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On C_2-cofiniteness of $\mathbb{Z}_2$-permutation orbifold models of vertex operator algebras[1]

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1 Introduction

The notion of C_2-cofiniteness was introduced by Zhu in [Z] and the structure of vertex operator algebra (VOA shortly) satisfying this condition has been studied by many researchers (see [M], [Ar] for example). However, the verfication of the condition is very difficult in general, and it is a task when we treat a VOA which is expected to be C_2-cofinite.

There are some conjectures on the C_2-cofiniteness condition of VOAs, and as one of them, the C_2-cofiniteness of an orbifold model of a C_2-cofinite VOA has been believed to be true for many years. A permutation orbifold model is an orbifold model of a VOA given as a tensor product of d-copies of a VOA V by a natural action of a permutation group Ω in S_d, which is denoted by $V \wr \Omega$. We try to prove that if V is a simple C_2-cofinite VOA, then $V \wr \Omega$ is C_2-cofinite for any permutation group Ω. We show that this is true for $d=2$ in this report. A part of the results are in [Ab] in which we consider for the Virasoro VOAs in the case $d=2$.

2 VOA and related notions

A *vertex operator algebra* (VOA) V is a $\mathbb{N}$-graded vector space $V=\bigoplus_{n=0}^{\infty} V_n$ over $\mathbb{C}$ equipped with bilinear maps $V \times V \ni (a,b) \mapsto a_{(m)}b \in V$ called m-th product, and there are distinguished vectors $\mathbf{1} \in V_0$ and $\omega \in V_2$ called the vacuum vector and the Virasoro vector of V rexpectively. These products satisfy the following axioms:

(1) For any $a,b \in V$, $a_{(n)}b=0$ for sufficiently large integer n.

(2) (Borcherds identity) For any $a,b \in V$,

$$\begin{aligned}&\sum_{i=0}^{\infty}\binom{q}{i}(a_{(p+i)}b)_{(q+r-i)}c\\ &\quad=\sum_{i=0}^{\infty}(-1)^i\binom{p}{i}(a_{(p+q-i)}b_{(r+i)}c-(-1)^p b_{(p+r-i)}a_{(q+i)}c).\end{aligned} \tag{2.1}$$

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(3) $\mathbf{1}_{(n)} = \delta_{n,-1}\mathrm{id}_V$ for $n \in \mathbb{Z}$.

If we set $p = 0$, we have Commutativity formula: for any $q, r \in \mathbb{Z}$, $a, b, c \in V$,

$$\sum_{i=0}^{\infty} \binom{q}{i} (a_{(i)}b)_{(q+r-i)}c = a_{(q)}b_{(r)}c - b_{(r)}a_{(q)}c. \tag{2.2}$$

If we set $q = 0$, we have Associativity formula: For any $p, r \in \mathbb{Z}$, $a, b, c \in V$,

$$(a_{(p)}b)_{(r)}c = \sum_{i=0}^{\infty}(-1)^i \binom{p}{i} (a_{(p-i)}b_{(r+i)}c - (-1)^p b_{(p+r-i)}a_{(i)}c). \tag{2.3}$$

It is known that Associativity formula and Comutativity formula impliy the Borcherds identity.

The Virasoro vector ω satisfies the following axioms when we denote $\omega_{(m)}$ by L_{m-1} for $m \in \mathbb{Z}$:

$$[L_m, L_n] := (m-n)L_{m+n} + \frac{m^3 - m}{12} c_V \delta_{m+n,0}$$

for some $c_V \in \mathbb{C}$ called the *central charge* of V, and

$$L_{-1}a = a_{(-2)}\mathbf{1} \quad \text{for } a \in V, \quad L_0 a = ka \quad \text{for } a \in V_k, \quad \dim V_k < \infty.$$

An *automorphism* of a VOA V is a linear isomorphism g satisfying that $g(a_{(m)}b) = g(a)_{(m)}g(b)$ for $a, b \in V$, $g(\mathbf{1}) = \mathbf{1}$ and $g(\omega) = \omega$. For a finite automorphism group G, $V^G = \{a \in V \,|\, g(a) = a\}$ has naturally a VOA structure. This VOA is called an orbifold model of V.

