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Higher Order Generalized Convexity and Duality in Multiobjective Programming involving Cones

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1 Introduction and Preliminaries

We consider the nonlinear programming problem

$$\textbf{(P)} \qquad \text{Minimize} \quad f(x) \qquad \text{subject to} \quad g(x) \geqq 0,$$

where f and g are twice differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}$ and $\mathbb{R}^m$, respectively. Higher order duality in nonlinear programming has been studied by many researchers. By introducing two differentiable functions $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, Mangasarian [4] formulated the higher order dual

$$\begin{array}{lll} \textbf{(HD1)} & \text{Maximize} & f(u) + h(u,p) - y^T g(u) - y^T k(u,p) \\ & \text{subject to} & \nabla_p h(u,p) = \nabla_p y^T k(u,p), \\ & & y \geqq 0, \end{array}$$

where $\nabla_p h(u,p)$ denotes the $n \times 1$ gradient of h with respect to p and $\nabla_p y^T k(u,p)$ denotes the $n \times 1$ gradient of $y^T k$ with respect to p. Later, in [8], Mond and Weir formulated the conditions for which duality holds between **(P)** and **(HD1)**. They considered other higher order duals to **(P)**, for instance,

$$\begin{array}{lll} \textbf{(HD)} & \text{Maximize} & f(u) + h(u,p) - p^T \nabla_p h(u,p) \\ & \text{subject to} & \nabla_p h(u,p) = \nabla_p y^T k(u,p), \\ & & y_i g_i(u) + y_i k_i(u,p) - p^T \nabla_p y_i k_i(u,p) \leq 0, \\ & & \quad i = 1, 2, \cdots, m, \\ & & y \geqq 0, \end{array}$$

Also, Mond and Zhang [9] gave more general invexity type conditions under which duality holds between **(P)** and **(HD1)**, and **(P)** and **(HD)**. The duality between **(P)** and a general higher order Mond-Weir dual was established. In [6], Mishra and Rueda introduced the concepts of higher-order type I, pseudo-type I and quasi-type I functions and established various higher-order duality results involving these functions.

Recently, Mishra and Rueda [5] considered higher order duality for nondifferentiable mathematical programming problem. They formulated a number of higher order duals to a nondifferentiable programming problem and established duality under the higher order generalized invexity conditions introduced in [6].

In [11], Yang et al. extended the results in [5] to a class of nondifferentiable multiobjective programming programs. A unified higher order dual model for nondifferentiable multiobjective programs was presented, where every component of the objective function contains a term involving the support function of a compact convex set.

Very recently, Kim et al.[2] formulated Mond-Weir and Wolfe type higher order dual models with cone constraints. Weak, strong and converse duality theorems are established for an efficient solution by using higher order generalized invexity conditions.

We consider the following nondifferentiable multiobjective programming problem. we introduce the nondifferentiable multiobjective problem involving cone constraints, where every component of the objective function contains a term involving the support function of a compact convex set.

$$\begin{array}{lll} \textbf{(MCP)} & \text{Minimize} & f(x) + s(x|D) \\ & & = (f_1(x) + s(x|D_1), f_2(x) + s(x|D_2), \cdots, f_l(x) + s(x|D_l)) \\ & \text{subject to} & -g(x) \in C_2^*,\ x \in C_1, \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}^l$, $g : \mathbb{R}^n \to \mathbb{R}^m$, C_1 and C_2 are closed convex cones with nonempty interiors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively and C_2^* is polar cone of C_2.

Definition 1.1 *(1)For $i = 1, \cdots, l$ and $j = 1, \cdots, m, (f_i, g_j)$ are said to be higher order type I at u with respect to η, if for all x, the following inequalities hold:*

$$f_i(x) - f_i(u) \geqq \eta(x,u)^T \nabla_p h_i(u,p) + h_i(u,p) - p^T \nabla_p h_i(u,p)$$
and
$$-g_j(u) \leqq \eta(x,u)^T \nabla_p k_j(u,p) + k_j(u,p) - p^T \nabla_p k_j(u,p).$$

