A topology in a vector lattice and fixed point theorems for nonexpansive mappings

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Abstract

In the previous paper [4] we show Takahashi’s and Fan-Browder’s fixed point theorems in a vector lattice and in the previous paper [5] we show Schauder-Tychonoff’s fixed point theorem using Fan-Browder’s fixed point theorem. The purpose of this paper is to introduce a topology in a vector lattice and to show a fixed point theorem for a nonexpansive mapping and also common fixed point theorems for commutative family of nonexpansive mappings in a vector lattice.

1 Introduction

There are many fixed point theorems in a topological vector space, for instance, Kirk’s fixed point theorem in a Banach space, and so on; see for example [8].

In this paper we consider fixed point theorems in a vector lattice. As known well every topological vector space has a linear topology. On the other hand, although every vector lattice does not have a topology, it has two lattice operators, which are the supremum V and the infimum ∧, and also an order is introduced from these operators; see also [6, 9] about vector lattices. There are some methods how to introduce a topology to a vector lattice. One method is to assume that the vector lattice has a linear topology [1]. On the other hand, there is another method to make up a topology in a vector lattice, for instance, in [2] one method is introduced in case of the vector lattice with unit.

In the previous paper [4] we show Takahashi’s and Fan-Browder’s fixed point theorems in a vector lattice and in the previous paper [5] we show Schauder-Tychonoff’s fixed point theorem using Fan-Browder’s fixed point theorem. The purpose of this paper is to introduce a topology in a vector lattice and to show a fixed point theorem for a nonexpansive mapping and also common fixed point theorems for commutative family of nonexpansive mappings in a vector lattice.

2 Topology in a vector lattice

First we introduce a topology in a vector lattice introduced by [2]; see also [4, 5].
Let $X$ be a vector lattice. $e \in X$ is said to be an unit if $e \land x > 0$ for any $x \in X$ with $x > 0$. Let $\mathcal{K}_X$ be the class of units of $X$. In case where $X$ is the set of real numbers $\mathbb{R}$, $\mathcal{K}_\mathbb{R}$ is the set of positive real numbers. Let $X$ be a vector lattice with unit and let $Y$ be a subset of $X$. $Y$ is said to be open if for any $x \in Y$ and for any $e \in \mathcal{K}_X$ there exists $\varepsilon \in \mathcal{K}_\mathbb{R}$ such that $[x - \varepsilon e, x + \varepsilon e] \subset Y$. Let $\mathcal{O}_X$ be the class of open subsets of $X$. $Y$ is said to be closed if $Y^c \in \mathcal{O}_X$. For $e \in \mathcal{K}_X$ and for an interval $[a, b]$ we consider the following subset

$$[a, b]^e = \{x \mid \text{there exists some } \varepsilon \in \mathcal{K}_\mathbb{R} \text{ such that } x - a \geq \varepsilon e \text{ and } b - x \geq \varepsilon e\}.$$  

By the definition of $[a, b]^e$ it is easy to see that $[a, b]^e \subset [a, b]$. Every mapping from $X \times \mathcal{K}_X$ into $(0, \infty)$ is said to be a gauge. Let $\Delta_X$ be the class of gauges in $X$. For $x \in X$ and $\delta \in \Delta_X$, $O(x, \delta)$ is defined by

$$O(x, \delta) = \bigcup_{e \in \mathcal{K}_X} [x - \delta(x, e)e, x + \delta(x, e)e]^e.$$  

$O(x, \delta)$ is said to be a $\delta$-neighborhood of $x$. Suppose that for any $x \in X$ and for any $\delta \in \Delta_X$ there exists $U \in \mathcal{O}_X$ such that $x \in U \subset O(x, \delta)$.

For a subset $Y$ of $X$ we denote by cl($Y$) and int($Y$), the closure and the interior of $Y$, respectively. Let $X$ and $Y$ be vector lattices with unit, $x_0 \in Z \subset X$ and $f$ a mapping from $Z$ into $Y$. $f$ is said to be continuous in the sense of topology at $x_0$ if for any $V \in \mathcal{O}_Y$ with $f(x_0) \in V$ there exists $U \in \mathcal{O}_X$ with $x_0 \in U$ such that $f(U \cap Z) \subset V$.

Let $X$ be a vector lattice with unit. $X$ is said to be Hausdorff if for any $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exists $O_1, O_2 \in \mathcal{O}_X$ such that $x_1 \in O_1$, $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$. A subset $Y$ of $X$ is said to be compact if for any open covering of $Y$ there exists a finite sub-covering. A subset $Y$ of $X$ is said to be normal if for any closed subsets $F_1$ and $F_2$ with $F_1 \cap F_2 \cap Y = \emptyset$ there exists $O_1, O_2 \in \mathcal{O}_X$ such that $F_1 \subset O_1$, $F_2 \subset O_2$ and $O_1 \cap O_2 \cap Y = \emptyset$.

