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The Bosonic Vertex Operator Algebra on a Genus $g$ Riemann Surface

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Abstract

We discuss the partition function for the Heisenberg vertex operator algebra on a genus $g$ Riemann surface formed by sewing $g$ handles to a Riemann sphere. In particular, it is shown how the partition can be computed by means of the MacMahon Master Theorem from classical combinatorics.

1 Introduction

In this paper we briefly sketch recent progress in defining and computing the partition function for the Heisenberg Vertex Operator Algebra (VOA) on a genus $g$ Riemann surface. The partition function and $n$-point correlation functions are familiar concepts at genus one and have recently been computed on genus two Riemann surfaces formed from sewing tori together [MT1],[MT2]. Here we discuss an alternative approach for computing these objects on a general genus $g$ Riemann surface formed by sewing $g$ handles onto a Riemann sphere. This approach includes the classical Schottky parameterisation and a related simpler canonical parameterisation for which we obtain the partition function for rank 2 Heisenberg VOA in terms of an explicit infinite determinant. This determinant is computed by means of the MacMahon Master Theorem in classical combinatorics [MM].

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2 A Generalized MacMahon Master Theorem

We begin with a review of the MacMahon Master Theorem and a recent generalization. We will provide a proof of this which gives some flavour of the combinatorial graph theory methods developed to compute higher genus partition functions [MT2], [TZ].

Let $A = (A_{ij})$ be an $n \times n$ matrix indexed by $i, j \in \{1, \ldots, n\}$. Consider the cycle decomposition of $\pi \in \Sigma_n$, the symmetric group on $\{1, \ldots, n\}$,

$$\pi = \sigma_1 \ldots \sigma_{C(\pi)}.$$  \hfill (1)

The $\beta$-extended Permanent of the matrix $A$ is defined by [FZ]

$$\text{perm}_\beta A \equiv \sum_{\pi \in \Sigma_n} \beta^{C(\pi)} \prod_i A_{i \pi(i)}.$$  \hfill (2)

The standard permanent and determinant are the particular cases:

$$\text{perm} A = \text{perm}_{+1} A, \quad \det A = (-1)^n \text{perm}_{-1} A.$$  \hfill (3)

Consider a multiset $\{k_1, \ldots, k_m\}$ with $1 \leq k_1 \leq \ldots \leq k_m \leq n$ i.e. index repetition is allowed. We notate the multiset as the unrestricted partition

$$k = \{1^{r_1}2^{r_2}\ldots n^{r_n}\},$$  \hfill (4)

i.e. the index $i$ occurs $r_i \geq 0$ times and where $m = \sum_i^{n} r_i$. Let $A(k)$ denote the $m \times m$ matrix indexed by $k$ for a given matrix $A$ indexed by $\{1, \ldots, n\}$. We now describe a generalisation of the classic MacMahon Master Theorem (MMT) of combinatorics [MM]. Let $A$ be an $n \times n$ matrix indexed by $\{1, \ldots, n\}$. Let $A(k)$ denote the $m \times m$ matrix indexed by a multiset $k$ (4).

Theorem 2.1 (Generalized MMT - Foata and Zeilberger [FZ])

$$\sum_k \frac{\text{perm}_\beta A(k)}{r_1!r_2!\ldots r_n!} = \frac{1}{\det(I - A)^\beta},$$  \hfill (5)

where the (infinite) sum ranges over all multisets $k = \{1^{r_1}2^{r_2}\ldots n^{r_n}\}$. 

For $\beta = 1$, Theorem 2.1 reduces to the classical MMT [MM]. For $\beta = -1$ we use (3) to find that the sum is restricted to proper subsets of \{1, 2, \ldots, n\} resulting in the determinant identity

$$\det(I + B) = \sum_{1 \leq k_1 < \ldots < k_m \leq n} \det B(k),$$

for $B = -A$.

