<table>
<thead>
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<th>Title</th>
<th>Some geometric constants related with the modulus of convexity of a Banach space (Nonlinear Analysis and Convex Analysis)</th>
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<tbody>
<tr>
<td>Author(s)</td>
<td>Takahashi, Yasuji; Kato, Mikio</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2011), 1755: 147-151</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2011-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/171211">http://hdl.handle.net/2433/171211</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
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Kyoto University
Some geometric constants related with the modulus of convexity of a Banach space

Yasuji Takahashi  岡山県立大学 名誉教授
ym-takahashi@clear.ocn.ne.jp

Mikio Kato  九州工業大学 工学研究院
katom@mns.kyutech.ac.jp

We shall consider the constant $C_f(X)$ for a Banach space $X$, where $f(u,v)$ is a real valued continuous function which is non-decreasing in $u$ and $v$ in $[0,2]$. Some geometric constants of $X$ are unifyingly described by this constant $C_f(X)$ with a suitable $f$ and some previos results are derived.

Let $X$ be a real Banach space with $\dim X \geq 2$. The modulus of convexity of $X$ is defined by

$$\delta_X(\epsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in S_X, \|x - y\| = \epsilon \right\} \quad (0 \leq \epsilon \leq 2),$$

where $S_X$ is the unit sphere of $X$. $S_X$ may be replaced by the unit ball $B_X$. The function $\delta_X$ is continuous on $[0,2)$, increasing on $[0,2]$ and strictly increasing on $[\epsilon_0,2]$, where $\epsilon_0 = \epsilon_0(X) = \sup \{ \epsilon \in [0,2] : \delta_X(\epsilon) = 0 \}$ is the coefficient of convexity of $X$. The function $\delta_X(\epsilon)/\epsilon$ is also increasing on $[0,2]$ (Figiel, 1976).

The James constant of $X$ is defined by

$$J(X) = \sup \{ \min(\|x + y\|, \|x - y\|) : x, y \in S_X \}.$$  

$X$ is called uniformly non-square if $J(X) < 2$. It is well-known that $X$ is uniformly non-square if and only if $\epsilon_0(X) < 2$. If $J(X) < 2$, we have

$$J(X) = 2(1 - \delta_X(J(X)))$$

(Casini [4]).
In this note we shall consider the following constant: Let \( f(u, v) \) be a real valued continuous function satisfying \( f(u_1, v_1) \leq f(u_2, v_2) \) for all \( 0 \leq u_1 \leq u_2 \leq 2 \) and \( 0 \leq v_1 \leq v_2 \leq 2 \). We define the constant \( C_f(X) \) to be

\[
C_f(X) = \sup \left\{ f(\|x - y\|, \|x + y\|) : x, y \in S_X \right\}. \tag{1}
\]

One should note that

\[
\begin{align*}
J(X) &= C_f(X) \quad \text{if } f(u, v) = \min(u, v), \\
A_2(X) &= C_f(X) \quad \text{if } f(u, v) = (u + v)/2, \\
T(X) &= C_f(X) \quad \text{if } f(u, v) = \sqrt{uv}, \\
C_{N,J}'(X) &= C_f(X) \quad \text{if } f(u, v) = (u^2 + v^2)/4.
\end{align*}
\]

We recall the definitions of these constants. The constant \( A_2(X) \) ([3]) is given by

\[
A_2(X) := \rho_X(1) + 1,
\]

where \( \rho_X(\tau) \) is the modulus of smoothness of \( X \),

\[
\rho_X(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in S_X \right\} \quad (\tau > 0).
\]

The constant \( T(X) \) is defined in [1] by

\[
T(X) := \sup \{ \sqrt{\|x - y\|}\|x + y\| : x, y \in S_X \}.
\]

The von Neumann-Jordan constant of \( X \) is

\[
C_{N,J}(X) := \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \text{ are not both 0} \right\}, \tag{2}
\]

where the supremum can be taken over all \( x \in S_X \) and \( y \in B_X \). The constant defined by taking supremum over all \( x, y \in S_X \) in (2) is denoted by \( C_{N,J}'(X) \) ([2]). We have \( C_{N,J}'(X) \leq C_{N,J}(X) \) and they do not coincide in general.

