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Some geometric constants related with the modulus of convexity of a Banach space

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We shall consider the contant $C_f(X)$ for a Banach space X, where $f(u,v)$ is a real valued continuous function which is non-decreasing in u and v in $[0,2]$. Some geometric constants of X are unifyingly described by this constant $C_f(X)$ with a suitable f and some previos results are derived.

Let X be a real Banach space with dim $X \geq 2$. The *modulus of convexity* of X is defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \ x, y \in S_X, \ \|x-y\| = \epsilon\right\} \quad (0 \leq \epsilon \leq 2),$$

where S_X is the unit sphere of X. S_X may be replaced by the unit ball B_X. The function δ_X is continuous on [0,2), increasing on [0,2] and strictly increasing on $[\epsilon_0, 2]$, where $\epsilon_0 = \epsilon_0(X) = \sup\{\epsilon \in [0,2] : \ \delta_X(\epsilon) = 0\}$ is the *coefficient of convexity* of X. The function $\delta_X(\epsilon)/\epsilon$ is also increasing on (0,2] (Figiel, 1976).

The *James constant* of X is defined by

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : \ x, y \in S_X\}.$$

X is called *uniformly non-square* if $J(X) < 2$. It is well-known that X is uniformly non-square if and only if $\epsilon_0(X) < 2$. If $J(X) < 2$, we have

$$J(X) = 2(1 - \delta_X(J(X))$$

(Casini [4]).

In this note we shall consider the following constant: Let $f(u,v)$ is a real valued continuous function satisfying $f(u_1,v_1) \le f(u_2,v_2)$ for all $0 \le u_1 \le u_2 \le 2$ and $0 \le v_1 \le v_2 \le 2$. We define the constant $C_f(X)$ to be

$$C_f(X) = \sup\left\{ f(\|x-y\|, \|x+y\|) :\ x, y \in S_X \right\}. \tag{1}$$

One should note that

$$\begin{aligned} J(X) &= C_f(X) \quad \text{if } f(u,v) = \min(u,v), \\ A_2(X) &= C_f(X) \quad \text{if } f(u,v) = (u+v)/2, \\ T(X) &= C_f(X) \quad \text{if } f(u,v) = \sqrt{uv}, \\ C'_{NJ}(X) &= C_f(X) \quad \text{if } f(u,v) = (u^2+v^2)/4. \end{aligned}$$

We recall the definitions of these constants. The constant $A_2(X)$ ([3]) is given by

$$A_2(X) := \rho_X(1) + 1,$$

where $\rho_X(\tau)$ is the modulus of smoothness of X,

$$\rho_X(\tau) = \sup\left\{ \frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 :\ x, y \in S_X \right\} \quad (\tau > 0).$$

The constant $T(X)$ is defined in [1] by

$$T(X) := \sup\{\sqrt{\|x-y\|\|x+y\|} :\ x, y \in S_X\}.$$

The *von Neumann-Jordan constant* of X is

$$C_{NJ}(X) := \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} :\ x, y \text{ are not both } 0 \right\}, \tag{2}$$

where the supremum can be taken over all $x \in S_X$ and $y \in B_X$. The constant defined by taking supremum over all $x, y \in S_X$ in (2) is denoted by $C'_{NJ}(X)$ ([2]). We have $C'_{NJ}(X) \le C_{NJ}(X)$ and they do not conincide in general.

It is readily seen that

$$C_f(X) = \sup\left\{ f(\varepsilon, 2(1-\delta_X(\varepsilon))) :\ 0 < \varepsilon < 2 \right\}. \tag{3}$$

With regard to a lower bound of $C_f(X)$ we easily have

$$C_f(X) \geq \max\left\{ f(J(X), J(X)), f(\epsilon_0(X), 2)\right\}. \tag{4}$$

In particular we have $C_f(X) = f(2,2)$ if $J(X) = 2$. It follows from (4) that $T(X) \geq \sqrt{2\epsilon_0(X)}$ ([1]) and $C'_{NJ}(X) \geq 1 + \epsilon_0(X)^2/4$ ([2]), where we have equality in both inequalities if X is not uniformly non-square.

Theorem 1. *Let $J(X) < 2$ and assume that $f(u,v) = f(v,u)$ for all $u, v \in [0,2]$. Then*

$$C_f(X) = \sup\left\{ f(\varepsilon, 2(1-\delta_X(\varepsilon)) :\ J(X) \leq \varepsilon < 2\right\}. \tag{5}$$

We shall present some applications of (5): Let $J(X) < 2$. Then

$$\rho_X(1) = \sup\left\{ \frac{\epsilon}{2} - \delta_X(\epsilon) :\ J(X) \leq \epsilon < 2\right\} \leq 2\left(1 - \frac{1}{J(X)}\right) \tag{6}$$

and

$$C'_{NJ}(X) = \sup\left\{ \frac{\epsilon^2}{4} + (1-\delta_X(\epsilon))^2 :\ J(X) \leq \epsilon < 2\right\} \leq 1 + 4\left(1 - \frac{1}{J(X)}\right)^2. \tag{7}$$

We shall give simple proofs of (6) and (7). We write J and $\delta(\epsilon)$ for $J(X)$ and $\delta_X(\epsilon)$ respectively. Since $\delta(\epsilon)/\epsilon$ is increasing, $\delta(\epsilon) \geq \delta(J)\epsilon/J$ for all $J \leq \epsilon < 2$. Noting $2\delta(J) = 2 - J$ we have

$$\frac{\epsilon}{2} - \delta(\epsilon) \leq \frac{\epsilon}{2} - \delta(J)\epsilon/J \leq 1 - 2\delta(J)/J = 1 - (2-J)/J = 2(1-1/J),$$

which proves (6). Similarly we have

$$\frac{\epsilon^2}{4} + (1-\delta_X(\epsilon))^2 \leq \frac{\epsilon^2}{4} + (1-\delta(J)\epsilon/J)^2 \leq 1 + (1-2\delta(J)/J)^2 = 1 + 4(1-1/J)^2,$$

which proves (7).

In 2008 Alonso et al. [2] showed that

$$C'_{NJ}(X) \leq J(X),$$

which is useful to estimate the von Neumann-Jordan constant $C_{NJ}(X)$ by $J(X)$. It was shown in [2] that

$$C_{NJ}(X) \leq 1 + (\sqrt{2C'_{NJ}(X)} - 1)^2 \leq 1 + (\sqrt{2J(X)} - 1\ )^2,$$

while by using (7) we easily have

$$C'_{NJ}(X) \le 1 + 4(1 - 1/J(X))^2 \le (1 + \sqrt{J(X) - 1}\,)^2/2,$$

which yields that

$$C_{NJ}(X) \le 1 + (\sqrt{2C'_{NJ}(X)} - 1)^2 \le J(X)$$

(Kato-Takahashi [6]; see also [8], [9]). The simple inequality

$$C_{NJ}(X) \le J(X) \tag{8}$$

concerning the von Neumann-Jordan and James constants was first proved by Takahashi and Kato [7] in 2009, which answered affirmatively a question posed in Alonso et al. [2]. In [7] they proved (8) as

$$C_{NJ}(X) \le \frac{2}{2 - \rho_X(1)} \le J(X),$$

where the second inequality is equivalent to (6).

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