**Proof of Theorem 2.1.** We use a graph theory method applied in [MT2]. Define a set of oriented graphs $\Gamma$ with elements $\gamma_\pi$ whose vertices are labelled by multisets $k = \{1^{r_1} \ldots n^{r_n}\}$ and directed edges $\{e_{ij}\}$ determined by permutations $\pi \in \Sigma(k)$ as follows

$$e_{ij} = \bullet k_i \rightarrow \bullet k_j \text{ for } k_j = \pi(k_i)$$

Define a $\beta$ dependent weight for each $\gamma_\pi$

$$w_\beta(e_{ij}) = A_{k_i k_j}, \quad w_\beta(\gamma_\pi) = \beta^{C(\pi)} \prod_{e_{ij} \in \gamma_\pi} w_\beta(e_{ij}), \quad \text{(6)}$$

where $C(\pi)$ is the number of disjoint cycles in $\pi$. Then we may write

$$\perm_\beta A(k) = \sum_{\pi \in \Sigma(k)} w_\beta(\gamma_\pi).$$

$\gamma_\pi$ is invariant under permutations of the identical labels of $k$. Hence the left hand side of (5) can be rewritten as

$$\sum_k \frac{\perm_\beta A(k)}{r_1! r_2! \ldots r_n!} = \sum_{\gamma \in \Gamma} \frac{w_\beta(\gamma)}{|\Aut(\gamma)|},$$

where we sum over all inequivalent graphs in $\Gamma$. Each $\gamma \in \Gamma$ can be decomposed into disjoint connected cycle graphs $\gamma_\sigma \in \Gamma$

$$\gamma = \gamma_{\sigma_1}^{m_1} \ldots \gamma_{\sigma_K}^{m_K}.$$

Each cycle $\sigma$ corresponds to a disjoint connected cycle graph $\gamma_\sigma \in \Gamma$ with weight

$$w_\beta(\gamma_\pi) = \prod_i w_\beta(\gamma_{\sigma_i})^{m_i}.$$
Furthermore

$$|\text{Aut}(\gamma_\pi)| = \prod_i |\text{Aut}(\gamma_{\sigma_i})|^{m_i} m_i!$$

Let $\Gamma_\sigma$ denote the set of inequivalent cycles. Then

$$\sum_{g \in \Gamma} \frac{w_\beta(g)}{|\text{Aut}(g)|} = \prod_{\gamma_\sigma \in \Gamma_\sigma} \sum_{m \geq 0} \frac{w_\beta(\gamma_\sigma)^m}{|\text{Aut}(\gamma_\sigma)|^m m!} = \exp \left( \sum_{\gamma_\sigma \in \Gamma_\sigma} \frac{w_\beta(\gamma_\sigma)}{|\text{Aut}(\gamma_\sigma)|} \right). \quad (7)$$

For a cycle $\sigma$ of order $|\sigma| = r$ then $\text{Aut}(\gamma_\sigma) = \langle \sigma^r \rangle$, a cyclic group of order $|\text{Aut}(\gamma_\sigma)| = \frac{r}{s}$. Using the trace identity

$$\sum_{\gamma_\sigma, |\sigma| = r} s w_\beta(\gamma_\sigma) = \beta \text{Tr}(A^r),$$

we find

$$\sum_{\gamma_\sigma \in \Gamma_\sigma} \frac{w_\beta(\gamma_\sigma)}{|\text{Aut}(\gamma_\sigma)|} = \beta \sum_{r \geq 1} \frac{1}{r} \text{Tr}(A^r)
= -\beta \text{Tr}(\log(I - A))
= -\beta \log \det(I - A).$$

Thus

$$\sum_k \frac{\text{perm}_\beta A(k)}{r_1! r_2! \ldots r_n!} = \det(I - A)^{-\beta}. \quad \square$$