(2)For $i = 1, \cdots, l$ and $j = 1, \cdots, m, (f_i, g_j)$ are said to be higher order pseudo quasi type I at u with respect to η, if for all x, the following inequalities hold:

$$\eta(x,u)^T \nabla_p h_i(u,p) \geqq 0 \Rightarrow f_i(x) - f_i(u) - h_i(u,p) + p^T \nabla_p h_i(u,p) \geqq 0$$
and
$$-g_j(u) \geqq k_j(u,p) - p^T \nabla_p k_j(u,p) \Rightarrow \eta(x,u)^T \nabla_p k_j(u,p) \geqq 0.$$

Definition 1.2 *Let $F : S \times S \times \mathbb{R}^n \to \mathbb{R}$ be a sublinear functional, $\rho = (\rho_1, \rho_2)$ and $d(\cdot,\cdot)$ be a metric on $\mathbb{R}$.*
(1)For $i = 1, \cdots, l$ and $j = 1, \cdots, m, (f_i, g_j)$ are said to be higher order (F, ρ) type I at u, if for all x, the following inequalities hold:

$$f_i(x) - f_i(u) \geqq F(x,u; \nabla_p h_i(u,p)) + h_i(u,p) - p^T \nabla_p h_i(u,p) + \rho_{1i} d(x,u)$$
and
$$g_j(u) \geqq F(x,u; -\nabla_p k_j(u,p)) - k_j(u,p) + p^T \nabla_p k_j(u,p) + \rho_{2j} d(x,u).$$

(2)For $i = 1, \cdots, l$ and $j = 1, \cdots, m, (f_i, g_j)$ are said to be higher order (F, ρ) pseudo quasi type I at u, if for all x, the following inequalities hold:

$$F(x,u; \nabla_p h_i(u,p)) \geqq -\rho_{1i} d(x,u)$$
$$\Rightarrow f_i(x) - f_i(u) - h_i(u,p) + p^T \nabla_p h_i(u,p) \geqq 0$$
and
$$g_j(u) + k_j(u,p) - p^T \nabla_p k_j(u,p) \leqq 0 \Rightarrow F(x,u; -\nabla_p k_j(u,p)) \leqq -\rho_{2j} d(x,u).$$

Definition 1.3 *[7] Let B be a compact convex set in $\mathbb{R}^n$. The support function $s(x|B)$ of B is defined by*

$$s(x|B) := max\{x^T y : y \in B\}.$$

The support function $s(x|B)$, being convex and everywhere finite, has a subdifferential, that is, there exists z such that

$$s(y|B) \geq s(x|B) + z^T(y - x) \ \ for\ all\ y \in B.$$

2 Duality Results

We propose the following dual problem (**MMCD**) to (**MCP**):

$$\text{(MMCD)} \quad \text{Maximize} \quad f(u) + u^T w + (\lambda^T h(u,p))e - p^T \nabla_p (\lambda^T h(u,p))e$$

$$\text{subject to} \quad \lambda^T[\nabla_p h(u,p) + w] = \nabla_p y^T k(u,p), \tag{1}$$

$$g(u) + k(u,p) - p^T \nabla_p k(u,p) \in C_2^*, \tag{2}$$

$$w_i \in D_i, \; i = 1, \cdots, l,$$

$$y \in C_2, \;\; \lambda > 0, \;\; \lambda^T e = 1,$$

where
$(i) f : \mathbb{R}^n \to \mathbb{R}^l$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable functions,
$(ii) C_1$ and C_2 are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,
$(iii) C_1^*$ and C_2^* are polar cones of C_1 and C_2, respectively ,
$(iv) e = (1, \cdots, 1)^T$ is vector in $\mathbb{R}^l$,
$(v) w_i (i = 1, \cdots, l)$ is vector in $\mathbb{R}^n$ and $D_i (i = 1, \cdots, l)$ is compact convex set in $\mathbb{R}^n$, respectively,
$(vi) h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^l$ and $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ are differentiable functions;
$\nabla_p h_j(u,p)$ and $\nabla_p y^T k(u,p)$ denote the $n \times 1$ gradient of h_j and $y^T k$ with respect to p, respectively.

Now we establish the duality theorems between (**MCP**) and (**MMCD**).