A vector lattice is said to be Archimedean if it holds that $x = 0$ whenever there exists $y \in X$ with $y \geq 0$ such that $0 \leq rx \leq y$ for any $r \in \mathcal{K}_\mathbb{R}$.

Let $X$ be a vector lattice with unit and $Y$ a vector lattice, $x_0 \in Z \subset X$ and $f$ a mapping from $Z$ into $Y$. $f$ is said to be continuous at $x_0$ if there exists $\{v_e \mid e \in \mathcal{K}_X\}$ satisfying the conditions (U1), (U2)$^d$ and (U3)$^d$ such that for any $e \in \mathcal{K}_X$ there exists $\delta \in \mathcal{K}_\mathbb{R}$ such that for any $x \in Z$ if $|x - x_0| \leq \delta e$, then $|f(x) - f(x_0)| \leq v_e$; where

(U1) $\; v_e \in Y$ with $v_e > 0$;

(U2)$^d$ $\; v_{e_1} \geq v_{e_2}$ if $e_1 \geq e_2$;

(U3)$^d$ For any $e \in \mathcal{K}_X$ there exists $\theta(e) \in \mathcal{K}_\mathbb{R}$ such that $v_{\theta(e)e} \leq \frac{1}{2} v_e$.

Let $X$ be an Archimedean vector lattice. Then there exists a positive homomorphism $f$ from $X$ into $\mathbb{R}$, that is, $f$ satisfies the following conditions:

(H1) $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for any $x, y \in X$ and for any $\alpha, \beta \in \mathbb{R}$;  

(H2) $f(x) \geq 0$ for any $x \in X$ with $x \geq 0$;
see [5, Example 3.1]. Suppose that there exists a homomorphism $f$ from $X$ into $\mathbb{R}$ satisfying the following condition instead of (H2):

$$(H2)^* \quad f(x) > 0 \text{ for any } x \in X \text{ with } x > 0.$$

**Example 2.1.** We consider of a sufficient condition to satisfy $(H2)^*$. Let $X$ be a Hilbert lattice with unit, that is, $X$ has an inner product $(\cdot, \cdot)$ and for any $x, y \in X$ if $|x| \leq |y|$, then $\langle x, x \rangle \leq \langle y, y \rangle$. For any $e \in \mathcal{K}_X$ let $f$ be a function from $X$ into $\mathbb{R}$ defined by $f(x) = \langle x, e \rangle$. Then $f$ satisfies (H1) and $(H2)^*$ clearly.

### 3 Fixed point theorem for a nonexpansive mapping

Let $X$ be a vector lattice and $Y$ a subset of $X$. A mapping $f$ from $Y$ into $Y$ is said to be nonexpansive if $|f(x) - f(y)| \leq |x - y|$ for any $x, y \in Y$. In this section we consider a fixed point theorem for a nonexpansive mapping.

**Lemma 3.1.** Let $X$ be a Hausdorff Archimedean vector lattice with unit and $K$ a non-empty compact convex subset of $X$. Then

$$c(K) = \left\{ x \left| x \in K, \bigvee_{y \in K} |x - y| = \bigwedge_{z \in K} \bigvee_{y \in K} |x - y| \right\}$$

is non-empty compact convex.

**Proof.** For any $x \in K$ and for any $e \in \mathcal{K}_X$ let

$$F(x, e) = \left\{ y \left| y \in K, |x - y| \leq \bigwedge_{z \in K} \bigvee_{y \in K} |x - y| + e \right\}.$$  

Then $F(x, e)$ is non-empty compact convex. Let $C(e) = \bigcap_{x \in K} F(x, e)$. Since $\bigcap_{x=1}^n F(x_i, e) \neq \emptyset$ for any $x_1, \cdots, x_n \in K$, $C(e)$ is non-empty compact convex. Since $C(e_1) \supset C(e_2)$ for any $e_1, e_2 \in \mathcal{K}_X$ with $e_1 \geq e_2$, $\bigcap_{e \in \mathcal{K}_X} C(e)$ is non-empty compact convex. Moreover $c(K) = \bigcap_{e \in \mathcal{K}_X} C(e)$. Indeed $c(K) \subset \bigcap_{e \in \mathcal{K}_X} C(e)$ is clear. Let $x \in C(e)$ for any $e \in \mathcal{K}_X$. Then

$$|x - y| \leq \bigwedge_{z \in K} \bigvee_{y \in K} |x - y| + e$$

for any $y \in K$. Therefore

$$\bigvee_{y \in K} |x - y| \leq \bigwedge_{z \in K} \bigvee_{y \in K} |x - y| + \bigwedge_{e \in \mathcal{K}_X} e = \bigwedge_{z \in K} \bigvee_{y \in K} |x - y|.$$  