Define a cycle to be primitive (or rotationless) if $|\text{Aut}(\gamma_\sigma)| = 1$. For a general cycle $\sigma$ with $|\text{Aut}(\gamma_\sigma)| = s$ we have for $\beta = 1$

$$w_1(\gamma_\sigma) = w_1(\gamma_\rho)^s,$$

for some primitive cycle $\rho$. Let $\Gamma_\rho$ denote the set of all primitive cycles. Then

$$\sum_{\gamma_\sigma \in \Gamma_\sigma} \frac{w_1(\gamma_\sigma)}{|\text{Aut}(\gamma_\sigma)|} = \sum_{\gamma_\sigma \in \Gamma_\rho} \sum_{s \geq 1} \frac{1}{s} w_1(\gamma_\rho)^s
= \sum_{\gamma_\rho \in \Gamma_\rho} \log \det(1 - w_1(\gamma_\rho)).$$

Combining this with (7) implies [MT2]
Theorem 2.2
\[ \det(I - A) = \prod_{\gamma_{\rho} \in \Gamma_{\rho}} (1 - w_{1}(\gamma_{\rho})). \]

3 Riemann Surfaces from a Sewn Sphere

3.1 The Riemann torus

Consider the construction of a torus by sewing a handle to the Riemann sphere \( \hat{\mathbb{C}} \) by identifying annular regions centred at \( A_{\pm 1} \in \hat{\mathbb{C}} \) via a sewing condition with complex sewing parameter \( \rho \)

\[
(z - A_{-1})(z' - A_{1}) = \rho. \tag{8}
\]

We call \( \rho, A_{\pm} \) canonical parameters. The annuli do not intersect provided

\[ |\rho| < \frac{1}{4} |A_{-1} - A_{1}|^2. \tag{9} \]

Inequivalent tori depend only on

\[ \chi = -\frac{\rho}{(A_{-1} - A_{1})^2}, \tag{10} \]

where (9) implies \( |\chi| < \frac{1}{4} \) [MT1].

Equivalently, we define \( q, a_{\pm 1} \), known as Schottky parameters, by

\[
a_{i} = \frac{A_{i} + qA_{-i}}{1 + q}, \quad \quad \frac{q}{(1 + q)^2} = \chi. \tag{11}
\]
for $i = \pm 1$. Inequivalent tori depend only on $q$ with $|q| < 1$. The canonical sewing condition (8) is equivalent to:

$$\left( \frac{z - a_{-1}}{z - a_1} \right) \left( \frac{z' - a_1}{z' - a_{-1}} \right) = q. \quad (12)$$

Inverting (11) we find that $q = C(\chi)$ for Catalan series

$$C(\chi) = \frac{1 - (1 - 4\chi)^{1/2}}{2\chi} - 1 = \sum_{n \geq 1} \frac{1}{n} \binom{2n}{n+1} \chi^n. \quad (13)$$

### 3.2 Genus $g$ Riemann Surfaces

We may similarly construct a general genus $g$ Riemann surface by identifying $g$ pairs of annuli centred at $A_{\pm i} \in \hat{\mathbb{C}}$ for $i = 1, \ldots, g$ and sewing parameters $\rho_i$ satisfying

$$(z - A_{-i})(z' - A_i) = \rho_i, \quad (14)$$

provided no two annuli intersect. Equivalently, for $i = 1, \ldots, g$ we define Schottky parameters $a_{\pm i}, q_i$ by

$$a_{\pm i} = \frac{A_{\pm i} + q A_{\mp i}}{1 + q_i},$$

$$q_i = \frac{\frac{\rho_i}{(1 + q_i)^2}}{(A_{-i} - A_i)^2}, \quad (15)$$

where $|q_i| < 1$ is again related to the Catalan series (13)

$$q_i = C(\chi_i), \quad \chi_i = -\frac{\rho_i}{(A_i - A_{-i})^2}. \quad (16)$$

The canonical sewing condition can then be rewritten as a standard Schottky sewing condition:

$$\left( \frac{z - a_{-i}}{z - a_i} \right) \left( \frac{z' - a_i}{z' - a_{-i}} \right) = q_i. \quad (16)$$

The Schottky sewing condition (16) determines a M"{o}bius map $z' = \gamma_i(z)$ where

$$\gamma_i = \sigma_i^{-1} \begin{pmatrix} q_i & 0 \\ 0 & 1 \end{pmatrix} \sigma_i, \quad (17)$$
for Möbius map
\[ \sigma_i(z) = \frac{z - a_i}{z - a_{-i}}. \] (18)

We define the Schottky group \( \Gamma = \langle \gamma_i \rangle \) as the Kleinian group freely generated by \( \gamma_i \) for \( i = 1, \ldots, g \).