It is readily seen that

\[
C_f(X) = \sup \left\{ f(\varepsilon, 2(1 - \delta_X(\varepsilon)) : 0 < \varepsilon < 2 \right\}. \tag{3}
\]
With regard to a lower bound of $C_f(X)$ we easily have
\[
C_f(X) \geq \max \left\{ f(J(X), J(X)), f(\epsilon_0(X), 2) \right\}.
\]
(4)

In particular we have $C_f(X) = f(2, 2)$ if $J(X) = 2$. It follows from (4) that 
$T(X) \geq \sqrt{2\epsilon_0(X)}$ ([1]) and $C_{NJ}'(X) \geq 1 + \epsilon_0(X)^2/4$ ([2]), where we have equality in both inequalities if $X$ is not uniformly non-square.

**Theorem 1.** Let $J(X) < 2$ and assume that $f(u, v) = f(v, u)$ for all $u, v \in [0, 2]$. Then
\[
C_f(X) = \sup \left\{ f(\epsilon, 2(1 - \delta_X(\epsilon)) : J(X) \leq \epsilon < 2 \right\}.
\]
(5)

We shall present some applications of (5): Let $J(X) < 2$. Then
\[
\rho_X(1) = \sup \left\{ \frac{\epsilon}{2} - \delta_X(\epsilon) : J(X) \leq \epsilon < 2 \right\} \leq 2 \left( 1 - \frac{1}{J(X)} \right)
\]
(6)

and
\[
C_{NJ}'(X) = \sup \left\{ \frac{\epsilon^2}{4} + (1 - \delta_X(\epsilon))^2 : J(X) \leq \epsilon < 2 \right\} \leq 1 + 4 \left( 1 - \frac{1}{J(X)} \right)^2.
\]
(7)

We shall give simple proofs of (6) and (7). We write $J$ and $\delta(\epsilon)$ for $J(X)$ and $\delta_X(\epsilon)$ respectively. Since $\delta(\epsilon)/\epsilon$ is increasing, $\delta(\epsilon) \geq \delta(J)\epsilon/J$ for all $J \leq \epsilon < 2$. Noting $2\delta(J) = 2 - J$ we have
\[
\frac{\epsilon}{2} - \delta(\epsilon) \leq \frac{\epsilon}{2} - \delta(J)\epsilon/J \leq 1 - 2\delta(J)/J = 1 - (2 - J)/J = 2(1 - 1/J),
\]
which proves (6). Similarly we have
\[
\frac{\epsilon^2}{4} + (1 - \delta_X(\epsilon))^2 \leq \frac{\epsilon^2}{4} + (1 - \delta(J)\epsilon/J)^2 \leq 1 + (1 - 2\delta(J)/J)^2 = 1 + 4(1 - 1/J)^2,
\]
which proves (7).

In 2008 Alonso et al. [2] showed that
\[
C_{NJ}'(X) \leq J(X),
\]
which is useful to estimate the von Neumann-Jordan constant $C_{NJ}(X)$ by $J(X)$. It was shown in [2] that
\[
C_{NJ}(X) \leq 1 + (\sqrt{2C_{NJ}'(X)} - 1)^2 \leq 1 + (\sqrt{2J(X)} - 1)^2,
\]
while by using (7) we easily have
\[
C'_{NJ}(X) \leq 1 + 4(1 - 1/J(X))^2 \leq (1 + \sqrt{J(X) - 1})^2/2,
\]
which yields that
\[
C_{NJ}(X) \leq 1 + (\sqrt{2C'_{NJ}(X) - 1})^2 \leq J(X)
\]
(Kato-Takahashi [6]; see also [8], [9]). The simple inequality
\[
C_{NJ}(X) \leq J(X)
\]
concerning the von Neumann-Jordan and James constants was first proved by Takahashi and Kato [7] in 2009, which answered affirmatively a question posed in Alonso et al. [2]. In [7] they proved (8) as
\[
C_{NJ}(X) \leq \frac{2}{2 - \rho_X(1)} \leq J(X),
\]
where the second inequality is equivalent to (6).

References


