Title
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(Research into Vertex Operator Algebras, Finite Groups and Combinatorics)

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Citation
数理解析研究所講究録 2011, 1756: 101-105

Issue Date
2011-08

URL
http://hdl.handle.net/2433/171289

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
Intertwining operator and $C_2$-cofiniteness of modules

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Abstract

Let $V$ be a vertex operator algebra and $T$ a $V$-module. We show that if there are $C_2$-cofinite $V$-modules $U$ and $W$ and a surjective (logarithmic) intertwining operator $\mathcal{Y}$ of type $(U \ T \ W)$, then $T$ is also $C_2$-cofinite. So, when $V$ is simple and $V' \cong V$, then if one of $V$-modules is $C_2$-cofinite, then so is $V$.

1 Introduction

A vertex algebra was introduced by axiomatizing the concept of a Chiral algebra in conformal field theory by Borcherds [1]. It is a triple $(V, Y, 1)$ satisfying the several axioms, where $V$ is a graded vector space $V = \oplus_{i \in \mathbb{Z}} V_i$ over the complex number field $\mathbb{C}$, $Y(v, z) = \sum_{m \in \mathbb{Z}} v_m z^{-m-1} \in \text{End}(V)[[z, z^{-1}]]$ denotes a vertex operator of $v \in V$ on $V$, $1 \in V_0$ is a specified element called the vacuum. When $V$ has another specified element $\omega \in V_2$ and $V$ has a lower bound of weights and all homogeneous subspaces are of finite dimensional, then we call $V$ a vertex operator algebra. We set $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-1}$.

For a VOA $V$-module $W$, we define $C_2(W) = \{v_{-2}u \mid v, u \in V, \text{wt}(v) \geq 1\}$. When $C_2(W)$ has a finite co-dimension in $W$, $W$ is called to be $C_2$-cofinite. A concept of $C_2$-cofiniteness is originally introduced by Zhu [8] as a technical assumption to prove a modular invariance property of the space of the trace functions on modules. However, we are now recognizing the real meaning and the importance of $C_2$-cofiniteness. For example, $V$ is $C_2$-cofinite if and only if all $V$-modules are $\mathbb{N}$-gradable. (See [2] and [7] for the proof.) We will use this fact frequently in this paper.

Our main result in this paper is the following:

**Theorem 1** Let $U$ be a vertex operator algebra of CFT-type. Let $A$, $B$, $C$ be simple $\mathbb{N}$-graded $U$-modules and $\mathcal{I}$ a surjective (formal power series) intertwining operator of type $(A \ C \ B)$. If both of $A$ and $B$ are $C_h$-cofinite as $U$-modules for $h = 1, 2$, then so is $C$.

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2 Preliminary

From the axiom of VOAs, for \( v \in V_r \) and \( u \in V_n \), we have \( v_m u \in V_{r - m - 1 + n} \). Hence there is an integer \( N \) such that \( v_u u = 0 \) for any \( s > N \). This property is called a truncation property. In this paper, we will say that \( v \) is truncated at \( u \) to simplify the terminology.

Set \( V^* = \text{Hom}(V, \mathbb{C}) \) and define a pairing \( \langle \cdot, \cdot \rangle \) on \( V^* \times V \) by \( \langle \xi, v \rangle = \xi(v) \) for \( \xi \in V^* \) and \( v \in V \). For \( T \subseteq V \), Annh\( (T) \) denotes an annihilator of \( T \), that is, Annh\( (T) = \{ \xi \in V^* \mid \langle \xi, t \rangle = 0 \text{ for all } t \in T \} \). For \( v \in V \) and \( m \in \mathbb{Z} \), an action \( v_m^* \) on \( V^* \) is defined by

\[
\langle (\sum_{m \in \mathbb{Z}} v_m z^{-m-1}) \xi, w \rangle = \langle \xi, Y(e^{L(1)z}(-z^{-2})^{L(0)}v, z^{-1})w \rangle
\]