Theorem 2.1 (Weak Duality) *Let x and (u, y, λ, w, p) be feasible solutions of* (**MCP**) *and* (**MMCD**), *respectively. Assume that*
$(i)(\lambda^T[f(\cdot) + (\cdot)^T w], y^T g(\cdot))$ is higher order pseudo quasi type I with respect to η or
$(ii)(f_i(\cdot) + (\cdot)^T w_i, y^T g(\cdot)), i = 1, 2, \cdots, l$, is higher order (F, ρ) type I with $\rho_1 + \rho_2 \geqq 0$ or
$(iii)(\lambda^T[f(\cdot) + (\cdot)^T w], y^T g(\cdot))$ is higher order (F, ρ) pseudo quasi type I with $\rho_1 + \rho_2 \geqq 0$.
Then,

$$f_i(x) + s(x|D_i) \leqq f_i(u) + u^T w_i + (\lambda^T h(u,p)) - p^T \nabla_p (\lambda^T h(u,p)), \text{ for all } i$$
$$\text{and } f_i(x) + s(x|D_i) < f_i(u) + u^T w_i + (\lambda^T h(u,p)) - p^T \nabla_p (\lambda^T h(u,p)), \text{ for some } i.$$

Proof. Assume to the contrary that

$$f(x) + s(x|D) < f(u) + u^T w + (\lambda^T h(u,p))e - p^T \nabla_p (\lambda^T h(u,p))e.$$

Since $\lambda > 0$,

$$\lambda^T[f(x) + s(x|D)] < \lambda^T[f(u) + u^T w] + \lambda^T h(u,p) - p^T \nabla_p \lambda^T h(u,p). \tag{3}$$

(i)Since $y \in C_2$ and the constraint (2), we obtain $y^T[g(u) + k(u,p) - p^T \nabla_p k(u,p)] \leqq 0$. By the assumption (i), we get

$$\eta(x,u)^T \nabla_p y^T k(u,p) \geqq 0.$$

From the constraint (1), the above inequality implies

$$\eta(x,u)^T \lambda^T [\nabla_p h(u,p) + w] \geqq 0.$$

Also, by the assumption (i), we have

$$\lambda^T[f(x)+x^Tw] \geqq \lambda^T[f(u)+u^Tw]+\lambda^T h(u,p)-p^T\nabla_p\lambda^T h(u,p).$$

Using the fact that $f(x)+s(x|D) \geqq f(x)+x^Tw$, it becomes

$$\lambda^T[f(x)+s(x|D)] \geqq \lambda^T[f(u)+u^Tw]+\lambda^T h(u,p)-p^T\nabla_p\lambda^T h(u,p).$$

which contradicts (3).
(ii)By the assumption (ii), we have

$$\begin{aligned}
&\lambda^T[f(x)+x^Tw]-\lambda^T[f(u)+u^Tw]-\lambda^T h(u,p)+p^T\nabla_p\lambda^T h(u,p)\\
&\quad \geqq F(x,u;\nabla_p\lambda^T h(u,p)+\lambda^T w)+\lambda^T\rho_1 d(x,u) \quad \text{and} \qquad (4)\\
&y^T g(u)+y^T k(u,p)-p^T\nabla_p y^T k(u,p)\\
&\quad \geqq F(x,u;-\nabla_p y^T k(u,p))+\rho_2 d(x,u). \qquad (5)
\end{aligned}$$

Summing (4) and (5), and using sublinearity of $F(x,u;\cdot)$, we have

$$\begin{aligned}
&\Big(\lambda^T[f(x)+x^Tw]-\lambda^T[f(u)+u^Tw]-\lambda^T h(u,p)+p^T\nabla_p\lambda^T h(u,p)\Big)\\
&+\Big(y^T g(u)+y^T k(u,p)-p^T\nabla_p y^T k(u,p)\Big)\\
&\geqq F(x,u;\nabla_p\lambda^T h(u,p)+\lambda^T w-\nabla_p y^T k(u,p))\\
&\qquad +(\lambda^T\rho_1+\rho_2)d(x,u).
\end{aligned}$$

Using the fact that $s(x|D) \geqq x^Tw$ and (1), above inequality becomes

$$\begin{aligned}
&\lambda^T[f(x)+s(x|D)]-\lambda^T[f(u)+u^Tw]-\lambda^T h(u,p)+p^T\nabla_p\lambda^T h(u,p)\\
&\quad \geqq -y^T g(u)-y^T k(u,p)+p^T\nabla_p y^T k(u,p).\\
&\quad \geqq 0, \qquad (\text{by } (2))
\end{aligned}$$

which contradicts (3).
(iii)Since $s(x|D) \geqq x^Tw$, (3) implies,

$$\lambda^T[f(x)+x^Tw] < \lambda^T[f(u)+u^Tw]+\lambda^T h(u,p)-p^T\nabla_p\lambda^T h(u,p).$$

