower continuous on $S$ then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous on $S$. Moreover, if $F$ is $(-C)$-upper continuous on $S$ then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on $S$.*
(ii) *For each $j = 2, 3$, if $F$ is $(-C)$-lower continuous on $S$ then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on $S$. Moreover, if $F$ is $C$-upper continuous on $S$ then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous on $S$.*

Let $V' \in 2^Y \setminus \{\emptyset\}$ and direction $k \in \mathrm{int}C$. To show sufficient conditions for the existence of solutions of ($j$-SVOP) by using properties of $I^{(j)}_{k,V'}$, we consider the following two kinds of scalar optimization problems:

$$\inf_{x \in S} (I^{(j)}_{k,V'} \circ F)(x) \quad \text{and} \quad \sup_{x \in S} (I^{(j)}_{k,V'} \circ F)(x).$$

**Lemma 3.5.** (See [7].) *Assume that $F$ is $C$-closed on $S$ and $x_0 \in S$. Let $k \in \mathrm{int}C$. For each $j = 1, 2, 3$, the following statements hold:*

(i) *If $x_0$ is a solution of $\inf_{x \in S} (I^{(j)}_{k,V'} \circ F)(x)$, then $x_0$ is a weak minimal solution of ($j$-SVOP).*
(ii) *If $x_0$ is a solution of $\sup_{x \in S} (I^{(j)}_{k,V'} \circ F)(x)$, then $x_0$ is a weak maximal solution of ($j$-SVOP).*

**Lemma 3.6.** (See [7].) *Assume that $F$ is $(-C)$-closed on $S$ and $x_0 \in S$. Let $k \in \mathrm{int}C$. For each $j = 4, 5$, the following statements hold:*

(i) *If $x_0$ is a solution of $\inf_{x \in S} (I^{(j)}_{k,V'} \circ F)(x)$, then $x_0$ is a weak minimal solution of ($j$-SVOP).*
(ii) *If $x_0$ is a solution of $\sup_{x \in S} (I^{(j)}_{k,V'} \circ F)(x)$, then $x_0$ is a weak maximal solution of ($j$-SVOP).*

## 4 Existence Theorems for Saddle Points of Set-Valued Maps

At first, we introduce definitions of saddle points for set-valued maps proposed in [8]. For each $j = 1, \ldots, 5$, if $(x_0, y_0) \in X \times Y$ satisfies the following properties:

$$\text{(i)} \quad F(x, y_0) \leq_C^{(j)} F(x_0, y_0) \quad \text{implies} \quad F(x_0, y_0) \leq_C^{(j)} F(x, y_0),$$

$$\text{(ii)} \quad F(x_0, y_0) \leq_C^{(j)} F(x_0, y) \quad \text{implies} \quad F(x_0, y) \leq_C^{(j)} F(x_0, y_0),$$

for any $x \in X$ and $y \in Y$, then we call it type $(j)$ *C-saddle point* of $F$. It is equivalent to

$$F(x_0,\, y_0) \in \left\{\mathrm{Min}_{(j)} F(X, y_0)\right\} \cap \left\{\mathrm{Max}_{(j)} F(x_0, Y)\right\}.$$

If $C$ is replaced by int$C$ then we call it type $(j)$ *weak C-saddle point* of $F$.

In this section, we give three types of existence theorems for type $(j)$ cone saddle points of set-valued maps. At first, we introduce the first existence theorems which are natural extensions of Sion's minimax theorem (see [9]).

**Theorem 4.1.** (See [7].) *Let $X$ and $Y$ be nonempty compact convex subsets of two real topological vector spaces, respectively, $Z$ a real topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that $F$ is $C$-closed and $(-C)$-closed on $X \times Y$. If $F$ satisfies the following conditions:*

(i) *$x \to F(x, y)$ is $C$-lower continuous and type (1) naturally quasi $C$-convex on $X$ for every $y \in Y$,*
(ii) *$x \to F(x, y)$ is $(-C)$-upper continuous and type (1) naturally quasi $C$-concave on $Y$ for every $x \in X$,*

*then $F$ has at least one type (1)-weak saddle point.*

**Theorem 4.2.** (See [7].) *Let $X$ and $Y$ be nonempty compact convex subsets of two real topological vector spaces, respectively, $Z$ a real topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that $F$ is $C$-closed on $X \times Y$. For each $j = 2, 3$, if $F$ satisfies that*

(i) *$x \to F(x, y)$ is $C$-upper continuous and type $(j)$ naturally quasi $C$-convex on $X$ for every $y \in Y$,*
(ii) *$x \to F(x, y)$ is $(-C)$-lower continuous and type $(j)$ naturally quasi $C$-concave on $Y$ for every $x \in X$,*

*then $F$ has at least one type $(j)$-weak saddle point.*

**Theorem 4.3.** (See [7].) *Let $X$ and $Y$ be nonempty compact convex subsets of two real topological vector spaces, respectively, $Z$ a real topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that $F$ is $(-C)$-closed on $X \times Y$ and $V'$ is $(-C)$-convex. For each $j = 4, 5$, if $F$ satisfies that*

(i) *$x \to F(x,\, y)$ is $C$-lower continuous and type $(j)$ naturally quasi $C$-convex on $X$ for every $y \in Y$,*
(ii) *$x \to F(x,\, y)$ is $(-C)$-upper continuous and type $(j)$ naturally quasi $C$-concave on $Y$ for every $x \in X$,*

*then $F$ has at least one type $(j)$-weak saddle point.*

Next, we introduce the second existence theorems which are natural extensions of Fan type minimax theorem (see [1]).