for \( w \in V \) and \( \xi \in \text{Hom}(V, \mathbb{C}) \), where \( Y^*(v, z) = \sum_{m \in \mathbb{Z}} v_m^* z^{-m-1} \) is called an adjoint operator of \( v \). An important fact is that \((\oplus_{m \in \mathbb{Z}} \text{Hom}(V_m, \mathbb{C}), Y^*)\) becomes a \( V \)-module as they proved in [3]. This module is called a restricted dual of \( V \) and denoted by \( V' \). In particular, \( Y^*(\cdot, z) \) satisfy the Borcherds identity:

\[
\sum_{i=0}^{\infty} \binom{m}{i} (u_{r+i}^* u_{n-i}^* \xi)_{m+n-i} = \sum_{i=0}^{\infty} (-1)^{i} \binom{r}{i} \{ u_{r+m-i}^* v_{n+i}^* \xi - (-1)^r v_{r+n-i}^* u_{m+i}^* \xi \} \quad (2.1)
\]

for any \( m, n, r \in \mathbb{Z} \), \( v, u \in V, \xi \in V' \). We note \( V' = \oplus_{n \in \mathbb{Z}} V_n \) and \( V^* = \prod_{n \in \mathbb{Z}} V_n \). Therefore we can express \( \xi \in V^* \) by \( \prod_n \xi_n \) with \( \xi_n \in \text{Hom}(V_n, \mathbb{C}) \). We call that \( \xi \in V^* \) is "\( L(0) \)-free" if \( \dim \mathbb{C}[L(0)] \xi = \infty \), that is, \( \xi_m \neq 0 \) for infinitely many \( m \). We note that any \( N \)-gradable module does not contain any \( L(0) \)-free elements.

Let go back to (2.1). If \( \xi \in \text{Hom}(V, \mathbb{C}) \), then all terms in (2.1) have the same weight \( \text{wt}(a) + \text{wt}(b) - r - m - n - 2 + t \) and so the Borcherds' identity is also well-defined on \( V^* \), as Li has pointed out in [5]. However, \( V^* \) is not a \( V \)-module because of failure of truncation properties. In order to find a \( V \)-module in \( V^* \), we will start our arguments from one point \( \xi \) in \( V^* \).

Lemma 2 If \( u \) and \( v \) are truncated at \( \xi \), then \( v_m u \) is also truncated at \( \xi \) for any \( m \). In particular, if all elements in \( \Omega \) of \( V \) are truncated at \( \xi \) and \( \langle \Omega \rangle_{VA} = V \), then all elements in \( V \) are truncated at \( \xi \), where \( \langle \Omega \rangle_{VA} \) denotes a vertex subalgebra generated by \( \Omega \).

[Proof] By the assumption, there is an integer \( N \) such that \( u_n \xi = v_n \xi = u_n v = 0 \) for \( n \geq N \). We assert that for \( s \in \mathbb{N} \) and \( n \geq 2N + s \), we have \((u_{N-s} v)_{n} \xi = 0 \). Suppose false and let \( s \) be a minimal counterexample. Substituting \( r = N - s \), \( n = N + s + p \), \( m = N + q \) in (2.1) with \( p, q \geq 0 \), we have

\[
\text{[LeftSide]} = \sum_{i=0}^{\infty} \binom{N+s}{i} (u_{N-s+i} v)_{2N+q+s+p-i} \xi = \sum_{i=0}^{\infty} \binom{N+s}{i} (u_{N-(s-i)} v)_{2N+s-i+p+q} \xi
\]

by the minimality of \( s \). On the other hand, we have:

\[
\text{[RightSide]} = \sum_{i=0}^{\infty} (-1)^i \binom{N-s}{i} (u_{2N-s+i} v)_{2N+s+p-i} \xi - (-1)^{N-s} v_{2N-s+p-i} u_{N+q+i} \xi = 0,
\]
which contradicts the choice of $s$.

