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On $C_2$-cofiniteness of $\mathbb{Z}_2$-permutation orbifold models of vertex operator algebras¹
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1 Introduction

The notion of $C_2$-cofiniteness was introduced by Zhu in [Z] and the structure of vertex operator algebra (VOA shortly) satisfying this condition has been studied by many researchers (see [M], [Ar] for example). However, the verification of the condition is very difficult in general, and it is a task when we treat a VOA which is expected to be $C_2$-cofinite.

There are some conjectures on the $C_2$-cofiniteness condition of VOAs, and as one of them, the $C_2$-cofiniteness of an orbifold model of a $C_2$-cofinite VOA has been believed to be true for many years. A permutation orbifold model is an orbifold model of a VOA given as a tensor product of $d$-copies of a VOA $V$ by a natural action of a permutation group $\Omega$ in $S_d$, which is denoted by $V \bowtie \Omega$. We try to prove that if $V$ is a simple $C_2$-cofinite VOA, then $V \bowtie \Omega$ is $C_2$-cofinite for any permutation group $\Omega$. We show that this is true for $d = 2$ in this report. A part of the results are in [Ab] in which we consider for the Virasoro VOAs in the case $d = 2$.

2 VOA and related notions

A vertex operator algebra (VOA) $V$ is a $\mathbb{N}$-graded vector space $V = \bigoplus_{n=0}^{\infty} V_n$ over $\mathbb{C}$ equipped with bilinear maps $V \times V \ni (a, b) \mapsto a_{(m)} b \in V$ called $m$-th product, and there are distinguished vectors $1 \in V_0$ and $\omega \in V_2$ called the vacuum vector and the Virasoro vector of $V$ respectively. These products satisfy the following axioms:

(1) For any $a, b \in V$, $a_{(n)} b = 0$ for sufficiently large integer $n$.

(2) (Borcherds identity) For any $a, b \in V$,

$$
\sum_{i=0}^{\infty} \binom{q}{i} (a_{(p+i)} b)_{(q+r-i)} c
= \sum_{i=0}^{\infty} (-1)^i \binom{p}{i} (a_{(p+q-i)} b_{(r+i)} c - (-1)^p b_{(p+r-i)} a_{(q+i)} c).
$$

¹16 Dec. 2010, "Vertex Operator Algebras, Finite Groups and Combinatorics" at Faculty of Science, Kyoto university
$(3) \ 1_{(n)} = \delta_{n,-1} \text{id}_V$ for $n \in \mathbb{Z}$.

If we set $p = 0$, we have Commutativity formula: for any $q, r \in \mathbb{Z}$, $a, b, c \in V$,

$$\sum_{i=0}^{\infty} \binom{q}{i} (a_{(i)}b)_{(q+r-i)}c = a_{(q)}b_{(r)}c - b_{(r)}a_{(q)}c.$$ (2.2)

If we set $q = 0$, we have Associativity formula: For any $p, r \in \mathbb{Z}$, $a, b, c \in V$,

$$(a_{(p)}b)_{(r)}c = \sum_{i=0}^{\infty} (-1)^{i} \binom{p}{i} (a_{(p-i)}b_{(r+i)}c - (-1)^{p}b_{(r-p+i)}a_{(i)}c).$$ (2.3)

It is known that Associativity formula and Comutativity formula imply the Borcherds identity.

The Virasoro vector $\omega$ satisfies the following axioms when we denote $\omega_{(m)}$ by $L_{m-1}$ for $m \in \mathbb{Z}$:

$$[L_{m}, L_{n}] := (m - n)L_{m+n} + \frac{m^3 - m}{12} c_v \delta_{m+n,0}$$

for some $c_v \in \mathbb{C}$ called the central charge of $V$, and

$$L_{-1}a = a_{(-2)}1 \quad \text{for} \ a \in V, \quad L_{0}a = ka \quad \text{for} \ a \in V_k, \quad \dim V_k < \infty.$$

An automorphism of a VOA $V$ is a linear isomorphism $g$ satisfying that $g(a_{(m)}b) = g(a)_{(m)}g(b)$ for $a, b \in V$, $g(1) = 1$ and $g(\omega) = \omega$. For a finite automorphism group $G$, $V^G = \{a \in V \mid g(a) = a\}$ has naturally a VOA structure. This VOA is called an orbifold model of $V$.

Now we recall the notion of $C_2$-cofiniteness. Let $C_2(V)$ be a subspace of $V$ defined by

$$C_2(V) = \langle a_{(-2)}b \mid a, b \in V \rangle_c.$$

A VOA $V$ is called $C_2$-cofinite if $R(V) := V/C_2(V)$ is finite dimensional. The following theorem is useful to verify the $C_2$-cofiniteness.

**Theorem 2.1.** Let $V$ be a VOA and $U$ a its subVOA with same Virasoro vector. If $U$ is $C_2$-cofinite then so is $V$.