One can find explicit formulas for various objects defined on the Riemann surface such as the bilinear form of the second kind, a basis of \( g \) holomorphic 1-forms and the genus \( g \) period matrix in terms of either the Canonical or Schottky parametrizations \([TZ]\). In the Schottky case, these involve sums or products over the Schottky group or subsets thereof.

### 4 Vertex Operator Algebras

Consider a simple VOA with \( \mathbb{Z} \)-graded vector space \( V = \oplus_{n \geq 0} V^{(n)} \) and local vertex operators \( Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \) for \( a \in V \) e.g. \([Ka],[FLM],[MN],[MT3]\]. We assume that \( V \) is of CFT type (i.e. \( V_0 = \mathbb{C}1 \)) with a unique symmetric invertible invariant bilinear form \( \langle , \rangle \) with normalization \( \langle 1, 1 \rangle = 1 \) where \([FHL],[Li]\]
\[ \langle Y(a, z)b, c \rangle = \langle b, Y(e^{zL_1}(-\frac{1}{z^2})^{L_0}a, \frac{1}{z})c \rangle \] (19)

For a \( V \)-basis \( \{u^\alpha\} \), we let \( \{\overline{u}^\alpha\} \) denote the dual basis. If \( a \in V^{(k)} \) is quasi-primary \( (L_1 a = 0) \) then (19) implies
\[ \langle a_n b, c \rangle = (-1)^k \langle b, a_{2k-n-2}c \rangle. \]

In particular:
\[ \langle a_n b, c \rangle = -\langle b, a_{-n}c \rangle \text{ for } a \in V^{(1)} \]
\[ \langle L_n b, c \rangle = \langle b, L_{-n}c \rangle \text{ for } \omega \in V^{(2)}, \] (20)

so that \( b, c \) with unequal weights are orthogonal.

#### 4.1 Genus Zero Correlation Functions

For \( u_1, u_2, \ldots, u_n \in V \) define the \( n \)-point (correlation) function by
\[ \langle 1, Y(u_1, z_1)Y(u_2, z_2) \ldots Y(u_n, z_n)1 \rangle. \] (21)
The locality property of vertex operators implies that this formal expression (21) coincides with the analytic expansion of a rational function of \( z_1, z_2, \ldots, z_n \) in the domain \( |z_1| > |z_2| > \ldots > |z_n| \). Thus the \( n \)-point function can taken to be a rational function of \( z_1, z_2, \ldots, z_n \in \hat{\mathbb{C}} \), the Riemann sphere in the domain. For example [HT]

**Theorem 4.1** For a VOA of central charge \( C \), the Virasoro \( n \)-point function is a \( \beta \)-extended permanent

\[
(1, Y(\omega, z_1) \ldots Y(\omega, z_n)1) = \text{perm}_{q} B,
\]

for \( B_{ij} = \frac{1}{(z_i - z_j)^2} \), \( i \neq j \) and \( B_{ii} = 0 \).

### 4.2 Rank Two Heisenberg VOA \( M_2 \)

Consider the VOA generated by two Heisenberg vectors \( a^{\pm} \in V^{(1)} \) whose modes satisfy non-trivial commutator

\[
[a^{+}_m, a^{-}_n] = m\delta_{m,-n}. \tag{22}
\]