Now we recall the notion of C_2-cofiniteness. Let $C_2(V)$ be a subspace of V defined by

$$C_2(V) = \langle a_{(-2)}b | a, b \in V\rangle_{\mathbb{C}}.$$

A VOA V is called *C_2-cofinite* if $R(V) := V/C_2(V)$ is finite dimensional. The following theorem is useful to verify the C_2-cofiniteness.

Theorem 2.1. *Let V be a VOA and U a its subVOA with same Virasoro vector. If U is C_2-cofinite then so is V.*

It is well known that -1-th product induces a commutative associative algebra structure on $R(V)$ and 0-th product indiuces a Lie algebra structure on it. By these two algebra sturctures, $R(V)$ becomes a Poisson algebra. We write $\bar{a} = a + C_2(V)$, $\bar{a} \cdot \bar{b} = \overline{a_{(-1)}b}$ and $[\bar{a}, \bar{b}] = \overline{a_{(0)}b}$ for $a, b \in V$.

Let S be a set of V. If $V = \langle a^1_{(-n_1)} \cdots a^r_{(-n_r)}\mathbf{1} | a^i \in S, n_i \in \mathbb{Z}_{>0}\rangle_{\mathbb{C}}$, then V is called to be **strongly generated** by S (see [Ar] for more properties). If V is strongly generated by a subset S, $R(V)$ is generated by $\{\bar{a} | a \in S\}$ as an algebra.

3 Permutation orbifold models

Let V be a VOA of central charge c_V and $V^{\otimes d}$ the tensor product of d-copies of the vector space V. Then $V^{\otimes d}$ cannonically has a VOA structure: For $a^1, \cdots, a^d, b^1, \cdots, b^d \in V$ and $m \in \mathbb{Z}$,

$$(a^1 \otimes \cdots \otimes a^d)_{(m)}(b^1 \otimes \cdots \otimes b^d) = \sum_{i_1, \cdots, i_d \in \mathbb{Z}, \sum i_j = m-d+1} a^1_{(i_1)}b^1 \otimes \cdots \otimes a^d_{(i_d)}b^d.$$

The vaccum vector and the Virasoro vecotor are given by $\mathbf{1}^{\otimes d}$ and

$$\sum_{i=1}^{d} \mathbf{1}^{\otimes(i-1)} \otimes \omega \otimes \mathbf{1}^{\otimes(d-i)},$$

where $\mathbf{1}^{\otimes k}$ denotes the tensor product of k copies of the vacuum $\mathbf{1}$. The central charge of $V^{\otimes d}$ is dc_V.

The symmetric group S_d of degree d acts on $V^{\otimes d}$ as permutations of tensor factors; for each permutation $\sigma \in S_d$, $\sigma(\otimes_{i=1}^d a^i) = \otimes_{i=1}^d a^{\sigma^{-1}(i)}$ for $a^i \in V$. For any subgroup $\Omega \subset S_d$, we define

$$V \wr \Omega := (V^{\otimes d})^{\Omega} = \{u \in V^{\otimes d} | \sigma(u) = u, \text{ for } \sigma \in S_d\}.$$

Then $V \wr \Omega$ is a subVOA of $V^{\otimes d}$ with same Virasoro vector.

Here we introduce a linear map $\eta : V \to V \wr S_d$ defined by

$$\eta(a) = \sum_{i=1}^{d} \mathbf{1}^{\otimes i-1} \otimes a \otimes \mathbf{1}^{\otimes d-i}$$

for $a \in V$. We see that $\eta(\omega)$ is the Virasoro vector of $V \wr S_d$. Since $V \wr \Omega$ has $\eta(\omega)$ as its Virasoro vector and $V \wr S_d \subset V \wr \Omega$, Theorem 2.1 shows that $V \wr \Omega$ is C_2-cofinite if $V \wr S_d$ is C_2-cofinite. Therefore we only consider the permutation orbifold model $V \wr S_d$.