**Theorem 4.4.** (See [8].) *Let $X$ be a nonempty compact subset of real topological space, $Y$ any space, $Z$ a real topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that $F$ is $C$-closed and $(-C)$-closed on $X \times Y$. If $F$ satisfies that*

(i) *$x \to F(x, y)$ is type (1) $C$-convexlike on $X$ for every $y \in Y$,*
(ii) *$x \to F(x, y)$ is $(-C)$-upper continuous and type (1) $C$-concavelike on $Y$ for every $x \in X$,*

*then $F$ has at least one type (1)-weak saddle point.*

**Theorem 4.5.** (See [8].) *Let $X$ be a nonempty compact subset of real topological space, $Y$ any space, $Z$ a real topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that $F$ is $C$-closed on $X \times Y$. For each $j = 2, 3$, if $F$ satisfies that*

(i) *$x \to F(x, y)$ is type $(j)$ $C$-convexlike on $X$ for every $y \in Y$,*
(ii) *$x \to F(x, y)$ is $(-C)$-lower continuous and type $(j)$ $C$-concavelike on $Y$ for every $x \in X$,*

*then $F$ has at least one type $(j)$-weak saddle point.*

**Theorem 4.6.** (See [8].) *Let $X$ be a nonempty compact subset of real topological space, $Y$ any space, $Z$ a real topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that $F$ is $(-C)$-closed on $X \times Y$. For each $j = 4, 5$, if $F$ satisfies that*

(i) *$x \to F(x, y)$ is type $(j)$ $C$-convexlike on $X$ for every $y \in Y$,*
(ii) *$x \to F(x, y)$ is $(-C)$-upper continuous and type $(j)$ $C$-concavelike on $Y$ for every $x \in X$,*

*then $F$ has at least one type $(j)$-weak saddle point.*

Finally, we give the third existence theorems for type $(j)$ cone saddle points of set-valued maps with separated form.

**Theorem 4.7.** (See [7].) *Let $X$ and $Y$ be nonempty compact subsets of two real topological spaces, respectively, $Z$ a real ordered topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. If $F$ satisfies that*

(i) *$F(x, y) := G_1(x) \cup G_2(y)$,*
(ii) *$G_1$ is $C$-closed and $C$-lower continuous on $X$,*
(iii) *$G_2$ is $(-C)$-closed and $(-C)$-upper continuous on $Y$,*

*where $G_1 : X \to 2^Z \setminus \{\emptyset\}$ and $G_2 : Y \to 2^Z \setminus \{\emptyset\}$, then $F$ has at least one type (1) $C$-saddle point.*

**Theorem 4.8.** (See [7].) *Let $X$ and $Y$ be nonempty compact subsets of two real topological spaces, respectively, $Z$ a real topological vector space with the partial ordering $\leq_C$, $k \in \mathrm{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. For each $j = 2, 3$, if $F$ satisfies that*

(i) $F(x,y) := G_1(x) \cup G_2(y)$,
(ii) $G_1$ *is* $C$*-closed and* $C$*-upper continuous on* $X$,
(iii) $G_2$ *is* $C$*-closed and* $(-C)$*-lower continuous on* $Y$,

*where* $G_1 : X \to 2^Z \setminus \{\emptyset\}$ *and* $G_2 : Y \to 2^Z \setminus \{\emptyset\}$, *then* $F$ *has at least one type* $(j)$ *weak* $C$*-saddle point.*

**Theorem 4.9.** (See [7].) *Let* $X$ *and* $Y$ *be nonempty compact subsets of two real topological spaces, respectively,* $Z$ *a real topological vector space with the partial ordering* $\leq_C$, $k \in \mathrm{int}C$, $V'$ *a nonempty subset of* $Z$ *and* $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. *For each* $j = 4, 5$, *if* $F$ *satisfies that*

(i) $F(x,y) := G_1(x) \cup G_2(y)$,
(ii) $G_1$ *is* $(-C)$*-closed and* $C$*-lower continuous on* $X$,
(iii) $G_2$ *is* $(-C)$*-closed and* $(-C)$*-upper continuous on* $Y$,

*where* $G_1 : X \to 2^Z \setminus \{\emptyset\}$ *and* $G_2 : Y \to 2^Z \setminus \{\emptyset\}$, *then* $F$ *has at least one type* $(j)$ *weak* $C$*-saddle point.*

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