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Associating quantum vertex algebras to quantum affine algebras

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Abstract

We give a summary account of the recent development on a particular theory of quantum vertex algebras and the association of quantum affine algebras with quantum vertex algebras.

1 Introduction

In the general field of vertex algebras, a fundamental problem has been to establish a theory of quantum vertex algebras so that quantum affine algebras can be canonically associated with quantum vertex algebras (see [FJ]; cf. [EFK]). In the past, several notions of quantum vertex (operator) algebra have been introduced and studied with various purposes (see [eFR], [EK], [B3], [Li2], [AB], [Li7]). With solving the very problem as one of the main goals, in a series of papers (see [Li2], [Li5], [Li6], [Li7]) we have developed certain theories of (weak) quantum vertex algebras. Indeed, using some of such theories we have obtained partial solutions while complete solutions are emerging.

The main theme of this series of studies is to investigate the algebraic structures that the generating functions of the generators in the Drinfeld realization (see [Dr]) could possibly "generate." Let W be a general vector space and set $\mathcal{E}(W)=\mathrm{Hom}(W,W((x)))$. In [Li2], we studied certain vertex algebra-like structures generated by various types of subsets of $\mathcal{E}(W)$, where the most general type consists of what we called quasi compatible subsets. It was proved therein (cf. [Li1]) that any quasi compatible subset of $\mathcal{E}(W)$ generates a nonlocal vertex algebra with W as a quasi module in a certain sense (cf. [Li3]). (Nonlocal vertex algebras are analogs of noncommutative associative algebras, in contrast to that vertex algebras are analogs of commutative and associative algebras.) It follows from this general result that a wide variety of algebras can be associated with nonlocal vertex algebras. In particular, nonlocal vertex algebras can be associated to quantum affine algebras by taking W to be a highest weight module for a quantum affine algebra and U the set of the generating functions.

We also formulated in [Li2] a notion of (weak) quantum vertex algebra, which was mostly motivated by Etingof-Kazhdan's notion of quantum vertex operator algebra, especially by the $\mathcal{S}$-locality axiom (see [EK]). A weak quantum vertex algebra was defined to be a nonlocal vertex algebra that satisfies (a small variation of) $\mathcal{S}$-locality, while a quantum vertex algebra was defined to be a weak quantum vertex algebra equipped with a unitary rational quantum Yang-Baxter operator governing the $\mathcal{S}$-locality. This notion of quantum vertex algebra came out as a variation of Etingof-Kazhdan's notion of quantum vertex operator algebra. What is more important is a conceptual result; we studied a notion of "$\mathcal{S}$-local subset" of $\mathcal{E}(W)$ (with W a vector space), which singles out a family of quasi compatible subsets, and we proved that every $\mathcal{S}$-local subset of $\mathcal{E}(W)$ generates a weak quantum vertex algebra with W as a canonical module. In a sequel [Li5] we have successfully associated quantum vertex algebras to certain versions of double Yangians. This makes the particular theory of quantum vertex algebras more interesting, though it was still a question whether one can associate (weak) quantum vertex algebras to quantum affine algebras.

An association of weak quantum vertex algebras to quantum affine algebras was obtained later in [Li8], where a new construction of weak quantum vertex algebras was established and a theory of what were called ϕ-coordinated quasi modules for weak quantum vertex algebras was developed. In this new theory, the parameter ϕ is a formal series $\phi(x,z) \in \mathbb{C}((x))[[z]]$ satisfying

$$\phi(x,0) = x, \qquad \phi(\phi(x,x_0),x_2) = \phi(x,x_0+x_2).$$

Particular examples are $\phi(x,z) = x+z$ and $\phi(x,z) = xe^z$. Given such a ϕ, for a nonlocal vertex algebra V we defined a notion of ϕ-coordinated quasi V-module for which the main axiom is an associativity

$$(Y(u,x_1)Y(v,x_2))\,|_{x_1=\phi(x_2,x_0)} = Y(Y(u,x_0)v,x_2)$$

(an unrigorous version). In the case $\phi(x,z) = x+z$, this notion reduces to that of an ordinary V-module. On the other hand, we generalized the conceptual construction in [Li2]. Given a general vector space W, we defined a (partial) vertex operation $Y_{\mathcal{E}}^{\phi}$ on $\mathcal{E}(W)$ by