By definition

$$\bigvee_{y \in K} |x - y| \geq \bigwedge_{z \in K} \bigvee_{y \in K} |x - y|.$$
Therefore
\[ \bigvee_{y \in K} |x - y| = \bigwedge_{x \in K} \bigvee_{y \in K} |x - y|, \]
that is, \( x \in c(K) \).

Let \( X \) be a Hausdorff Archimedean vector lattice with unit and \( Y \) a subset of \( X \). We say that \( Y \) has the normal structure if for any compact convex subset \( K \), which contains two points at least, of \( Y \) there exists \( x \in K \) such that
\[ \bigvee_{y \in K} |x - y| < \bigvee_{x, y \in K} |x - y|. \]

\textbf{Lemma 3.2.} Let \( X \) be a Hausdorff Archimedean vector lattice with unit and \( K \) a non-empty compact convex subset, which contains two points at least, of \( X \). Suppose that \( K \) has the normal structure. Then
\[ \bigvee_{z, y \in (K')} |x - y| < \bigvee_{x, y \in K} |x - y|. \]

\textbf{Proof.} Since \( K \) has the normal structure, there exists \( z \in K \) such that
\[ |x - y| \leq \bigvee_{y \in K} |x - y| = \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| \leq \bigvee_{y \in K} |x - y| < \bigvee_{x, y \in K} |x - y| \]
for any \( x, y \in c(K) \). Therefore
\[ \bigvee_{x, y \in c(K)} |x - y| < \bigvee_{x, y \in K} |x - y|. \]

\textbf{Theorem 3.3.} Let \( X \) be a Hausdorff Archimedean vector lattice with unit and \( K \) a non-empty compact convex subset of \( X \). Suppose that \( K \) has the normal structure. Then every nonexpansive mapping from \( K \) into \( K \) has a fixed point.

\textbf{Proof.} Let \( f \) be a nonexpansive mapping from \( K \) into \( K \) and \( \{K_\lambda \mid \lambda \in \Lambda\} \) the family of non-empty compact convex subsets of \( K \) satisfying that \( f(K_\lambda) \subset K_\lambda \). By Zorn’s lemma there exists a minimal element \( K_0 \) of \( \{K_\lambda \mid \lambda \in \Lambda\} \). Assume that \( K_0 \) contains two points at least. By Lemma 3.1 \( c(K_0) \) is non-empty compact convex. Let \( x \in c(K_0) \). For any \( y \in K_0 \)
\[ |f(x) - f(y)| \leq |x - y| \leq \bigvee_{y \in K_0} |x - y| = \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y|. \]

Let
\[ M = \left\{ y \mid y \in K, |f(x) - y| \leq \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y| \right\}. \]
Then \( f(K_0) \subset M \) and hence \( f(K_0 \cap M) \subset K_0 \cap M \). Since \( K_0 \) is a minimal element, it holds that \( K_0 \subset M \). Therefore
\[
\bigvee_{y \in K_0} |f(x) - y| \leq \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y|.
\]
By definition
\[
\bigvee_{y \in K_0} |f(x) - y| \geq \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y|.
\]
Therefore
\[
\bigvee_{y \in K_0} |f(x) - y| = \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y|,
\]
that is, \( f(x) \in c(K_0) \). Since \( K_0 \) is a minimal element, it holds that \( c(K_0) = K_0 \) and hence
\[
\bigvee_{x,y \in c(K_0)} |x - y| = \bigwedge_{x,y \in K_0} |x - y|.
\]
However by Lemma 3.2
\[
\bigvee_{x,y \in c(K_0)} |x - y| < \bigwedge_{x,y \in K_0} |x - y|.
\]
It is a contradiction. Therefore \( K_0 \) only contains a unique point. The point is a fixed point. \( \square \)

4 \textbf{Fixed point theorem for the commutative family of nonexpansive mappings}

For any nonexpansive mapping \( f \) from \( K \) into \( K \) let \( F_K(f) \) be the set of fixed points of \( f \).