It is well known that $-1$-th product induces a commutative associative algebra structure on $R(V)$ and 0-th product induces a Lie algebra structure on it. By these two algebra structures, $R(V)$ becomes a Poisson algebra. We write $\overline{a} = a + C_2(V)$, $\overline{a} \cdot \overline{b} = \overline{a_{(-1)}b}$ and $[\overline{a}, \overline{b}] = \overline{a_{(0)}b}$ for $a, b \in V$.

Let $S$ be a set of $V$. If $V = \langle a_{(-n_1)} \cdots a_{(-n_r)}1 \mid a_i \in S, n_i \in \mathbb{Z}_{>0} \rangle_c$, then $V$ is called to be strongly generated by $S$ (see [Ar] for more properties). If $V$ is strongly generated by a subset $S$, $R(V)$ is generated by $\{\overline{a} \mid a \in S\}$ as an algebra.
3 Permutation orbifold models

Let $V$ be a VOA of central charge $c_V$ and $V^\otimes d$ the tensor product of $d$-copies of the vector space $V$. Then $V^\otimes d$ canonically has a VOA structure: For $a^1, \ldots, a^d, b^1, \ldots, b^d \in V$ and $m \in \mathbb{Z}$,

$$(a^1 \otimes \cdots \otimes a^d)(m)(b^1 \otimes \cdots \otimes b^d) = \sum_{i_1, \ldots, i_d \in \mathbb{Z}, \sum i_j = m-d+1} a^1_{(i_1)} b^1 \otimes \cdots \otimes a^d_{(i_d)} b^d.$$

The vacuum vector and the Virasoro vector are given by $1^\otimes d$ and

$$\sum_{i=1}^{d} 1^{i-1} \otimes \omega \otimes 1^{d-i},$$

where $1^\otimes k$ denotes the tensor product of $k$ copies of the vacuum $1$. The central charge of $V^\otimes d$ is $dc_V$.

The symmetric group $S_d$ of degree $d$ acts on $V^\otimes d$ as permutations of tensor factors; for each permutation $\sigma \in S_d$, $\sigma(\otimes_{i=1}^{d} a^i) = \otimes_{i=1}^{d} a^{\sigma^{-1}(i)}$ for $a^i \in V$. For any subgroup $\Omega \subset S_d$, we define

$$V \lhd \Omega := (V^\otimes d)^{\Omega} = \{ u \in V^\otimes d | \sigma(u) = u \mbox{ for } \sigma \in S_d \}.$$

Then $V \lhd \Omega$ is a subVOA of $V^\otimes d$ with same Virasoro vector.

Here we introduce a linear map $\eta : V \rightarrow V \lhd S_d$ defined by

$$\eta(a) = \sum_{i=1}^{d} 1^{(i-1)} \otimes a \otimes 1^{(d-i)},$$

for $a \in V$. We see that $\eta(\omega)$ is the Virasoro vector of $V \lhd S_d$. Since $V \lhd \Omega$ has $\eta(\omega)$ as its Virasoro vector and $V \lhd S_d \subset V \lhd \Omega$, Theorem 2.1 shows that $V \lhd \Omega$ is $C_2$-cofinite if $V \lhd S_d$ is $C_2$-cofinite. Therefore we only consider the permutation orbifold model $V \lhd S_d$.

We also have

$$\eta(a)^{(i)} \eta(b) = \eta(a_{(i)} b) \mbox{ for } i \in \mathbb{Z}_{\geq 0},$$

$$\eta(a)^{(-1)} \eta(b) = \eta(a_{(-1)} b) + \phi_2(a, b) \mbox{ for } i \in \mathbb{Z}_{\geq 0},$$

where we define $\phi_k : V^k \rightarrow V \lhd S_d$ by

$$\phi_k(a^1, \ldots, a^k) = \frac{1}{(d-k)!} \sum_{\sigma \in S_d} \sigma(a^1 \otimes \cdots \otimes a^k \otimes 1 \otimes \cdots \otimes 1)$$

for $a^i \in V$. For $k \geq 1$, $\phi_k(a^1, \ldots, a^k)$ can be expressed as a sum of $-1$-th products of vectors in $\text{Im} \eta$ and $\text{Im} \phi_{k-1}$. This implies the following proposition.
Proposition 3.1. $V \triangleleft S_d$ is strongly generated by $\text{Im} \eta$. Hence $R(V \triangleleft S_d)$ is generated by $\{\eta(a) + C_2(V \triangleleft S_d)|a \in V\}$.

Now we denote the images of $\eta(a)$ and $\phi_k(a^1, \cdots, a^k)$ in $R(V \triangleleft S_d)$ by
\[
\overline{\eta}(a) = \eta(a) + C_2(V \triangleleft S_d),
\overline{\phi}_k(a^1, \cdots, a^k) = \phi_k(a^1, \cdots, a^k) + C_2(V \triangleleft S_d)
\]
for $a, a^i \in V$ and $k \geq 2$. Then we can show that the following theorem.