By assumption (iii), it yields

$$F(x,u;\nabla_p\lambda^T h(u,p)+\lambda^T w) < -\rho_1 d(x,u). \qquad (6)$$

Since $y \in C_2$ and (2), we get

$$y^T[g(u)+k(u,p)-p^T\nabla_p k(u,p)] \leqq 0.$$

By assumption (iii), it yields

$$F(x,u;-\nabla_p y^T k(u,p)) \leqq -\rho_2 d(x,u). \qquad (7)$$

Hence (6), (7), sublinearity of F and $\rho_1+\rho_2 \geqq 0$, then we have

$$F(x,u;\nabla_p\lambda^T h(u,p)+\lambda^T w-\nabla_p y^T k(u,p)) < 0,$$

which is a contradiction, since $F(x,u;0)=0$. □

We obtain the following lemma from [1] and [7] in order to prove strong duality theorem.

Lemma 2.1 *If $\overline{x}$ is a weakly efficient solution of* (**MCP**) *at which constraint qualification [3] be satisfied. Then there exist $\overline{w}_i \in D_i (i = 1, \cdots, l), \overline{\lambda} > 0$ and $\overline{y} \in C_2$ with $(\overline{\lambda}, \overline{y}) \neq 0$ such that*

$$
\begin{aligned}
&[\overline{\lambda}^T (\nabla f(\overline{x}) + \overline{w}) - \nabla \overline{y}^T g(\overline{x})]^T (x - \overline{x}) \geqq 0, \quad \textit{for all} \quad x \in C_1, \\
&\overline{y}^T g(\overline{x}) = 0, \\
&\overline{w}_i \in D_i, \ s(\overline{x}|D_i) = \overline{x}^T \overline{w}_i, \ i = 1, \cdots, l.
\end{aligned}
$$

Theorem 2.2 (Strong Duality) *Let $\overline{x}$ be a weakly efficient solution of* (**MCP**) *at which constraint qualification [3] be satisfied. Let*

$$h(\overline{x}, 0) = 0, \ k(\overline{x}, 0) = 0, \ \nabla_p h(\overline{x}, 0) = \nabla f(\overline{x}), \ \nabla_p k(\overline{x}, 0) = \nabla g(\overline{x}). \tag{8}$$

Then there exist $\overline{\lambda} > 0$, $\overline{y} \in C_2$ and $\overline{w}_i \in D_i (i = 1, \cdots, l)$ such that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ is feasible for (**MMCD**) *and the objective values of* (**MCP**) *and* (**MMCD**) *are equal. If the assumptions of Theorem 2.1 are satisfied, then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ is a weakly efficient solution of* (**MMCD**).

Proof. Since $\overline{x}$ is a weakly efficient solution of (**MCP**), by Lemma 2.1, then there exist $\overline{w}_i \in D_i, i = 1, \cdots, l$, $\overline{\lambda} > 0$ and $\overline{y} \in C_2$ with $(\overline{\lambda}, \overline{y}) \neq 0$ such that

$$(\overline{\lambda}^T (\nabla f(\overline{x}) + \overline{w}) - \overline{y}^T \nabla g(\overline{x}))^T (x - \overline{x}) \geqq 0, \quad \text{for all} \quad x \in C_1, \tag{9}$$

$$\overline{y}^T g(\overline{x}) = 0, \tag{10}$$

$$s(\overline{x}|D_i) = \overline{x}^T \overline{w}_i, \ i = 1, \cdots, l. \tag{11}$$

Since $x \in C_1$, $\overline{x} \in C_1$ and C_1 is a closed convex cone, we have $x + \overline{x} \in C_1$ and thus the inequality (9) implies

$$
\begin{aligned}
&(\overline{\lambda}^T (\nabla f(\overline{x}) + \overline{w}) - \overline{y}^T \nabla g(\overline{x}))^T x \geqq 0, \quad \text{for all} \quad x \in C_1, \\
&\text{i.e.,} \\
&\overline{\lambda}^T (\nabla f(\overline{x}) + \overline{w}) - \overline{y}^T \nabla g(\overline{x}) = 0.
\end{aligned}
$$