\textbf{Lemma 4.1.} Let \( X \) be a Hausdorff Archimedean vector lattice with unit, \( Y \) a subset of \( X \) and \( f \) a nonexpansive mapping from \( Y \) into \( Y \). Suppose that there exists a homomorphism from \( X \) into \( \mathbb{R} \) satisfying the condition \((\text{H2})^*\). Then \( F_Y(f) \) is closed.

\textit{Proof.} Assume that \( F_Y(f) \) is not closed. Then for any \( \delta \in \Delta_X \) there exists \( x \in F_Y(f)^C \) such that \( O(x, \delta) \not\subset F_Y(f)^C \). Take \( y_\delta \in O(x, \delta) \cap F_Y(f) \). Then \( f(y_\delta) = y_\delta \). Note that every nonexpansive mapping is continuous and hence by [5, Lemma 3.2] it is also continuous in the sense of topology. Since \( \{y_\delta \mid \delta \in \Delta_X\} \) is convergent to \( x \) in the sense of topology, \( \{f(y_\delta) \mid \delta \in \Delta_X\} \) is convergent to \( f(x) \) in the sense of topology. Since \( X \) is Hausdorff, \( f(x) = x \). It is a contradiction. Therefore \( F_Y(f) \) is closed. \( \square \)
Lemma 4.2. Let $X$ be a vector lattice. If $|x-z| = |x-w|$, $|y-z| = |y-w|$ and $|x-z| + |y-z| = |x-y|$, then $z = w$.

Proof. Note that $|a + b| = |a - b|$ if and only if $|a| \wedge |b| = 0$. Since

$$|x-z| = \left|x - \frac{1}{2}(z+w) - \frac{1}{2}(z-w)\right|$$

and

$$|x-w| = \left|x - \frac{1}{2}(z+w) + \frac{1}{2}(z-w)\right|,$$

it holds that $\left|x - \frac{1}{2}(z+w)\right| \wedge \frac{1}{2}|z-w| = 0$. In the same way it holds that $|y - \frac{1}{2}(z+w)| \wedge \frac{1}{2}|z-w| = 0$. Note that $(a + b) \wedge c \leq a \wedge c + b \wedge c$ for any $a, b, c \geq 0$. Therefore

$$|x - y| \wedge \frac{1}{2}|z-w| \leq \left(|x - \frac{1}{2}(z-w) + \frac{1}{2}(z-w)| - \frac{1}{2}|z-w|\right) \wedge \frac{1}{2}|z-w|$$

$$= 0.$$

Assume that $z \neq w$. Note that, if $|b| \wedge |c| = 0$, then $||a| - |b|\wedge |c| = |a| \wedge |c|$. Therefore

$$((|x-z| + |y-z|) \wedge \frac{1}{2}|z-w| \geq |x-z| \wedge \frac{1}{2}|z-w|$$

$$\geq \left|x - \frac{1}{2}|z-w| - \frac{1}{2}|z-w|\right| \wedge \frac{1}{2}|z-w|$$

$$= \frac{1}{2}|z-w| > 0.$$ It is a contradiction. Therefore $z = w$. □

Lemma 4.3. Let $X$ be a Hausdorff Archimedean vector lattice with unit, $Y$ a subset of $X$ and $f$ a nonexpansive mapping from $Y$ into $Y$. Then $F_Y(f)$ is convex.

Proof. Let $x, y \in F_Y(f)$ and $0 \leq \alpha \leq 1$. Then

$$|x - f((1-\alpha)x + \alpha y)| = |f(x) - f((1-\alpha)x + \alpha y)|$$

$$\leq |x - ((1-\alpha)x + \alpha y)| = \alpha|x-y|,$$

$$|y - f((1-\alpha)x + \alpha y)| = |f(y) - f((1-\alpha)x + \alpha y)|$$

$$\leq |y - ((1-\alpha)x + \alpha y)| = (1-\alpha)|x-y|.$$ Since

$$|x - y| \leq |x - f((1-\alpha)x + \alpha y)| + |y - f((1-\alpha)x + \alpha y)|$$

$$\leq |x - ((1-\alpha)x + \alpha y)| + |y - ((1-\alpha)x + \alpha y)| = |x - y|,$$
it holds that
\[ |x - f((1 - \alpha)x + \alpha y)| = |x - ((1 - \alpha)x + \alpha y)|, \]
\[ |y - f((1 - \alpha)x + \alpha y)| = |y - ((1 - \alpha)x + \alpha y)|, \]
and hence
\[ |x - f((1 - \alpha)x + \alpha y)| + |y - f((1 - \alpha)x + \alpha y)| = |x - y|. \]
By Lemma 4.2 \( f((1 - \alpha)x + \alpha y) = (1 - \alpha)x + \alpha y \), that is, \( F_Y(f) \) is convex.

