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ON ALMOST CONVERGENCE FOR VECTOR-VALUED FUNCTIONS AND ITS APPLICATION

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1. INTRODUCTION

In 1948, Lorentz [11] introduced a notion of almost convergence for bounded sequences of real numbers: Let \( \{x_n\} \) be a bounded sequence of real numbers. Then, \( \{x_n\} \) is said to be almost convergent if

\[
\mu_n(x_n) = \nu_n(x_n)
\]

for any Banach limits \( \mu \) and \( \nu \). Day [6] defined a notion of almost convergence for bounded real-valued functions defined on an amenable semigroup.

On the other hand, von Neumann [15] introduced a notion of almost periodicity for bounded real-valued functions defined on a group and proved the existence of the mean values for those functions. Later, Bochner and von Neumann [3] proved the existence of the mean values for vector-valued almost periodic functions defined on a group with values in a locally convex space. Recently, Miyake and Takahashi [13, 14] proved the existence of the mean values for vector-valued almost periodic functions defined on an amenable semigroup and obtained non-linear mean ergodic theorems for transformation semigroups of various types.

In this paper, we announce some results recently obtained in studying on almost convergence for vector-valued functions defined on an amenable semigroup with values in a locally convex space. First, motivated by the work of Lorentz, we introduce a notion of almost convergence for those functions and obtain characterizations of vector-valued almost convergent functions. Next, we introduce a notion of the mean values for those functions defined on a semigroup without assumption of amenability and prove characterizations of the space of bounded real-valued functions defined on a semigroup. Finally, by study on almost convergence for commutative semigroups of non-linear mappings, we prove mean ergodic theorems for non-Lipschitzian asymptotically isometric semigroups of continuous self-mappings of a compact convex subset of a general Banach space.
2. Preliminaries

Throughout this paper, we denote by $S$ a semigroup with identity and by $E$ a locally convex topological vector space (or l.c.s.). We also denote by $\mathbb{R}_+$ and $\mathbb{N}_+$ the set of non-negative real numbers and the set of non-negative integers, respectively. Let $\langle E, F \rangle$ be the duality between vector spaces $E$ and $F$. For each $y \in F$, we define a linear functional $f_y$ on $E$ by $f_y(x) = \langle x, y \rangle$. We denote by $\sigma(E, F')$ the weak topology on $E$ generated by $\{f_y : y \in F\}$. $E_\sigma$ denotes a l.c.s. $E$ with the weak topology $\sigma(E, E')$. If $X$ is a l.c.s., we denote by $X'$ the topological dual of $X$. We also denote by $\langle \cdot, \cdot \rangle$ the canonical bilinear form between $E$ and $E'$, that is, for $x \in E$ and $x' \in E'$, $\langle x, x' \rangle$ is the value of $x'$ at $x$.

We denote by $l^\infty(S)$ the Banach space of bounded real-valued functions on $S$. For each $s \in S$, we define operators $l(s)$ and $r(s)$ on $l^\infty(S)$ by

$$(l(s)f)(t) = f(st) \quad \text{and} \quad (r(s)f)(t) = f(ts)$$

for each $t \in S$ and $f \in l^\infty(S)$, respectively. A subspace $X$ of $l^\infty(S)$ is said to be translation invariant if $l(s)X \subseteq X$ and $r(s)X \subseteq X$ for each $s \in S$. Let $X$ be a subspace of $l^\infty(S)$ which contains constants. A linear functional $\mu$ on $X$ is said to be a mean on $X$ if $\|\mu\| = \mu(e) = 1$, where $e(s) = 1$ for each $s \in S$. We often write $\mu_s f(s)$ instead of $\mu(f)$ for each $f \in X$. For $s \in S$, we define a point evaluation $\delta_s$ by $\delta_s(f) = f(s)$ for each $f \in X$. A convex combination of point evaluations is called a finite mean on $S$. As is well known, $\mu$ is a mean on $X$ if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$$