And (10) implies $\overline{y}^T g(\overline{x}) \leqq 0$, then $g(\overline{x}) \in C_2^*$. Clearly, using (8) and (11), $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ is feasible for (**MMCD**) and corresponding values of (**MCP**) and (**MMCD**) are equal. If the assumptions of Theorem 2.1 are satisfied, then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ is a weakly efficient solution of (**MMCD**). □

Theorem 2.3 (Converse Duality) *Let $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p})$ be a weakly efficient solution of* (**MMCD**). *Assume that*
(i)$h(\overline{u}, 0) = 0$, $k(\overline{u}, 0) = 0$, $\nabla_p h(\overline{u}, 0) = \nabla f(\overline{u})$, $\nabla_p k(\overline{u}, 0) = \nabla g(\overline{u})$,
(ii)$\nabla_p [\nabla_u \overline{\lambda}^T h(\overline{u}, \overline{p}) - \nabla_u \overline{y}^T k(\overline{u}, \overline{p})]$ is positive or negative definite and
(iii)the set of vectors $\{[\nabla_{pp} \overline{\lambda}^T h(\overline{u}, \overline{p})]_j, \ [\nabla_{pp} k_i(\overline{u}, \overline{p})]_j, \ i = 1, \cdots, m, \ j = 1, \cdots, n\}$ are linearly independent, where $[\nabla_{pp} \overline{\lambda}^T h(\overline{u}, \overline{p})]_j$ is the j-th row of the matrix $\nabla_{pp} \overline{\lambda}^T h(\overline{u}, \overline{p})$ and $[\nabla_{pp} k_i(\overline{u}, \overline{p})]_j$ is the j-th row of the matrix $\nabla_{pp} k_i(\overline{u}, \overline{p})$.
Then $\overline{u}$ is feasible for (**MCP**) *and the objective values of* (**MCP**) *and* (**MMCD**) *are equal. If the assumptions of Theorem 2.1 are satisfied, then $\overline{u}$ is a weakly efficient solution of* (**MCP**).

Proof. Since $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p})$ is a weakly efficient solution of (**MMCD**), by modifying the Fritz John optimality condition, then there exist $\alpha \in \mathbb{R}^l_+, \beta \in \mathbb{R}^n_+, \mu \in C_2, \delta \in C_2^*$ and $\rho \in \mathbb{R}^l_+$ such that

$$\begin{aligned}
&\alpha^T[\nabla_u f(\bar{u}) + \bar{w} + \nabla_u \bar{\lambda}^T h(\bar{u},\bar{p})e - \bar{p}^T \nabla_{pu} \bar{\lambda}^T h(\bar{u},\bar{p})e] \\
&\quad -\beta^T[\nabla_{pu}\bar{\lambda}^T h(\bar{u},\bar{p}) - \nabla_{pu}\bar{y}^T k(\bar{u},\bar{p})] \\
&\quad -\mu^T[\nabla_u g(\bar{u}) + \nabla_u k(\bar{u},\bar{p}) - \bar{p}^T\nabla_{pu}k(\bar{u},\bar{p})] = 0, && (12)\\
&\beta^T \nabla_p k(\bar{u},\bar{p}) - \delta = 0, && (13)\\
&\alpha^T[h(\bar{u},\bar{p})e - \bar{p}^T\nabla_p h(\bar{u},\bar{p})e] - \beta^T[\nabla_p h(\bar{u},\bar{p}) + \bar{w}] + \rho = 0, && (14)\\
&\alpha_i^T \bar{u} - \beta^T \bar{\lambda}_i \in N_{D_i}(\bar{w}_i),\ i = 1,\cdots,l, && (15)\\
&[(\alpha^T e)\bar{p} + \beta]^T \nabla_{pp}\bar{\lambda}^T h(\bar{u},\bar{p}) - [(\beta^T\bar{y}) + (\mu^T\bar{p})]^T\nabla_{pp}k(\bar{u},\bar{p}) = 0, && (16)\\
&\beta^T[\bar{\lambda}^T(\nabla_p h(\bar{u},\bar{p}) + \bar{w}) - \nabla_p \bar{y}^T k(\bar{u},\bar{p})] = 0, && (17)\\
&\mu^T[g(\bar{u}) + k(\bar{u},\bar{p}) - \bar{p}^T\nabla_p k(\bar{u},\bar{p})] = 0, && (18)\\
&\delta^T\bar{y} = 0, && (19)\\
&\rho^T\bar{\lambda} = 0, && (20)\\
&(\alpha,\beta,\mu,\delta,\rho) \neq 0. && (21)
\end{aligned}$$