**Theorem 4.4.** Let \( X \) be a Hausdorff Archimedean vector lattice with unit, \( K \) a compact convex subset of \( X \) and \( \{f_i \mid i = 1, \cdots, n\} \) the finite commutative family of nonexpansive mappings from \( K \) into \( K \). Suppose that there exists a homomorphism from \( X \) into \( \mathbb{R} \) satisfying the condition \((H2)^{s}\) and \( K \) has the normal structure. Then \( \bigcap_{i=1}^{n} F_K(f_i) \) is non-empty.

**Proof.** Let \( \{K_\lambda \mid \lambda \in \Lambda\} \) be the family of non-empty compact convex subsets of \( K \) satisfying that \( f_i(K_\lambda) \subset K_\lambda \) for any \( i \). By Zorn's lemma there exists a minimal element \( K_0 \) of \( \{K_\lambda \mid \lambda \in \Lambda\} \). Assume that \( K_0 \) contains two points at least. By Theorem 3.3 \( F_{K_0}(f_1 \cdots f_n) \) is non-empty. Moreover by Lemma 4.1 and Lemma 4.3 \( F_{K_0}(f_1 \circ \cdots \circ f_n) \) is compact convex.

It holds that \( f(F_{K_0}(f_1 \circ \cdots \circ f_n)) = F_{K_0}(f_1 \circ \cdots \circ f_n) \) for any \( i \). It is shown as follows. Let \( x \in F_{K_0}(f_1 \circ \cdots \circ f_n) \). Since
\[ f_i(x) = f_i((f_1 \circ \cdots \circ f_n)(x)) = (f_1 \circ \cdots \circ f_n)(f_i(x)) \]
for any \( i, f_i(x) \in F_{K_0}(f_1 \circ \cdots \circ f_n) \), that is, \( f_i(F_{K_0}(f_1 \circ \cdots \circ f_n)) \subset F_{K_0}(f_1 \circ \cdots \circ f_n) \). Next let \( x_i = (f_1 \circ \cdots \circ f_{i-1} \circ f_{i+1} \circ \cdots \circ f_n)(x) \). Since
\[ (f_1 \circ \cdots \circ f_n)(x_i) = (f_1 \circ \cdots \circ f_{i-1} \circ f_{i+1} \circ \cdots \circ f_n)(x) = x_i, \]
it holds that \( x_i \in F_{K_0}(f_1 \circ \cdots \circ f_n) \). Moreover \( f_i(x_i) = x_i \). Therefore \( F_{K_0}(f_1 \circ \cdots \circ f_n) \subset f_i(F_{K_0}(f_1 \circ \cdots \circ f_n)) \). Since \( K \) has the normal structure, there exists \( x_0 \in K_0 \) such that
\[ \bigvee_{y \in K_0} |x_0 - y| < \bigvee_{x, y \in K_0} |x - y|. \]
Let
\[ A = \left\{ x \mid x \in K_0, \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y| \leq \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x_0 - y| \right\}. \]

\( A \) is non-empty and convex clearly. Moreover since \( X \) is Archimedean, \( A \) is closed and hence compact. Let \( x \in A \). Then for any \( i \) and for any \( y \in F_{K_0}(f_1 \circ \cdots \circ f_n) \)
\[ |f_i(x) - y| = |f_i(x) - f_i(y_i)| \leq |x - y| \leq \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y| \leq \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x_0 - y|. \]
and hence $f_i(a) \in A$, that is, $f_i(A) \subset A$. Since $K_0$ is minimal, $A = K_0$. Therefore

$$\bigvee_{x,y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y| \leq \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x_0 - y| < \bigvee_{x,y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y|.$$  

It is a contradiction. Therefore $K_0$ only contains a unique point. The point is a common fixed point of $\{f_i | i = 1, \cdots, n\}$. \qed

**Theorem 4.5.** Let $X$ be a Hausdorff Archimedean vector lattice with unit, $K$ a compact convex subset of $X$ and $\{f_i | i \in I\}$ the commutative family of nonexpansive mappings from $K$ into $K$. Suppose that there exists a homomorphism from $X$ into $\mathbb{R}$ satisfying the condition (H2)$^e$ and $K$ has the normal structure. Then $\bigcap_{i \in I} F_K(f_i)$ is non-empty.

**Proof.** By Theorem 4.4 $\bigcap_{i=1}^n F_K(f_{i_k})$ is non-empty for any finite set $i_1, \cdots, i_n \in I$. Since $K$ is compact, $\bigcap_{i \in I} F_K(f_i)$ is non-empty. \qed

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**References**