for each $f \in X$; see Day [6] and Takahashi [22] for more details. Let $X$ be also translation invariant. Then, a mean $\mu$ on $X$ is said to be left (or right) invariant if $\mu(l(s)f) = \mu(f)$ (or $\mu(r(s)f) = \mu(f)$) for each $s \in S$ and $f \in X$. A mean $\mu$ on $X$ is said to be invariant if $\mu$ is both left and right invariant. If there exists a left (or right) invariant mean on $X$, then $X$ is said to be left (or right) amenable. If $X$ is also left and right amenable, then $X$ is said to be amenable. We know from Day [6] that if $S$ is commutative, then $X$ is amenable. Let $\{\mu_\alpha\}$ be a net of means on $X$. Then $\{\mu_\alpha\}$ is said to be asymptotically invariant (or strongly regular) if for each $s \in S$, both $l(s)' \mu_\alpha - \mu_\alpha$ and $r(s)' \mu_\alpha - \mu_\alpha$ converge to $0$ in the weak topology $\sigma(X', X)$ (or the norm topology), where $l(s)'$ and $r(s)'$ are the adjoint operators of $l(s)$ and $r(s)$, respectively. Such nets were first studied by Day [6].

We denote by $l^\infty(S, E)$ the vector space of vector-valued functions defined on $S$ with values in $E$ such that for each $f \in l^\infty(S, E)$, $f(S) = \{f(s) : s \in S\}$.
\{f(s): s \in S\} is bounded. Let \(\mathcal{U}\) is a neighborhood base of 0 in \(E\) and let \(M(V) = \{f \in l^{\infty}(S, E): f(S) \subset V\}\) for each \(V \in \mathcal{U}\). A family \(\mathfrak{B} = \{M(V): V \in \mathcal{U}\}\) is a filter base in \(l^{\infty}(S, E)\). Then, \(l^{\infty}(S, E)\) is a l.c.s. with the topology \(\Xi\) of uniform convergence on \(S\) that has a neighborhood base \(\mathfrak{B}\) of 0. For each \(s \in S\), we define the operators \(R(s)\) and \(L(s)\) on \(l^{\infty}(S, E)\) by

\[(R(s)f)(t) = f(ts) \quad \text{and} \quad (L(s)f)(t) = f(st)\]

for each \(t \in S\) and \(f \in l^{\infty}(S, E)\), respectively. Let \(f \in l^{\infty}(S, E)\). We denote by \(\mathcal{R}\mathcal{O}(f)\) the right orbit of \(f\), that is, the set \(\{R(s)f \in l^{\infty}(S, E): s \in S\}\) of right translates of \(f\). Similarly, we also denote by \(\mathcal{L}\mathcal{O}(f)\) the left orbit of \(f\), that is, the set \(\{L(s)f \in l^{\infty}(S, E): s \in S\}\) of left translates of \(f\). A subspace \(\Xi\) of \(l^{\infty}(S, E)\) is said to be translation invariant if \(L(s)\Xi \subset \Xi\) and \(R(s)\Xi \subset \Xi\) for each \(s \in S\). Let \(\Xi\) be a subspace of \(l^{\infty}(S, E)\) which contains constant functions. For each \(s \in S\), we define a (vector-valued) point evaluation \(\Delta_s\) by \(\Delta_s(f) = f(s)\) for each \(f \in l^{\infty}(S, E)\). A convex combination of vector-valued point evaluations is said to be a (vector-valued) finite mean. A mapping \(M\) of \(\Xi\) into \(E\) is called a vector-valued mean on \(\Xi\) if \(M\) is contained in the closure of convex hull of \(\{\Delta_s: s \in S\}\) in the product space \((E_{\sigma})^{\Xi}\). Then, a vector-valued mean \(M\) on \(\Xi\) is a linear continuous mapping of \(\Xi\) into \(E\) such that (i) \(Mp = p\) for each constant function \(p\) in \(\Xi\), and (ii) \(M(f)\) is contained in the closure of convex hull of \(f(S)\) for each \(f \in \Xi\). We denote by \(\Phi_{\Xi}\) the set of vector-valued means on \(\Xi\). Let \(\Xi\) be also translation invariant. Then, a vector-valued mean \(M\) on \(\Xi\) is said to be left (or right) invariant if \(M(L(s)f) = M(f)\) (or \(M(R(s)f) = M(f)\)) for each \(s \in S\) and \(f \in \Xi\). A vector-valued mean \(M\) on \(\Xi\) is said to be invariant if \(M\) is both left and right invariant. Let \(f \in \Xi\) and let \(M\) be a vector-valued mean on \(\Xi\). We define a vector-valued function \(M.f \in l^{\infty}(S, E)\) by \((M.f)(s) = M(L(s)f)\) for each \(s \in S\). Then, \(\Xi\) is said to be introverted if for each \(f \in \Xi\) and vector-valued mean \(M\) on \(\Xi\), \(M.f\) is contained in \(\Xi\).

