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A topology in a vector lattice and fixed point theorems for nonexpansive mappings

川崎敏治

(Toshiharu Kawasaki, toshiharu.kawasaki@nifty.ne.jp)

Abstract

In the previous paper [4] we show Takahashi's and Fan-Browder's fixed point theorems in a vector lattice and in the previous paper [5] we show Schauder-Tychonoff's fixed point theorem using Fan-Browder's fixed point theorem. The purpose of this paper is to introduce a topology in a vector lattice and to show a fixed point theorem for a nonexpansive mapping and also common fixed point theorems for commutative family of nonexpansive mappings in a vector lattice.

1 Introduction

There are many fixed point theorems in a topological vector space, for instance, Kirk's fixed point theorem in a Banach space, and so on; see for example [8].

In this paper we consider fixed point theorems in a vector lattice. As known well every topological vector space has a linear topology. On the other hand, although every vector lattice does not have a topology, it has two lattice operators, which are the supremum $\vee$ and the infimum $\wedge$, and also an order is introduced from these operators; see also [6, 9] about vector lattices. There are some methods how to introduce a topology to a vector lattice. One method is to assume that the vector lattice has a linear topology [1]. On the other hand, there is another method to make up a topology in a vector lattice, for instance, in [2] one method is introduced in case of the vector lattice with unit.

In the previous paper [4] we show Takahashi's and Fan-Browder's fixed point theorems in a vector lattice and in the previous paper [5] we show Schauder-Tychonoff's fixed point theorem using Fan-Browder's fixed point theorem. The purpose of this paper is to introduce a topology in a vector lattice and to show a fixed point theorem for a nonexpansive mapping and also common fixed point theorems for commutative family of nonexpansive mappings in a vector lattice.

2 Topology in a vector lattice

First we introduce a topology in a vector lattice introduced by [2]; see also [4, 5].

Let X be a vector lattice. $e \in X$ is said to be an unit if $e \wedge x > 0$ for any $x \in X$ with $x > 0$. Let $\mathcal{K}_X$ be the class of units of X. In case where X is the set of real numbers $\mathbf{R}$, $\mathcal{K}_\mathbf{R}$ is the set of positive real numbers. Let X be a vector lattice with unit and let Y be a subset of X. Y is said to be open if for any $x \in Y$ and for any $e \in \mathcal{K}_X$ there exists $\varepsilon \in \mathcal{K}_\mathbf{R}$ such that $[x - \varepsilon e, x + \varepsilon e] \subset Y$. Let $\mathcal{O}_X$ be the class of open subsets of X. Y is said to be closed if $Y^C \in \mathcal{O}_X$. For $e \in \mathcal{K}_X$ and for an interval $[a, b]$ we consider the following subset

$$[a,b]^e = \{x \mid \text{ there exists some } \varepsilon \in \mathcal{K}_\mathbf{R} \text{ such that } x - a \geq \varepsilon e \text{ and } b - x \geq \varepsilon e\}.$$

By the definition of $[a,b]^e$ it is easy to see that $[a,b]^e \subset [a,b]$. Every mapping from $X \times \mathcal{K}_X$ into $(0, \infty)$ is said to be a gauge. Let Δ_X be the class of gauges in X. For $x \in X$ and $\delta \in \Delta_X$, $O(x, \delta)$ is defined by

$$O(x,\delta) = \bigcup_{e \in \mathcal{K}_X} [x - \delta(x,e)e, x + \delta(x,e)e]^e.$$

$O(x, \delta)$ is said to be a δ-neighborhood of x. Suppose that for any $x \in X$ and for any $\delta \in \Delta_X$ there exists $U \in \mathcal{O}_X$ such that $x \in U \subset O(x, \delta)$.

For a subset Y of X we denote by $\mathrm{cl}(Y)$ and $\mathrm{int}(Y)$, the closure and the interior of Y, respectively. Let X and Y be vector lattices with unit, $x_0 \in Z \subset X$ and f a mapping from Z into Y. f is said to be continuous in the sense of topology at x_0 if for any $V \in \mathcal{O}_Y$ with $f(x_0) \in V$ there exists $U \in \mathcal{O}_X$ with $x_0 \in U$ such that $f(U \cap Z) \subset V$.