Theorem 3.2. $R(V \triangleleft S_d)$ is finite dimensional if and only if $\text{Im} \overline{\eta}$ is finite dimensional.

Theorem 3.2 in the case $d = 2$ can be refer in [Ab]. Consequently it suffices to show the $C_2$-cofiniteness of $V \triangleleft S_d$ that $\text{Ker} \overline{\eta}$ is finite codimensional. It is easy to see that $L_{-1}V \subset \text{Ker} \overline{\eta}$ and
\[
\overline{\phi}_2(a_{(-n)}u, v) = -\overline{\phi}_2(u, a_{(-n)}v) - \overline{\phi}_3(a_{(-n)}1, u, v) \quad (3.1)
\]
for any $a, u, v \in V$ and $n \geq 2$. This identity plays an essential role in the $d = 2$ case because the second term in the right hand side need not to be considered.

4 $C_2$-cofiniteness of $V \triangleleft S_2$

We consider the case $d = 2$. In this case we have
\[
\overline{\eta}(a)\overline{\eta}(b) = \overline{\eta}(a_{(-1)}b)
\]
if $a$ or $b$ are in $\text{Ker} \overline{\eta}$. By using this fact and a slightly long argument, we have

Theorem 4.1. Let $V$ be a $C_2$-cofinite VOA with $V_0 = \mathbb{C}1$. Suppose that $V$ is strongly generated by a (finite) set $S$. Then $\text{Im} \overline{\eta}$ is finite dimensional if and only if the subspace $\langle \overline{\eta}(x_{(-n)}y)|x, y \in S, n \geq 0\rangle_C$ is finite dimensional.

We now set
\[
D(x, y) := \langle \overline{\eta}(x_{(-n)}y)|n \geq 0\rangle_C
\]
for any $x, y \in V$. By Theorems 3.2 and 4.1, we have the following theorem.

Theorem 4.2. Let $V$ be a $C_2$-cofinite VOA with $V_0 = \mathbb{C}1$. Suppose that $V$ is strongly generated by $S$. Then $V \triangleleft S_2$ is $C_2$-cofinite if and only if the subspace $D(x, y)$ is finite dimensional for each $x, y \in S$.

In fact we can show the following lemma (in the case $d = 2$).

Lemma 4.3. If $V$ is a (not necessarily $C_2$-cofinite) simple VOA with $V_0 = \mathbb{C}1$, then $\dim D(x, y) < \infty$ for any $x, y \in V$.

Therefore we have the desired result.

Theorem 4.4. Let $V$ be a $C_2$-cofinite, simple VOA with $V_0 = \mathbb{C}1$. Then $V \triangleleft S_2$ is $C_2$-cofinite.
5 Proof of Lemma 4.3

The proof of Lemm 4.3 is given by complicated calculations. So we can not write the detail of them in this report. Hence we explain how to show Lemma 4.3 by dividing 4-steps.

Firstly we have the following lemma with respect to the Virasoro vector $\omega$:

**Lemma 5.1.** ([Ab]) $\dim D(\omega, \omega) \leq 14$.

To get the lemma, we show $\overline{\eta}(L_{(-n)}\omega) = 0$ if $n \geq 30$ because $\overline{\eta}(L_{-n}\omega) = 0$ if $n$ is a positive odd integer. To prove this we calculate the difference of vectors

$$\overline{\eta}((L_{-m}L_{-n}1)_{(-1)}L_{-p}L_{-q}1)$$

and

$$\overline{\eta}((L_{-m}L_{-p}1)_{(-1)}L_{-n}L_{-q}1).$$

By Associativity formula, (5.2) is equal to a sum of $\overline{\eta}(L_{-m}L_{-n}L_{-n}L_{-q}1)$ and lower length terms, where we say a vector of the form $\overline{\eta}(L_{-m_1} \cdots L_{-m_k}1)$ to be a length $k$. But we see that

$$\overline{\eta}(L_{-m}L_{-p}L_{-n}L_{-q}1) = \overline{\eta}(L_{-m}L_{-n}L_{-p}L_{-q}1) + (p-n)\overline{\eta}(L_{-m}L_{-p-n}L_{-q}1).$$

Thus the difference of (5.1) and (5.2) is a sum of terms of length 2 and length 3.