We also have

$$\eta(a)_{(i)}\eta(b) = \eta(a_{(i)}b) \quad \text{for } i \in \mathbb{Z}_{\geq 0},$$
$$\eta(a)_{(-1)}\eta(b) = \eta(a_{(-1)}b) + \phi_2(a, b) \quad \text{for } i \in \mathbb{Z}_{\geq 0},$$

where we define $\phi_k : V^k \to V \wr S_d$ by

$$\phi_k(a^1, \cdots, a^k) = \frac{1}{(d-k)!} \sum_{\sigma \in S_d} \sigma(a^1 \otimes \cdots \otimes a^k \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1})$$

for $a^i \in V$. For $k \geq 1$, $\phi_k(a^1, \cdots, a^k)$ can be expressed as a sum of -1-th products of vectors in $\operatorname{Im} \eta$ and $\operatorname{Im} \phi_{k-1}$. This implies the following proposition.

Proposition 3.1. *$V \wr S_d$ is strongly generated by* $\mathrm{Im}\,\eta$. *Hence $R(V \wr S_d)$ is generated by $\{\eta(a) + C_2(V \wr S_d) | a \in V\}$.*

Now we denote the images of $\eta(a)$ and $\phi_k(a^1, \cdots, a^k)$ in $R(V \wr S_d)$ by

$$\overline{\eta}(a) = \eta(a) + C_2(V \wr S_d),$$
$$\overline{\phi}_k(a^1, \cdots, a^k) = \phi_k(a^1, \cdots, a^k) + C_2(V \wr S_d)$$

for $a, a^i \in V$ and $k \geq 2$. Then we can show that the following theorem.

Theorem 3.2. *$R(V \wr S_d)$ is finite dimensional if and only if* $\mathrm{Im}\,\overline{\eta}$ *is finite dimensional.*

Theorem 3.2 in the case $d = 2$ can be refer in [Ab]. Consequently it suffices to show the C_2-cofiniteness of $V \wr S_d$ that $\mathrm{Ker}\;\overline{\eta}$ is finite codimensional. It is easy to see that $L_{-1}V \subset \mathrm{Ker}\;\overline{\eta}$ and

$$\overline{\phi}_2(a_{(-n)}u, v) = -\overline{\phi}_2(u, a_{(-n)}v) - \overline{\phi}_3(a_{(-n)}\mathbf{1}, u, v) \tag{3.1}$$

for any $a, u, v \in V$ and $n \geq 2$. This identitiy plays an essential role in the $d = 2$ case because the scond term in the right hand side need not to be considered.

4 C_2-cofiniteness of $V \wr S_2$

We consider the case $d = 2$. In this case we have

$$\overline{\eta}(a)\overline{\eta}(b) = \overline{\eta}(a_{(-1)}b)$$

if a or b are in $\mathrm{Ker}\;\overline{\eta}$. By using this fact and a slightly long argument, we have

Theorem 4.1. *Let V be a C_2-cofinite VOA with $V_0 = \mathbb{C}\mathbf{1}$. Suppose that V is strongly generated by a (finite) set S. Then* $\mathrm{Im}\,\overline{\eta}$ *is finite dimensional if and only if the subspace $\langle \overline{\eta}(x_{(-n)}y) | x, y \in S, n \geq 0\rangle_{\mathbb{C}}$ is finite dimensoional.*

We now set

$$D(x, y) := \langle \overline{\eta}(x_{(-n)}y) | n \geq 0\rangle_{\mathbb{C}}$$

for any $x, y \in V$. By Theorems 3.2 and 4.1, we have the following thorem.