Let X be a vector lattice with unit. X is said to be Hausdorff if for any $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exists $O_1, O_2 \in \mathcal{O}_X$ such that $x_1 \in O_1$, $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$. A subset Y of X is said to be compact if for any open covering of Y there exists a finite sub-covering. A subset Y of X is said to be normal if for any closed subsets F_1 and F_2 with $F_1 \cap F_2 \cap Y = \emptyset$ there exists $O_1, O_2 \in \mathcal{O}_X$ such that $F_1 \subset O_1$, $F_2 \subset O_2$ and $O_1 \cap O_2 \cap Y = \emptyset$.

A vector lattice is said to be Archimedean if it holds that $x = 0$ whenever there exists $y \in X$ with $y \geq 0$ such that $0 \leq rx \leq y$ for any $r \in \mathcal{K}_\mathbf{R}$.

Let X be a vector lattice with unit and Y a vector lattice, $x_0 \in Z \subset X$ and f a mapping from Z into Y. f is said to be continuous at x_0 if there exists $\{v_e \mid e \in \mathcal{K}_X\}$ satisfying the conditions (U1), $(\mathrm{U2})^d$ and $(\mathrm{U3})^s$ such that for any $e \in \mathcal{K}_X$ there exists $\delta \in \mathcal{K}_\mathbf{R}$ such that for any $x \in Z$ if $|x - x_0| \leq \delta e$, then $|f(x) - f(x_0)| \leq v_e$; where

(U1) $v_e \in Y$ with $v_e > 0$;

$(\mathrm{U2})^d$ $v_{e_1} \geq v_{e_2}$ if $e_1 \geq e_2$;

$(\mathrm{U3})^s$ For any $e \in \mathcal{K}_X$ there exists $\theta(e) \in \mathcal{K}_\mathbf{R}$ such that $v_{\theta(e)e} \leq \frac{1}{2} v_e$.

Let X be an Archimedean vector lattice. Then there exists a positive homomorphism f from X into $\mathbf{R}$, that is, f satisfies the following conditions:

(H1) $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for any $x, y \in X$ and for any $\alpha, \beta \in \mathbf{R}$;

(H2) $f(x) \geq 0$ for any $x \in X$ with $x \geq 0$;

see [5, Example 3.1]. Suppose that there exists a homomorphism f from X into $\mathbf{R}$ satisfying the following condition instead of (H2):

(H2)^s $f(x) > 0$ for any $x \in X$ with $x > 0$.

Example 2.1. We consider of a sufficient condition to satisfy (H2)^s. Let X be a Hilbert lattice with unit, that is, X has an inner product $\langle \cdot, \cdot \rangle$ and for any $x, y \in X$ if $|x| \le |y|$, then $\langle x, x \rangle \le \langle y, y \rangle$. For any $e \in \mathcal{K}_X$ let f be a function from X into $\mathbf{R}$ defined by $f(x) = \langle x, e \rangle$. Then f satisfies (H1) and (H2)^s clearly.

3 Fixed point theorem for a nonexpansive mapping

Let X be a vector lattice and Y a subset of X. A mapping f from Y into Y is said to be nonexpansive if $|f(x) - f(y)| \le |x - y|$ for any $x, y \in Y$. In this section we consider a fixed point theorem for a nonexpansive mapping.

Lemma 3.1. *Let X be a Hausdorff Archimedean vector lattice with unit and K a non-empty compact convex subset of X. Then*

$$c(K) = \left\{ x \;\middle|\; x \in K, \bigvee_{y \in K} |x - y| = \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| \right\}$$

is non-empty compact convex.

Proof. For any $x \in K$ and for any $e \in \mathcal{K}_X$ let

$$F(x, e) = \left\{ y \;\middle|\; y \in K, |x - y| \le \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| + e \right\}.$$

Then $F(x, e)$ is non-empty compact convex. Let $C(e) = \bigcap_{x \in K} F(x, e)$. Since $\bigcap_{i=1}^n F(x_i, e) \neq \emptyset$ for any $x_1, \cdots, x_n \in K$, $C(e)$ is non-empty compact convex. Since $C(e_1) \supset C(e_2)$ for any $e_1, e_2 \in \mathcal{K}_X$ with $e_1 \ge e_2$, $\bigcap_{e \in \mathcal{K}_X} C(e)$ is non-empty compact convex. Moreover $c(K) = \bigcap_{e \in \mathcal{K}_X} C(e)$. Indeed $c(K) \subset \bigcap_{e \in \mathcal{K}_X} C(e)$ is clear. Let $x \in C(e)$ for any $e \in \mathcal{K}_X$. Then

$$|x - y| \le \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| + e$$

for any $y \in K$. Therefore

$$\bigvee_{y \in K} |x - y| \le \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| + \bigwedge_{e \in \mathcal{K}_X} e = \bigwedge_{x \in K} \bigvee_{y \in K} |x - y|.$$