We also denote by \(l^{\infty}_{\sigma}(S, E)\) the subspace of \(l^{\infty}(S, E)\) such that for each \(f \in l^{\infty}_{\sigma}(S, E)\), \(f(S)\) is relatively weakly compact in \(E\). Let \(X\) be a subspace of \(l^{\infty}(S)\) containing constants such that for each \(f \in l^{\infty}_{\sigma}(S, E)\) and \(x' \in E'\), a function \(s \mapsto \langle f(s), x' \rangle\) is contained in \(X\). Such an \(X\) is called admissible. Let \(\mu \in X'\). Then, for each \(f \in l^{\infty}_{\sigma}(S, E)\), we define a linear functional \(\tau(\mu)\) on \(E'\) by

\[\tau(\mu)f : x' \mapsto \mu(f(\cdot), x').\]

It follows from the bipolar theorem that \(\tau(\mu)f\) is contained in \(E\). A mapping \(\tau\) of \(X'\) onto \(\Phi_{l^{\infty}_{\sigma}(S, E)}\) is linear and continuous where \(X'\) is
equipped with the weak topology $\sigma(X', X)$. Then, for each mean $\mu$ on $X$, $\tau(\mu)$ is a vector-valued mean on $l^\infty_c(S, E)$ (generated by $\mu$). Conversely, every vector-valued mean on $l^\infty_c(S, E)$ is also a vector-valued mean in the sense of Goldberg and Irwin [8], that is, for each $M \in \Phi_{l^\infty_c(S, E)}$, there exists a mean $\mu$ on $X$ such that $\tau(\mu) = M$. Note that $\Phi_{l^\infty_c(S, E)}$ is compact and convex in $(E_a)^{l^\infty(S, E)}$; see also Day [6], Takahashi [20, 22] and Kada and Takahashi [10]. Let $X$ be also translation invariant and amenable. If $\mu$ is a left (or right) invariant mean on $X$, then $\tau(\mu)$ is also left (or right) invariant. Conversely, if $M$ is a left (or right) invariant vector-valued mean on $l^\infty_c(S, E)$, then there exists a left (or right) invariant mean $\mu$ on $X$ such that $\tau(\mu) = M$.

Let $C$ be a closed convex subset of a l.c.s. $E$ and let $\mathcal{F}$ be the semigroup of continuous self-mappings of $C$ under operator multiplication. If $T$ is a semigroup homomorphism of $S$ into $\mathcal{F}$, then $T$ is said to be a representation of $S$ as continuous self-mappings of $C$. Let $S = \{T(s) : s \in S\}$ be a representation of $S$ as continuous self-mappings of $C$ such that for each $x \in C$, the orbit $\mathcal{O}(x) = \{T(s)x : s \in S\}$ of $x$ under $S$ is relatively weakly compact in $C$ and let $X$ be a subspace of $l^\infty(S)$ containing constants such that for each $x \in C$ and $x' \in E'$, a function $s \mapsto \langle T(s)x, x' \rangle$ is contained in $X$. Such an $X$ is called admissible with respect to $S$. If no confusion will occur, then $X$ is simply called admissible. Let $\mu \in X'$. Then, there exists a unique point $x_0$ of $E$ such that $\mu(T(\cdot)x, x') = \langle x_0, x' \rangle$ for each $x' \in E'$. We denote such a point $x_0$ by $T(\mu)x$. Note that if $\mu$ is a mean on $X$, then for each $x \in C$, $T(\mu)x$ is contained in the closure of convex hull of the orbit $\mathcal{O}(x)$ of $x$ under $S$.

3. ON ALMOST CONVERGENCE FOR VECTOR-VALUED FUNCTIONS

Motivated by the work of Lorentz [11], we introduce a notion of almost convergence for vector-valued functions defined on a left amenable semigroup with values in a locally convex space and also obtain characterizations of almost convergence for those functions.