Theorem 4.2. *Let V be a C_2-cofinite VOA with $V_0 = \mathbb{C}\mathbf{1}$. Suppose that V is strongly generated by S. Then $V \wr S_2$ is C_2-cofinite if and only if the subspace $D(x, y)$ is finite dimensoional for each $x, y \in S$.*

In fact we can show the following lemma (in the case $d = 2$).

Lemma 4.3. *If V is a (not necessarily C_2-cofinite) simple VOA with $V_0 = \mathbb{C}\mathbf{1}$, then* $\dim D(x, y) < \infty$ *for any $x, y \in V$.*

Therefore we have the desired result.

Theorem 4.4. *Let V be a C_2-cofinite, simple VOA with $V_0 = \mathbb{C}\mathbf{1}$. Then $V \wr S_2$ is C_2-cofinite.*

5 Proof of Lemma 4.3

The proof of Lemm 4.3 is given by complicated calculations. So we can not write the detail of them in this report. Hence we explain how to show Lemma 4.3 by dividing 4-steps .

Firstly we have the following lemma with respect to the Virasoro vector ω:

Lemma 5.1. ([Ab]) $\dim D(\omega,\omega) \leq 14$.

To get the lemma, we show $\overline{\eta}(L_{(-n)}\omega) = 0$ if $n \geq 30$ because $\overline{\eta}(L_{-n}\omega) = 0$ if n is a positive odd integer. To prove this we calculate the difference of vectors

$$\overline{\eta}((L_{-m}L_{-n}\mathbf{1})_{(-1)}L_{-p}L_{-q}\mathbf{1}) \tag{5.1}$$

and

$$\overline{\eta}((L_{-m}L_{-p}\mathbf{1})_{(-1)}L_{-n}L_{-q}\mathbf{1}). \tag{5.2}$$

By Associativity formula, (5.2) is equal to a sum of $\overline{\eta}(L_{-m}L_{-p}L_{-n}L_{-q}\mathbf{1})$ and lower lengh terms, where we say a vector of the form $\overline{\eta}(L_{-m_1}\cdots L_{-m_k}\mathbf{1})$ to be a lenght k. But we see that

$$\overline{\eta}(L_{-m}L_{-p}L_{-n}L_{-q}\mathbf{1}) = \overline{\eta}(L_{-m}L_{-n}L_{-p}L_{-q}\mathbf{1}) + (p-n)\overline{\eta}(L_{-m}L_{-p-n}L_{-q}\mathbf{1}).$$

Thus the difference of (5.1) and (5.2) is a sum of terms of length 2 and length 3.

On the other hand, the vectors (5.1)–(5.2) are related to teo products $\overline{\eta}(L_{-m}L_{-n}\mathbf{1})\cdot\overline{\eta}(L_{-p}L_{-q}\mathbf{1})$ and $\overline{\eta}(L_{-m}L_{-p}\mathbf{1})\cdot\overline{\eta}(L_{-n}L_{-q}\mathbf{1})$ respectively. We here note that $\overline{\eta}(L_{-k}L_{-l}\mathbf{1}) = 0$ if $k,l \geq 3$ and $k+l$ is odd. Hence if $m+p$ and $m+n$ is odd, then we have $\overline{\eta}(L_{-m}L_{-n}\mathbf{1})\cdot\overline{\eta}(L_{-p}L_{-q}\mathbf{1}) = \overline{\eta}(L_{-m}L_{-p}\mathbf{1})\cdot\overline{\eta}(L_{-n}L_{-q}\mathbf{1}) = 0$. This fact, the difference of (5.1) and (5.2) and Identity (3.1) give us identities among terms of length 2 and length 3. Actually we can get enough identities to show $\overline{\eta}(L_{-s}\omega) = 0$ for $s = m+n+p+q \geq 30$.

Secondly we show Lemma 4.3 when $x, y \in V_1$.