By definition

$$\bigvee_{y \in K} |x - y| \ge \bigwedge_{x \in K} \bigvee_{y \in K} |x - y|.$$

Therefore

$$\bigvee_{y\in K}|x-y| = \bigwedge_{x\in K}\bigvee_{y\in K}|x-y|,$$

that is, $x \in c(K)$. □

Let X be a Hausdorff Archimedean vector lattice with unit and Y a subset of X. We say that Y has the normal structure if for any compact convex subset K, which contains two points at least, of Y there exists $x \in K$ such that

$$\bigvee_{y\in K}|x-y| < \bigvee_{x,y\in K}|x-y|.$$

Lemma 3.2. *Let X be a Hausdorff Archimedean vector lattice with unit and K a non-empty compact convex subset, which contains two points at least, of X. Suppose that K has the normal structure. Then*

$$\bigvee_{x,y\in c(K)}|x-y| < \bigvee_{x,y\in K}|x-y|.$$

Proof. Since K has the normal structure, there exists $z \in K$ such that

$$|x-y| \leq \bigvee_{y\in K}|x-y| = \bigwedge_{x\in K}\bigvee_{y\in K}|x-y| \leq \bigvee_{y\in K}|z-y| < \bigvee_{x,y\in K}|x-y|$$

for any $x, y \in c(K)$. Therefore

$$\bigvee_{x,y\in c(K)}|x-y| < \bigvee_{x,y\in K}|x-y|.$$

□

Theorem 3.3. *Let X be a Hausdorff Archimedean vector lattice with unit and K a non-empty compact convex subset of X. Suppose that K has the normal structure. Then every nonexpansive mapping from K into K has a fixed point.*

Proof. Let f be a nonexpansive mapping from K into K and $\{K_\lambda \mid \lambda \in \Lambda\}$ the family of non-empty compact convex subsets of K satisfying that $f(K_\lambda) \subset K_\lambda$. By Zorn's lemma there exists a minimal element K_0 of $\{K_\lambda \mid \lambda \in \Lambda\}$. Assume that K_0 contains two points at least. By Lemma 3.1 $c(K_0)$ is non-empty compact convex. Let $x \in c(K_0)$. For any $y \in K_0$

$$|f(x)-f(y)| \leq |x-y| \leq \bigvee_{y\in K_0}|x-y| = \bigwedge_{x\in K_0}\bigvee_{y\in K_0}|x-y|.$$

Let

$$M = \left\{ y \;\middle|\; y \in K, |f(x)-y| \leq \bigwedge_{x\in K_0}\bigvee_{y\in K_0}|x-y| \right\}.$$

Then $f(K_0) \subset M$ and hence $f(K_0 \cap M) \subset K_0 \cap M$. Since K_0 is a minimal element, it holds that $K_0 \subset M$. Therefore

$$\bigvee_{y \in K_0} |f(x) - y| \leq \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y|.$$

By definition

$$\bigvee_{y \in K_0} |f(x) - y| \geq \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y|.$$

Therefore

$$\bigvee_{y \in K_0} |f(x) - y| = \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y|,$$

that is, $f(x) \in c(K_0)$. Since K_0 is a minimal element, it holds that $c(K_0) = K_0$ and hence

$$\bigvee_{x,y \in c(K_0)} |x - y| = \bigvee_{x,y \in K_0} |x - y|.$$

However by Lemma 3.2

$$\bigvee_{x,y \in c(K_0)} |x - y| < \bigvee_{x,y \in K_0} |x - y|.$$

It is a contradiction. Therefore K_0 only contains a unique point. The point is a fixed point. □

4 Fixed point theorem for the commutative family of nonexpansive mappings

For any nonexpansive mapping f from K into K let $F_K(f)$ be the set of fixed points of f.