Since $v_n u_m \xi = u_m v_n \xi + \sum_{i=0}^{\infty} \binom{i}{i}(v_i u)_{n+m-i} \xi$, the above lemma also implies:

**Lemma 3** If $v$ and $u$ are truncated at $\xi$, then $v$ is truncated at $u_m \xi$ for any $m$. In particular, if all elements of $V$ are truncated at $\xi$, then $<v_1^m, \ldots, v_n^m \xi | u^i \in V, m_i \in \mathbb{Z} >_C$ is a $V$-module.

As Buhl has shown in [2], if $V$ is $C_2$-cofinite, then all $V$-modules are $N$-gradable and so there are no $L(0)$-free elements at which all elements in $V$ are truncated. Namely, we have proved the following, which we will frequently use.

**Lemma 4** Let $V$ be a $C_2$-cofinite vertex operator algebra and $\xi \in V^*$. If $\Omega \subseteq V$ generates $V$ as a vertex subalgebra and all elements of $\Omega$ are truncated on $\xi$, then $\xi$ is not $L(0)$-free.

For $A, B \subseteq V$, we will often use the notation $A_{[m]}B$ to denote a subspace spanned by $\{a_m b | a \in A, b \in B\}$. We note that if $A$ is a $\mathbb{C}[L(-1)]$-module, then $A_{(-2)}B \subseteq A_{(-2)}B$ for $m \in \mathbb{N}$ since $(L(-1)\alpha - m)b = ma_{-m-1}b$ for $a \in A$ and $b \in B$. Not only $V$, we use this notation for a pair $(U, W)$ of a VOA $U$ and its module $W$. For example, we set $C_2(W) = U_{(-2)}W$, where $U^+ = \bigoplus_{k=1}^{\infty} U_k$. We also set $C_1(W) = U_{(-1)}W$. We say that $W$ is $C_2$-cofinite as a $U$-module if dim $W/C_{h}(W) < \infty$ for $h = 1, 2$. We note any VOA $U$ is $C_1$-cofinite as a $U$-module and so this definition is not equal to the ordinary $C_1$-cofiniteness.

We start the proof of Theorem 1. Namely, we will prove:

**Theorem 1** Let $U$ be a vertex operator algebra of CFT-type. Let $A, B, C$ be simple $\mathbb{N}$-graded $U$-modules and $I$ a surjective (formal power series) intertwining operator of type $(A \overset{C}{\leftarrow} B)$. If both of $A$ and $B$ are $C_h$-cofinite as $U$-modules for $h = 1, 2$, then so is $C$.

We note that if $U$ is of CFT-type and an $\mathbb{N}$-graded $U$-module $A = \bigoplus_{k=0}^{\infty} A_{r+k}$ is $C_1$-cofinite, then dim $A_{r+k} < \infty$ for any $k$ since $A_{r+k} \cap C_1(A) = \sum_{s=1}^{k-1} (U_s)_{-1} A_{r+k-s}$ has a finite codimension in $A_{r+k}$.

In the remainder part of this section, we assume the hypotheses of Theorem 1. Since $A$ and $B$ are $C_h$-cofinite, there are finite dimensional subspaces $F^1 \subseteq A$ and $F^2 \subseteq B$ such that $A = U_{-h}^+ A + F^1$ and $B = U_{-h}^+ B + F^2$. Let $c_A$ and $c_B$ be conformal weights of $A$ and $B$, respectively. We may assume that there is an integer $N$ such that $F^1 = \bigoplus_{k=0}^{N} A_{c_A + k}$ and $F^2 = \bigoplus_{k=0}^{N} B_{c_B + k}$. Fix bases $\{p^i \mid i \in I\}$ of $F^1$ and $\{q^j \mid j \in J\}$ of $F^2$. In order to prove Theorem 1, we prove the following lemma by applying an idea in [4] to $(C/U_{-h})^* C'$.