By the assumption (iii), (16) implies that

$$(\alpha^T e)\bar{p} + \beta = 0 \ \text{ and } \ (\beta^T\bar{y}) + (\mu^T\bar{p}) = 0. \tag{22}$$

Also, using (22) in (12), we have

$$\alpha^T[\nabla_u f(\bar{u}) + \bar{w} + \nabla_u\bar{\lambda}^T h(\bar{u},\bar{p})e] - \mu^T[\nabla_u g(\bar{u}) + \nabla_u k(\bar{u},\bar{p})] = 0. \tag{23}$$

Multiplying (23) by $\bar{p}$ and using (22), we obtain

$$\beta^T[\nabla_u\bar{\lambda}^T f(\bar{u}) + \bar{\lambda}^T\bar{w} + \nabla_u\bar{\lambda}^T h(\bar{u},\bar{p})] - (\beta^T\bar{y})^T[\nabla_u g(\bar{u}) + \nabla_u k(\bar{u},\bar{p})] = 0,$$

that is,

$$\beta^T[\nabla_u\bar{\lambda}^T f(\bar{u}) + \bar{\lambda}^T\bar{w} + \nabla_u\bar{\lambda}^T h(\bar{u},\bar{p}) - \nabla_u\bar{y}^T g(\bar{u}) - \nabla_u\bar{y}^T k(\bar{u},\bar{p})] = 0. \tag{24}$$

Differentiating (24) with respect to $\bar{p}$, it follows that

$$\beta^T\nabla_p[\nabla_u\bar{\lambda}^T h(\bar{u},\bar{p}) - \nabla_u\bar{y}^T k(\bar{u},\bar{p})] = 0. \tag{25}$$

Multiplying (25) by β, it follows that

$$\beta^T\nabla_p[\nabla_u\bar{\lambda}^T h(\bar{u},\bar{p})) - \nabla_u\bar{y}^T k(\bar{u},\bar{p}))]\beta = 0.$$

By the assumption (ii), it implies that $\beta = 0$. So, (22) yields $(\alpha^T e)\bar{p} = 0$ and $\mu^T\bar{p} = 0$. If $\alpha = 0$ and $\mu = 0$, then from (13) and (14), $\delta = 0$ and $\rho = 0$. This contradicts (21). Hence $\bar{p} = 0$. Using $\bar{p} = 0$ and the assumption (i), (18) yields $\mu^T g(\bar{u}) = 0$, which implies $\mu^T g(\bar{u}) \geqq 0$. Since $\mu \in C_2$, we get $-g(\bar{u}) \in C_2^*$. Thus, $\bar{u}$ is a feasible solution of **(MCP)**. From (15), $\bar{u} \in N_{D_i}(\bar{w}_i)$, $i = 1,\cdots,l$, so that $\bar{u}^T\bar{w}_i = s(\bar{u}|D_i), i = 1,\cdots,l$. Therefore, The corresponding value of **(MCP)** and **(MMCD)** are equal, because of $\bar{p} = 0$ and the assumption (i). Moreover, By Theorem 2.1, it follows that $\bar{u}$ is a weakly efficient solution of **(MCP)**. □

Also, we consider the following Wolfe dual problem **(MWCD)** to **(MCP)**:

$$\begin{aligned}
\textbf{(MWCD)} \quad \text{Maximize} \quad & f(u) + u^T w + (\lambda^T h(u,p))e - p^T\nabla_p(\lambda^T h(u,p))e \\
& \quad -y^T[g(u) + k(u,p) - p^T\nabla_p k(u,p)]e \\
\text{subject to} \quad & \lambda^T[\nabla_p h(u,p) + w] = \nabla_p y^T k(u,p), && (26)\\
& w_i \in D_i,\ i = 1,\cdots,l, \\
& y \in C_2,\ \lambda > 0,\ \lambda^T e = 1,
\end{aligned}$$

By using the similar method, we can establish the weak, strong and converse duality theorems between **(MCP)** and **(MWCD)**.