Definition 1. Let $S$ be left amenable and let $f \in l^\infty_c(S, E)$. Then, $f$ is said to be almost convergent in the sense of Lorentz if

$$\tau(\mu)f = \tau(\nu)f$$

for any left invariant means $\mu$ and $\nu$ on $l^\infty(S)$. Note that $f$ is almost convergent in the sense of Lorentz if and only if $M(f) = N(f)$ for any left invariant vector-valued means on $M$ and $N$ on $l^\infty_c(S, E)$.

Theorem 1. Let $S$ be left amenable and let $f \in l^\infty_c(S, E)$. Then, the following are equivalent:

(i) $f$ is almost convergent in the sense of Lorentz;
(ii) the closure $\mathcal{K}$ of convex hull of $\mathcal{RO}(f)$ contains exactly one constant function in the topology $\tau_{wp}$ of weakly pointwise convergence on $S$;
(iii) for each function $g \in \mathcal{K}$, the $\tau_{wp}$-closure of convex hull of $\mathcal{RO}(g)$ contains exactly one constant function.

**Theorem 2.** Let $S$ be commutative, let $f \in l_c^\infty(S, E)$ and let $X$ be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constant functions. Then, the following are equivalent:

(i) $f$ is almost convergent in the sense of Lorentz;
(ii) there exists a strongly regular net $\{\lambda_\alpha\}$ of finite means such that $\{\tau(\lambda_\alpha).f\}$ converges in the topology $\tau_{wu}$ of weakly uniform convergence on $S$;
(iii) for each strongly regular net $\{\mu_\alpha\}$ of means on $X$, $\{\tau(\mu_\alpha).f\}$ converges in the topology $\tau_{wu}$.

Next, we introduce a notion of the mean value for bounded vector-valued functions defined on a semigroup without assumption of amenability and also obtain characterizations of the space of bounded real-valued functions defined on a semigroup which have the mean values.

**Definition 2.** Let $f \in l^\infty(S, E)$ and let $\mathcal{K}$ be the closure of convex hull of $\mathcal{RO}(f)$ in the topology $\tau_{wp}$ of weakly pointwise convergence on $S$. If for each function $g$ in $\mathcal{K}$, the $\tau_{wp}$-closure of convex hull of $\mathcal{RO}(g)$ contains exactly one constant function with value $p$, then $p$ is said to be the *mean value* of $f$; see also von Neumann [15], Bochner and von Neumann [3] and Miyake and Takahashi [13]. In particular, if $S$ is commutative, then it follows from Theorem 1 that $f \in l_c^\infty(S, E)$ has the mean value if and only if the $\tau_{wp}$-closure of convex hull of $\mathcal{RO}(f)$ contains exactly one constant function. We denote by $AC(S)$ the set of bounded real-valued functions defined on $S$ with the mean values.

As in similar arguments of Lemma 1 (the localization theorem) in [9], we obtain some characterizations of the space of bounded real-valued functions defined on a semigroup with the mean values.

**Proposition 1.** $AC(S)$ is a translation invariant and introverted subspace of $l^\infty(S)$ containing constant functions.

Note that it follows from Theorem 1 that if $S$ is left amenable, then $AC(S)$ is the subspace of $l^\infty(S)$ consisting of bounded real-valued functions defined on $S$ which are almost convergent in the sense of Lorentz.

**Theorem 3.** $AC(S)$ is amenable and has a unique invariant mean $\mu$. In this case, $\mu$ is also a unique left invariant mean on $AC(S)$. 
Theorem 4. $AC(S)$ is a maximum translation invariant and introverted subspace of $l^\infty(S)$ containing constant functions which has a unique left invariant mean, ordered by set inclusion.

Theorem 5. If $S$ is commutative, then $AC(S)$ is a maximum translation invariant subspace of $l^\infty(S)$ containing constant functions which has a unique invariant mean, ordered by set inclusion.

4. APPLICATIONS

By studying on almost convergence in the sense of Lorentz for commutative semigroups of non-linear mappings, we prove mean ergodic theorems for non-Lipschitzian asymptotically isometric semigroups of continuous mappings in general Banach spaces. The following lemma is crucial for proving our results.

Lemma 1. Let $S$ be commutative and let $f \in l^\infty_c(S, E)$. If the closure of convex hull of $RO(f)$ contains a constant function with value $p$ in the topology of uniform convergence on $S$, then $f$ is almost convergent in the sense of Lorentz (equivalently, $f$ has the mean value $p$.)