**Lemma 5** For $p \in A, q \in B$ and $\theta \in \text{Ann}(U_{(-h)}^+ C) \cap C'$,

$$F(\theta, p, q; z) := \langle \theta, I(p, z)q \rangle$$

is a linear combination of $\{F(\theta, p^i, q^j; z) \mid i \in I, j \in J\}$ with coefficients in $\mathbb{C}[z, z^{-1}]$ and we may choose these coefficients independently of the choice of $\theta$. 
[Proof] We will prove the assertion by the induction on the total weight $\text{wt}(p)+\text{wt}(q)$. If $\text{wt}(p) > N + c_B$, then $p = \sum_{k} u_k^h a^k$ for some $u_k^h \in U$ and $a^k \in A$. We note this expression does not depend on the choice of $\theta$. So we may assume $p = u_{-h} a$ with $u \in U$ and $a \in A$. Then for $\theta \in \text{Annh}(U_{(-h)}^+)C$, we have:

$$\langle \theta, \mathcal{I}(p, z) q \rangle = \langle \theta, \mathcal{I}(u_{-h} a, z) q \rangle$$
$$= \langle \theta, Y^-(L(-1)^{h-1} u, z) \mathcal{I}(a, z) q + \mathcal{I}(a, z) Y^+(L(-1)^{h-1} u, z) q \rangle$$
$$= \langle \theta, \mathcal{I}(a, z) Y^+(L(-1)^{h-1} u, z) q \rangle,$$

where $Y^-(u, z) = \sum_{m \leq 0} v_m z^{-m-1}$ and $Y^+(u, z) = \sum_{m \geq 0} v_m z^{-m-1}$. This is a reduction on the sum of weights because $Y^+(L(-1)^{h-1} u, z) q$ is a sum of finite terms and all weights of the coefficients are less than $\text{wt}(u) + \text{wt}(q)$.

Similarly, if $\text{wt}(q) > N + c_B$, then we may assume $q = u_{-h} b$ with $u \in U$ and $b \in B$ and

$$\langle \theta, \mathcal{I}(p, z) q \rangle = \langle \theta, \mathcal{I}(p, z) u_{-h} b \rangle$$
$$= \langle \theta, u_{-h} \mathcal{I}(p, z) b \rangle + \sum_{i=0}^{\infty} (\begin{array}{c}-hi \\\\\\\\\\end{array}) z^{-h-i} \mathcal{I}(u_i p, z) b$$
$$= \sum_{i=0}^{\infty} (\begin{array}{c}-hi \\\\\\\\\\end{array}) z^{-h-i} \langle \theta, \mathcal{I}(u_i p, z) b \rangle.$$  

Again, these process do not depend on the choice of $\theta$ and this is also a reduction on the weights because $\text{wt}(u_{-h} b) + \text{wt}(p)$ for $i \geq 0$. Therefore, $\langle \theta, \mathcal{I}(p, z) q \rangle$ is a linear combination of $\{ \langle \theta, \mathcal{I}(p^i, z) q^j \rangle | i \in I, j \in J \}$ with coefficients in $\mathbb{C}[z, z^{-1}]$. We note the coefficients do not depend on the choice of $\theta$.

Now we are able to prove Theorem 1. By the proof of the above lemma,

$$\frac{d}{dz} F(\theta, p^s, q^t; z) = F(\theta, L(-1)p^s, q^t; z)$$

is a linear combination of $\{ F(\theta, p^i, q^j; z) | i \in I, j \in J \}$ with coefficients in $\mathbb{C}[z, z^{-1}]$ for any $s \in I, t \in J$ and all coefficients do not depend on the choice of $\theta$. Therefore, there is a differential linear equation such that $F(\theta, p^i, q^j)$ are all its solutions for any $s \in I, t \in J$ and $\theta$. Furthermore, since $\{ \mathcal{I}(p, z) q | p \in A, q \in B, z \in \mathbb{Z} \}$ spans $C$ modulo $U_{(-h)}^+ C$ and $\langle \theta, Y(p, z) q \rangle$ are a linear sum of $\langle \theta, \mathcal{I}(p^i, z) q^j \rangle$, $\theta \in C' \cap \text{Annh}(U_{(-h)}^+ C) \rightarrow \prod_{i \in I, j \in J} \langle \theta, \mathcal{I}(p^i, z) q^j \rangle$ is injective. Therefore, we have $\dim C/U_{(-h)} C < \infty$. This completes the proof of Theorem 1.

References