Lemma 4.1. *Let X be a Hausdorff Archimedean vector lattice with unit, Y a subset of X and f a nonexpansive mapping from Y into Y. Suppose that there exists a homomorphism from X into $\mathbf{R}$ satisfying the condition* (H2)^s. *Then $F_Y(f)$ is closed.*

Proof. Assume that $F_Y(f)$ is not closed. Then for any $\delta \in \Delta_X$ there exists $x \in F_Y(f)^C$ such that $O(x,\delta) \not\subset F_Y(f)^C$. Take $y_\delta \in O(x,\delta) \cap F_Y(f)$. Then $f(y_\delta) = y_\delta$. Note that every nonexpansive mapping is continuous and hence by [5, Lemma 3.2] it is also continuous in the sense of topology. Since $\{y_\delta \mid \delta \in \Delta_X\}$ is convergent to x in the sense of topology, $\{f(y_\delta) \mid \delta \in \Delta_X\}$ is convergent to $f(x)$ in the sense of topology. Since X is Hausdorff, $f(x) = x$. It is a contradiction. Therefore $F_Y(f)$ is closed. □

Lemma 4.2. *Let X be a vector lattice. If $|x-z| = |x-w|$, $|y-z| = |y-w|$ and $|x-z|+|y-z| = |x-y|$, then $z = w$.*

Proof. Note that $|a+b| = |a-b|$ if and only if $|a| \wedge |b| = 0$. Since

$$|x-z| = \left|x - \frac{1}{2}(z+w) - \frac{1}{2}(z-w)\right|$$

and

$$|x-w| = \left|x - \frac{1}{2}(z+w) + \frac{1}{2}(z-w)\right|,$$

it holds that $\left|x - \frac{1}{2}(z+w)\right| \wedge \frac{1}{2}|z-w| = 0$. In the same way it holds that $\left|y - \frac{1}{2}(z+w)\right| \wedge \frac{1}{2}|z-w| = 0$. Note that $(a+b)\wedge c \le a\wedge c + b\wedge c$ for any $a, b, c \ge 0$. Therefore

$$\begin{aligned}
|x-y| \wedge \frac{1}{2}|z-w| &\le \left(\left|x - \frac{1}{2}(z-w)\right| + \left|\frac{1}{2}(z-w) - y\right|\right) \wedge \frac{1}{2}|z-w| \\
&\le \left|x - \frac{1}{2}(z-w)\right| \wedge \frac{1}{2}|z-w| + \left|y - \frac{1}{2}(z+w)\right| \wedge \frac{1}{2}|z-w| \\
&= 0.
\end{aligned}$$

Assume that $z \ne w$. Note that, if $|b| \wedge |c| = 0$, then $||a| - |b|| \wedge |c| = |a| \wedge |c|$. Therefore

$$\begin{aligned}
(|x-z| + |y-z|) \wedge \frac{1}{2}|z-w| &\ge |x-z| \wedge \frac{1}{2}|z-w| \\
&\ge \left|\left|x - \frac{1}{2}|z-w|\right| - \frac{1}{2}|z-w|\right| \wedge \frac{1}{2}|z-w| \\
&= \frac{1}{2}|z-w| > 0.
\end{aligned}$$

It is a contradiction. Therefore $z = w$. □

Lemma 4.3. *Let X be a Hausdorff Archimedean vector lattice with unit, Y a subset of X and f a nonexpansive mapping from Y into Y. Then $F_Y(f)$ is convex.*

Proof. Let $x, y \in F_Y(f)$ and $0 \le \alpha \le 1$. Then

$$\begin{aligned}
|x - f((1-\alpha)x + \alpha y)| &= |f(x) - f((1-\alpha)x + \alpha y)| \\
&\le |x - ((1-\alpha)x + \alpha y)| = \alpha|x-y|, \\
|y - f((1-\alpha)x + \alpha y)| &= |f(y) - f((1-\alpha)x + \alpha y)| \\
&\le |y - ((1-\alpha)x + \alpha y)| = (1-\alpha)|x-y|.
\end{aligned}$$

Since

$$\begin{aligned}
|x-y| &\le |x - f((1-\alpha)x + \alpha y)| + |y - f((1-\alpha)x + \alpha y)| \\
&\le |x - ((1-\alpha)x + \alpha y)| + |y - ((1-\alpha)x + \alpha y)| = |x-y|,
\end{aligned}$$

it holds that

$$\begin{aligned} |x - f((1-\alpha)x + \alpha y)| &= |x - ((1-\alpha)x + \alpha y)|, \\ |y - f((1-\alpha)x + \alpha y)| &= |y - ((1-\alpha)x + \alpha y)|, \end{aligned}$$

and hence

$$|x - f((1-\alpha)x + \alpha y)| + |y - f((1-\alpha)x + \alpha y)| = |x - y|.$$

By Lemma 4.2 $f((1-\alpha)x + \alpha y) = (1-\alpha)x + \alpha y$, that is, $F_Y(f)$ is convex. □