Lemma 5.2. *Suppose that V is simple. For $x, y \in V_1$,* $\dim D(x,y) < \infty$.

The argument is very similar as Lemma 5.1 but the caluculations are more easier. Thirdly we show the following lemma.

Lemma 5.3. *For $x \in V$,* $\dim D(\omega, x) < \infty$.

Tow show this lemma, we use induction on weight of x. The case $x \in V_1$, we use the same argument of Lemma 5.2. For the case of higer weight, we use the similar calculation of the proof Lemma 5.1. In both calculations we use the result in Lemmas 5.2 and 5.3.

Finally we can show Lemma 4.3 by using Lie algebra structure of $R(V \wr S_2)$. By Lemma 5.3, for any $y \in V$, there exists N such that $L_{-n}y \in \mathrm{Ker}\,\overline{\eta}$ for $n \geq N$. Therefore for any $x \in V$, we have

$$0 = [\overline{\eta}(x), \overline{\eta}(L_{-n}y)] = \overline{\eta(x)_{(0)}\eta(L_{-n}y)} = \overline{\eta}(x_{(0)}L_{-n}y).$$

Thus we see that $x_{(0)}L_{-n}y \in \mathrm{Ker}\,\overline{\eta}$. Here we see that

$$\begin{aligned} & x_{(0)}L_{-n}y \\ &= L_{-n}x_{(0)}y + (n-1)(|x|-1)x_{(-n)}y - \sum_{i=2}^{\infty} \binom{-n+1}{i} (L_{i-1}x)_{(-n-i+1)}y. \end{aligned}$$

Therefore by using induction on x and Lemmas 5.2–5.3, we have $x_{(-n)}y \in \mathrm{Ker}\,\overline{\eta}$ for sufficiently large n.

6 Conclusions and Considerations for general d

In this report we have shown that $V \wr S_2$ is C_2-cofinite if V is simple and C_2-cofinite. To show this we use Lemma 4.3, i.e., the fact that $D(x, y)$ is finite dimensional for any $x, y \in V$.

Our next aim is to prove the C_2-cofiniteness of $V \wr S_d$ for a simple C_2-cofinite VOA V and $d \geq 3$. In this case Lemma 4.3 is a weaker one for the C_2-cofiniteness of $V \wr S_d$ as explain below. We consider a subspace $C_N(V) := \langle a_{(-N)}b | a, b \in V \rangle_{\mathbb{C}}$ of V. A VOA V is called *C_N-cofinite* if $\dim V/C_N(V) < \infty$. It is well known that V is C_2-cofinite then V is C_N-cofinite for any $N \geq 2$.

Now we consider the case d is general. We see that under the assumption that V is C_N-cofinite, $C_N(V) \subset \mathrm{Ker}\,\overline{\eta}$ implies $\dim \mathrm{Im}\,\overline{\eta} < \infty$. Conversely if $\mathrm{Im}\,\overline{\eta}$ is finite dimensional and V is C_2-confinite then $C_N(V) \subset \mathrm{Ker}\,\overline{\eta}$ fo some $N \geq 2$ because both $C_N(V)$ and $\mathrm{Ker}\,\overline{\eta}$ are graded subspaces of V. Hnece by Theorem 4.1, $V \wr S_2$ is C_2-cofinite if and only if $C_N(V) \subset \mathrm{Ker}\,\overline{\eta}$ for some $N \geq 2$. We here note that $\dim D(x, y) \leq N$ for any $x, y \in V$ if $C_N(V) \subset \mathrm{Ker}\,\overline{\eta}$. Therefore Lemma 4.3 is a weaker condition than the C_2-cofiniteness of $V \wr S_d$, and they are equivalent in the case $d = 2$. To prove Lemma 4.3 in general case and Theorem 4.1 seems to be very hard problem. We expect that $C_N(V) \subset \mathrm{Ker}\,\overline{\eta}$ for some N is true in general and can be shown by another way.

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