\( V \) has a Fock basis spanned by

\[
a_{k,1} = a^{-}_{-k_{1}} \ldots a^{-}_{-k_{m}} a^{+}_{-l_{1}} \ldots a^{+}_{-l_{n}} 1, \tag{23}
\]

labelled by multisets \( k = \{k_1, \ldots, k_m\} = \{1^{r_1}.2^{r_2} \ldots\} \) and \( l = \{l_1, \ldots, l_n\} = \{1^{s_1}.2^{s_2} \ldots\} \). The Fock vectors are orthogonal with respect to the invariant bilinear form with dual basis

\[
\overline{a}_{k,1} = \prod_i \frac{1}{s_i!} \prod_j \frac{1}{j^2 s_j!} a_{l,k}. \tag{24}
\]

The basic Heisenberg 2-point function is

\[
\langle 1, Y(a^{+}, x)Y(a^{-}, y)1 \rangle = \frac{1}{(x - y)^2}. \tag{25}
\]

This function provides all the necessary data for computing the Heisenberg partition and correlation functions on a genus \( g \) surface! Thus the general rank 2 Heisenberg \( 2n \)-point function is

\[
(1, Y(a^{+}, x_1) \ldots Y(a^{+}, x_n)Y(a^{-}, y_1) \ldots Y(a^{-}, y_n)1) = \text{perm} \left( \frac{1}{(x_i - y_j)^2} \right). \tag{26}
\]
This is a generating function for all rank two Heisenberg correlation functions by associativity of the VOA.

Let $x_{-i} = x - A_{-i}$ and $y_{j} = y - A_{j}$ be local coordinates in the neighborhood of canonical sewing parameters $A_{-i}, A_{j}$ for $i, j \in \{ \pm 1, \ldots, \pm g \}$ with $i \neq -j$. The 2-point function has expansion

$$\frac{1}{(x - y)^2} = \sum_{k, l \geq 1} (-1)^{k+1} \frac{(k + l - 1)!}{(k-1)! (l-1)!} \frac{x_{-i}^{k-1} y_{j}^{l-1}}{(A_{-i} - A_{j})^{k+l}}.$$ 

Define the canonical moment matrix $R_{ij}^{Can}$, an infinite matrix indexed by $k, l = 1, 2, \ldots$ and $i, j \in \{ \pm 1, \ldots, \pm g \}$ where

$$R_{ij}^{Can}(k, l) = \left\{ \begin{array}{ll} \frac{(-1)^k \rho_i^{k/2} \rho_j^{l/2}}{\sqrt{kl}} \frac{(k+l-1)!}{(k-1)! (l-1)!} \frac{1}{(A_{-i} - A_{j})^{k+l}}, & i \neq -j \\ 0, & i = -j \end{array} \right.$$ 

$(I - R_{ij}^{Can})^{-1}$ plays a central role in computing the genus $g$ period matrix and other structures.

We similarly have expansions in the Schottky parameters. Let

$$x_{-i} = \sigma_{-i}(x) = \frac{x-a_{-i}}{x-a_{i}}$$
$$y_{j} = \sigma_{j}(x) = \frac{y-a_{j}}{y-a_{-j}}$$

for $i, j \in \{1, \ldots, g\}$ be local coordinates in the neighborhood of the Schottky points $a_{-i}$ and $a_{j}$ for $i \neq -j$. The 2-point function expansion leads to the Schottky moment matrix with

$$R_{ij}^{Sch}(k, l) = \left\{ \begin{array}{ll} q_i^{k/2} q_j^{l/2} D(k, l)(\sigma_i \sigma_j^{-1}), & i \neq -j \\ 0, & i = -j \end{array} \right.$$ 

where for $\gamma \in SL(2, \mathbb{C})$

$$D(k, l)(\gamma) = \frac{1}{l!} \left[ \frac{1}{k} \partial_z (\gamma(z)^k) \right]_{z=0}. $$

$D$ is an $SL(2, \mathbb{C})$ representation [Mo]. Then it follows

$$\sum_{s \geq 1} R_{ij}^{Sch}(r, s) R_{jk}^{Sch}(s, t) = q_i^{r/2} q_j^{t/2} D(r, t)(\sigma_i \gamma_j \sigma_k^{-1}),$$

for Schottky generator (17).
4.3 The Genus $g$ Partition Function - Canonical Parameters

We now define the genus $g$ partition function for a VOA $V$ in the canonical sewing scheme in terms of genus zero 2g-point correlation functions as follows:

$$Z_{V}^{(g)}(\rho_{i}, A_{\pm i}) = \langle 1, \prod_{i=1}^{g} \sum_{n_{i} \geq 0} \rho_{i}^{n_{i}} \sum_{v_{i} \in V(n)} Y(v_{i}, A_{-i})Y(\overline{v}_{i}, A_{i})1 \rangle,$$

(33)

where $\overline{v}_{i}$ is dual to $v_{i}$.