Theorem 4.4. *Let X be a Hausdorff Archimedean vector lattice with unit, K a compact convex subset of X and $\{f_i \mid i = 1, \cdots, n\}$ the finite commutative family of nonexpansive mappings from K into K. Suppose that there exists a homomorphism from X into* **R** *satisfying the condition* (H2)^s *and K has the normal structure. Then $\bigcap_{i=1}^n F_K(f_i)$ is non-empty.*

Proof. Let $\{K_\lambda \mid \lambda \in \Lambda\}$ be the family of non-empty compact convex subsets of K satisfying that $f_i(K_\lambda) \subset K_\lambda$ for any i. By Zorn's lemma there exists a minimal element K_0 of $\{K_\lambda \mid \lambda \in \Lambda\}$. Assume that K_0 contains two points at least. By Theorem 3.3 $F_{K_0}(f_1 \circ \cdots \circ f_n)$ is non-empty. Moreover by Lemma 4.1 and Lemma 4.3 $F_{K_0}(f_1 \circ \cdots \circ f_n)$ is compact convex. It holds that $f(F_{K_0}(f_1 \circ \cdots \circ f_n)) = F_{K_0}(f_1 \circ \cdots \circ f_n)$ for any i. It is shown as follows. Let $x \in F_{K_0}(f_1 \circ \cdots \circ f_n)$. Since

$$f_i(x) = f_i((f_1 \circ \cdots \circ f_n)(x)) = (f_1 \circ \cdots \circ f_n)(f_i(x))$$

for any i, $f_i(x) \in F_{K_0}(f_1 \circ \cdots \circ f_n)$, that is, $f_i(F_{K_0}(f_1 \circ \cdots \circ f_n)) \subset F_{K_0}(f_1 \circ \cdots \circ f_n)$. Next let $x_i = (f_1 \circ \cdots \circ f_{i-1} \circ f_{i+1} \circ \cdots \circ f_n)(x)$. Since

$$(f_1 \circ \cdots \circ f_n)(x_i) = (f_1 \circ \cdots \circ f_{i-1} \circ f_{i+1} \circ \cdots \circ f_n)(x) = x_i,$$

it holds that $x_i \in F_{K_0}(f_1 \circ \cdots \circ f_n)$. Moreover $f_i(x_i) = x$. Therefore $F_{K_0}(f_1 \circ \cdots \circ f_n) \subset f_i(F_{K_0}(f_1 \circ \cdots \circ f_n))$. Since K has the normal structure, there exists $x_0 \in K_0$ such that

$$\bigvee_{y \in K_0} |x_0 - y| < \bigvee_{x, y \in K_0} |x - y|.$$

Let

$$A = \left\{ x \;\middle|\; x \in K_0, \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y| \le \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x_0 - y| \right\}.$$

A is non-empty and convex clearly. Moreover since X is Archimedean, A is closed and hence compact. Let $x \in A$. Then for any i and for any $y \in F_{K_0}(f_1 \circ \cdots \circ f_n)$

$$\begin{aligned} |f_i(x) - y| = |f_i(x) - f_i(y_i)| &\le |x - y_i| \\ &\le \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y| \\ &\le \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x_0 - y| \end{aligned}$$

and hence $f_i(a) \in A$, that is, $f_i(A) \subset A$. Since K_0 is minimal, $A = K_0$. Therefore

$$\bigvee_{x,y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y| \leq \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x_0 - y| < \bigvee_{x,y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y|.$$

It is a contradiction. Therefore K_0 only contains a unique point. The point is a common fixed point of $\{f_i \mid i = 1, \cdots, n\}$. □

Theorem 4.5. *Let X be a Hausdorff Archimedean vector lattice with unit, K a compact convex subset of X and $\{f_i \mid i \in I\}$ the commutative family of nonexpansive mappings from K into K. Suppose that there exists a homomorphism from X into* **R** *satisfying the condition* (H2)^s *and K has the normal structure. Then $\bigcap_{i \in I} F_K(f_i)$ is non-empty.*

Proof. By Theorem 4.4 $\bigcap_{k=1}^{n} F_K(f_{i_k})$ is non-empty for any finite set $i_1, \cdots, i_n \in I$. Since K is compact, $\bigcap_{i \in I} F_K(f_i)$ is non-empty. □

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