Definition 3. Let $S$ be commutative and let $S = \{T(s) : s \in S\}$ be a representation of $S$ as continuous mappings of a closed convex subset $C$ of a Banach space $E$ into itself. Then, $S$ is said to be asymptotically isometric on $C$ if, for each $x \in C$,

$$\lim_{s \in S} \|T(s + k)x - T(s + h)x\|$$

exists uniformly in $k, h \in S$.

See Bruck [4] and Kada and Takahashi [10].

Definition 4. Let $S$ be left amenable and let $S = \{T(s) : s \in S\}$ be a representation of $S$ as continuous mappings of a weakly compact convex subset $C$ of $E$ into itself and define a mapping $\phi_S$ of $C$ into $l^\infty_c(S, E)$ by $(\phi_S(x))(s) = T(s)x$ for each $s \in S$. Then, a representation $S$ is said to be almost convergent in the sense of Lorentz if, for each $x \in C$, $\phi_S(x)$ has the mean value $p_x$. Such a point $p_x$ is also said to be the mean value of $x$ under $S$.

Theorem 6. Let $S$ be commutative, let $C$ be a compact convex subset of a Banach space $E$, let $S = \{T(s) : s \in S\}$ be an asymptotically isometric representation of $S$ as continuous mappings of $C$ into itself, let $X$ be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constants and let $\{\mu_\alpha\}$ be a strongly regular net of means on $X$. Then, $S$ is almost convergent in the sense of Lorentz, that is, for each $x \in C$, $\{T(l(h)\mu_\alpha)x\}$ converges to the mean value $p_x$ of $x$ under $S$ in $C$ uniformly in $h \in S$. In this case, $p = T(\mu)x$ for each invariant mean $\mu$ on $X$. 
Remark 1. Note that the mean value $T(\mu)x$ of $x$ under $S$ is not always a common fixed point for $S$. It is known in [19] that there exists a nonexpansive mapping $T$ of $C$ into itself such that for some $x \in C$, its Cesàro means $\{1/n \sum_{k=0}^{n-1} T^kx\}$ converge, but its limit point is not a fixed point of $T$; see also Edelstein [7], Bruck [5], Atsushiba and Takahashi [1], Atsushiba, Lau and Takahashi [2], Miyake and Takahashi [13] and Miyake and Takahashi [14]. We conjecture in Theorem 6 that if a Banach space $E$ is strictly convex, then the mean value $p_x$ of $x$ under $S$ is a common fixed point for $S$, that is, $T(s)p_x = p_x$ for each $s \in S$.

For example, the following corollaries are the case when $S$ is a set of the non-negative integers or real numbers.

**Corollary 1.** Let $C$ be a compact convex subset of a Banach space, let $T$ be a continuous mapping of $C$ into itself such that $\lim_{n \to \infty} \|T^{n+k}x - T^{n+h}x\|$ exists uniformly in $k, h \in \mathbb{N}_+$. Then, for each $x \in C$, the Cesàro means

$$\frac{1}{n} \sum_{i=0}^{n-1} T^{i+h}x$$

converge to the mean value of $x$ under $T$ in $C$ uniformly in $h \in \mathbb{N}_+$.

**Corollary 2.** Let $C$ be a compact convex subset of a Banach space and let $S = \{T(t) : t \in \mathbb{R}_+\}$ be an asymptotically isometric one-parameter semigroup of continuous mappings of $C$ into itself. Then, for each $x \in C$, the Bohr means

$$\frac{1}{t} \int_0^t T(t+h)x \, dt$$

converge to the mean value of $x$ under $S$ in $C$ uniformly in $h \in \mathbb{R}_+$ as $t \to +\infty$.

**Corollary 3.** Let $C$ be a compact convex subset of a Banach space and let $S = \{T(t) : t \in \mathbb{R}_+\}$ be an asymptotically isometric one-parameter semigroup of continuous mappings of $C$ into itself. Then, for each $x \in C$, the Abel means

$$r \int_0^\infty \exp(-rt)T(t+h)x \, dt$$

converge to the mean value of $x$ under $S$ in $C$ uniformly in $h \in \mathbb{R}_+$ as $r \to +\infty$.

**REFERENCES**