3 Special Cases

We give some special cases of our duality.
If $C_1 = \mathbb{R}^n_+$, $C_2 = \mathbb{R}^m_+$ and $D_i = \{0\}$, $i = 1, \cdots, l$,
(i)$h(u,p) = p^T\nabla f(u)$ and $k(u,p) = p^T\nabla g(u)$, then **(MWCD)** becomes the first order dual program in Wolfe [10],
(ii)$h(u,p) = p^T\nabla f(u) + \frac{1}{2}p^T\nabla^2 f(u)p$ and $k(u,p) = p^T\nabla g(u) + \frac{1}{2}p^T\nabla^2 g(u)p$, then we obtain second order dual programs which studied by Mangasarian [4].
(iii)then our primal and dual models become dual programs considered in Zhang [12].
(iv)then our primal and dual model **(MWCD)** become dual programs considered in Mangasarian [4].
(v)$l = 1$, then our dual programs become dual programs considered in Mond and Zhang [9].
(vi)Let $C_1 = \mathbb{R}^n_+$, $C_2 = \mathbb{R}^m_+$, $l = 1$ and $D \in \mathbb{R}^n \times \mathbb{R}^n$ be positive semidefinite symmetric matrix. If $s(x|D) = (x^TBx)^{\frac{1}{2}}$ where $D = \{Bw | w^TBw \leqq 1\}$, then we get higher order dual programs which studied by Mishra and Rueda [5].
(vii)If $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$, then our primal and dual model become dual programs considered in Yang et al. [11].
(viii)If $D_i = \{0\}$, $i = 1, \cdots, l$, then **(MMCD)** and **(MWCD)** reduced the pair of Mond-Weir and Wolfe type programs considered in D.S. Kim et al.[2].

References

[1] M.S. Bazaraa and J.J. Goode, On symmetric duality in nonlinear programming, *Oper. Res.* **21(1)** (1973), 1-9.

[2] D.S. Kim, H.S. Kang, Y.J. Lee and Y.Y. Seo, Higher Order Duality in Multiobjective Programming with Cone Constraints, *Optimization* **59** (2010), 29-43

[3] O.L. Mangasarian, Nonlinear Programming, MeGraw-Hill, New York, 1969.

[4] O.L. Mangasarian, Second and higher order duality in nonlinear programming, *J. Math. Anal. Appl.* **51** (1975), 607-620.

[5] S.K. Mishra and N.G. Rueda, Higher order generalized invexity and duality in nondifferentiable mathematical programming, *J. Math. Anal. Appl.* **272** (2002), 496-506.

[6] S.K. Mishra and N.G. Rueda, Higher order generalized invexity and duality in mathematical programming, *J. Math. Anal. Appl.* **247** (2000), 173-182.

[7] B. Mond and M. Schechter, Nondifferentiable symmetric duality, *Bull. Austral. Math. Soc.* **53** (1996), 177-188.

[8] B. Mond and T. Weir, Generalized convexity and higher order duality, *in: S. Schaible, W. T. Ziemba(Eds.), Generalized convexity in optimization and economics, Academic Press, New York, 1981,* 263-280.

[9] B. Mond and J. Zang, Higher order invexity and duality in mathematical programming, *in: J. P. Crouzeix et al.(Eds.), Generalized invexity, Generalized Monotonicity:Recent Results, Kluwer Academic, Dordrecht,* (1998), 357-372.

[10] P. Wolfe, A duality theorem for nonlinear programming, *Quart. Appl. Math.* **19** (1961), 239-244.

[11] X.M. Yang, K.L. Teo and X.Q. Yang, Higher order generalized convexity and duality in nondifferentiable multiobjective mathematical programming, *J. Math. Anal. Appl.* **297** (2004), 48-55.

[12] J. Zhang, Higher order convexity and duality in multiobjective programming problems, *in: A. Eberhard, R. Hill, D. Ralph, B. M. Glover(Eds.),"Progress in optimization", Contributions from Australasia, Applied Optimization, Vol. 30, Kluwer Academic Publishers, Dordrecht,* (1999), 101-117.