For genus one this reverts to the standard definition:

**Theorem 4.2 (Mason and T.)**

$$Z_{V}^{(1)}(\rho, A_{\pm 1}) = \text{Tr}_{V}(q^{L_{0}})$$

where $q = C(\chi)$, the Catalan series for $\chi = -\frac{\rho}{(A_{-1}-A_{1})^{2}}$.

4.4 $Z_{M_{2}}^{(g)}(\rho_{i}, A_{\pm i})$ for Heisenberg VOA $M_{2}$

The genus $g$ partition function can be computed for the rank 2 Heisenberg VOA by means of the MacMahon Master Theorem where, schematically, we have:

- Sum over $g$ Fock bases $\rightarrow$ Sum over multisets
- $2g$-point function $\rightarrow$ Permanent of matrix
- Dual vector factorials $\rightarrow$ Multiset factorials
- $\rho_{i}$ and other dual vector factors $\rightarrow$ Absorbed into matrix definition

We then find that [TZ]

**Theorem 4.3**

$$Z_{M_{2}}^{(g)}(\rho_{i}, A_{\pm i}) = \frac{1}{\det(I - R^{\text{Can}})},$$

where $R^{\text{Can}}$ is the canonical moment matrix. Furthermore, $\det(I - R^{\text{Can}})$ is holomorphic and non-vanishing. In general, the genus $g$ Heisenberg generating function is expressed in terms of a permanent of genus $g$ bilinear forms of the second kind.
We may repeat this by using an alternative definition of the genus $g$ partition function in terms of in Schottky parameters account must be taken of the Möbius maps $\sigma_i$ of (18). We then find [TZ]

**Theorem 4.4** The genus $g$ partition function is

$$Z_{M_2}^{(g)}(q_i, a_{\pm i}) = \frac{1}{\det(I - R^{Sch})},$$

where $R^{Sch}$ is the Schottky moment matrix. Furthermore, $\det(I - R^{Sch})$ is holomorphic and non-vanishing and the genus $g$ Heisenberg generating function is expressed in terms of a permanent of genus $g$ bilinear forms of the second kind.

Conjecture: $\det(I - R^{Can}) = \det(I - R^{Sch})$. This is true for $g = 1$ [MT2].

### 4.5 The Montonen-Zograf Product Formula

det($I - R^{Sch}$) can be also re-expressed in terms of an infinite product formula originally calculated in physics by Montonen in 1974 [Mo]. A similar product formula was subsequently found by Zograf [Z]. This has been recently related by McIntyre and Takhtajan [McT] to Mumford’s theorem concerning the absence of a global section on moduli space for the canonical line bundle [Mu].

Recall that $R^{Sch}_{ij}(k, l)$ is expressed in terms of an $SL(2, \mathbb{C})$ representation $D$. This leads to

$$\det(I - R^{Sch}) = \prod_{m \geq 1} \prod_{\gamma \in \Gamma} (1 - q_{\alpha}^m),$$

where the inner product ranges over the primitive elements $\gamma^\alpha$ of the Schottky group $\Gamma$ i.e. $\gamma^\alpha \neq \gamma^k$ for any $\gamma \in \Gamma$ for $k > 1$. Each such element has a multiplier $q_{\alpha}$ where

$$\gamma^\alpha \sim \begin{pmatrix} q_{\alpha} & 0 \\ 0 & 1 \end{pmatrix}. $$

(35)
References