On the other hand, the vectors (5.1)-(5.2) are related to two products $\overline{\eta}(L_{-m}L_{-n}1) \cdot \overline{\eta}(L_{-p}L_{-q}1)$ and $\overline{\eta}(L_{-m}L_{-p}1) \cdot \overline{\eta}(L_{-n}L_{-q}1)$ respectively. We here note that $\overline{\eta}(L_{-k}L_{-l}1) = 0$ if $k, l \geq 3$ and $k+l$ is odd. Hence if $m+p$ and $m+n$ is odd, then we have $\overline{\eta}(L_{-m}L_{-n}1) \cdot \overline{\eta}(L_{-p}L_{-q}1) = \overline{\eta}(L_{-m}L_{-p}1) \cdot \overline{\eta}(L_{-n}L_{-q}1) = 0$. This fact, the difference of (5.1) and (5.2) and Identity (3.1) give us identities among terms of length 2 and length 3. Actually we can get enough identities to show $\overline{\eta}(L_{-s}\omega) = 0$ for $s = m + n + p + q \geq 30$.

Secondly we show Lemma 4.3 when $x, y \in V_1$.

**Lemma 5.2.** Suppose that $V$ is simple. For $x, y \in V_1$, $\dim D(x, y) < \infty$.

The argument is very similar as Lemma 5.1 but the calculations are more easier. Thirdly we show the following lemma.

**Lemma 5.3.** For $x \in V$, $\dim D(\omega, x) < \infty$.

To show this lemma, we use induction on weight of $x$. The case $x \in V_1$, we use the same argument of Lemma 5.2. For the case of higher weight, we use the similar calculation of the proof Lemma 5.1. In both calculations we use the result in Lemmas 5.2 and 5.3.
Finally we can show Lemma 4.3 by using Lie algebra structure of $R(V \triangleright S_2)$. By Lemma 5.3, for any $y \in V$, there exists $N$ such that $L_{-n}y \in \text{Ker } \overline{\eta}$ for $n \geq N$. Therefore for any $x \in V$, we have
\[ 0 = [\overline{\eta}(x), \overline{\eta}(L_{-n}y)] = \overline{\eta}(x)\langle a_{(0)}b|a, b \in V \rangle_{\mathbb{C}} - \overline{\eta}(x_{(0)}L_{-n}y) = \overline{\eta}(x_{(0)}L_{-n}y). \]
Thus we see that $x_{(0)}L_{-n}y \in \text{Ker } \overline{\eta}$. Here we see that
\[
x_{(0)}L_{-n}y = L_{-n}x_{(0)}y + (n - 1)(|x| - 1)x_{(-n)}y - \sum_{i=2}^{\infty} \binom{-n + 1}{i}(L_{i-1}x)_{(-n-i+1)}y.
\]
Therefore by using induction on $x$ and Lemmas 5.2–5.3, we have $x_{(-n)}y \in \text{Ker } \overline{\eta}$ for sufficiently large $n$.

6 Conclusions and Considerations for general $d$

In this report we have shown that $V \triangleright S_2$ is $C_2$-cofinite if $V$ is simple and $C_2$-cofinite. To show this we use Lemma 4.3, i.e., the fact that $D(x, y)$ is finite dimensional for any $x, y \in V$.

Our next aim is to prove the $C_2$-cofiniteness of $V \triangleright S_d$ for a simple $C_2$-cofinite VOA $V$ and $d \geq 3$. In this case Lemma 4.3 is a weaker one for the $C_2$-cofiniteness of $V \triangleright S_d$ as explain below. We consider a subspace $C_N(V) := \langle a_{(-N)}b|a, b \in V \rangle_C$ of $V$. A VOA $V$ is called $C_N$-cofinite if $\dim V/C_N(V) < \infty$. It is well known that $V$ is $C_2$-cofinite then $V$ is $C_N$-cofinite for any $N \geq 2$.

Now we consider the case $d$ is general. We see that under the assumption that $V$ is $C_N$-cofinite, $C_N(V) \subset \text{Ker } \overline{\eta}$ implies $\dim \text{Im } \overline{\eta} < \infty$. Conversely if $\text{Im } \overline{\eta}$ is finite dimensional and $V$ is $C_2$-cofinite then $C_N(V) \subset \text{Ker } \overline{\eta}$ for some $N \geq 2$ because both $C_N(V)$ and $\text{Ker } \overline{\eta}$ are graded subspaces of $V$. Hence by Theorem 4.1, $V \triangleright S_2$ is $C_2$-cofinite if and only if $C_N(V) \subset \text{Ker } \overline{\eta}$ for some $N \geq 2$. We here note that $\dim D(x, y) \leq N$ for any $x, y \in V$ if $C_N(V) \subset \text{Ker } \overline{\eta}$. Therefore Lemma 4.3 is a weaker condition than the $C_2$-cofiniteness of $V \triangleright S_d$, and they are equivalent in the case $d = 2$. To prove Lemma 4.3 in general case and Theorem 4.1 seems to be very hard problem. We expect that $C_N(V) \subset \text{Ker } \overline{\eta}$ for some $N$ is true in general and can be shown by another way.

References