$$Y_{\mathcal{E}}^{\phi}(a(x),z)b(x) = (a(x_1)b(x))\,|_{x_1=\phi(x,z)}$$

(unrigorous) for $a(x), b(x) \in \mathcal{E}(W)$. It was proved that every quasi compatible subset of $\mathcal{E}(W)$ generates under the vertex operation $Y_{\mathcal{E}}^{\phi}$ a nonlocal vertex algebra with W as a ϕ-coordinated quasi module. We furthermore formulated a notion of quasi $\mathcal{S}_{trig}$-locality, to capture the main features of the set of generating functions for quantum affine algebras. It was proved that every quasi $\mathcal{S}_{trig}$-local subset U of $\mathcal{E}(W)$ generates a weak quantum vertex algebra with W as a ϕ-coordinated quasi module with $\phi(x,z) = xe^z$. Take W to be a highest weight module for a quantum affine algebra and U the set of the generating functions. Then U is a quasi $\mathcal{S}_{trig}$-local subset of $\mathcal{E}(W)$, and hence it generates a weak quantum vertex algebra with W as a ϕ-coordinated quasi module.

Having associated weak quantum vertex algebras to quantum affine algebras in a conceptual way, we have provided a rough solution to the aforementioned problem. Note that for the association of affine Lie algebras with vertex algebras, the underlying spaces of the associated vertex algebras are vacuum modules for the affine Lie algebras (see [FZ]; cf. [LL]). To complete this solution we shall have to construct the underlying spaces explicitly, preferably as (vacuum) modules for certain algebras, and show that the associated weak quantum vertex algebras are indeed quantum vertex algebras.

We mention that there are also two other closely related theories of quantum vertex algebras. In [Li6], a theory of $\hbar$-adic (weak) quantum vertex algebras was developed and $\hbar$-adic quantum vertex algebras were associated to a centrally extended double Yangian. In [Li7], a theory of (weak) quantum vertex $\mathbb{C}((t))$-algebras was developed and weak quantum vertex $\mathbb{C}((t))$-algebras were associated to quantum affine algebras.

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2 Weak quantum vertex algebras and quantum vertex algebras

In this section, following [Li2] we present the basic notions of weak quantum vertex algebra and quantum vertex algebra, including a conceptual construction.

First of all, we work on the field $\mathbb{C}$ of complex numbers and we use the formal variable notations and conventions as established in [FLM] and [FHL] (cf. [LL]). Letters such as $x, y, z, x_0, x_1, x_2, \ldots$ are mutually commuting independent formal variables. For a positive integer r, denote by $\mathbb{C}[[x_1, x_2, \ldots, x_r]]$ the algebra of formal nonnegative power series and by $\mathbb{C}((x_1, \ldots, x_r))$ the algebra of formal Laurent series which are globally truncated with respect to all the variables. Note that in the case $r = 1$, $\mathbb{C}((x))$ is in fact a field. By $\mathbb{C}(x_1, x_2, \ldots, x_r)$ we denote the field of rational functions.

For any permutation $(i_1, i_2, \ldots, i_r)$ on $\{1, \ldots, r\}$, $\mathbb{C}((x_{i_1}))\cdots((x_{i_r}))$ is a field containing $\mathbb{C}[x_1, \ldots, x_r]$ as a subalgebra, so there exists an algebra embedding

$$\iota_{x_{i_1},\ldots,x_{i_r}} : \mathbb{C}(x_1, x_2, \ldots, x_r) \to \mathbb{C}((x_{i_1}))\cdots((x_{i_r})), \tag{2.1}$$

extending uniquely the identity endomorphism of $\mathbb{C}[x_1, \ldots, x_r]$ (cf. [FHL]). Note that both $\mathbb{C}(x_1, \ldots, x_r)$ and $\mathbb{C}((x_{i_1}))\cdots((x_{i_r}))$ contain $\mathbb{C}((x_1, \ldots, x_r))$ as a subalgebra. We see that $\iota_{x_{i_1},\ldots,x_{i_r}}$ preserves $\mathbb{C}((x_1, \ldots, x_r))$ element-wise and is $\mathbb{C}((x_1, \ldots, x_r))$-linear.

Definition 2.1. A *nonlocal vertex algebra* is a vector space V, equipped with a linear map

$$\begin{aligned} Y(\cdot, x) : \quad & V \to \mathrm{Hom}(V, V((x))) \subset (\mathrm{End}V)[[x, x^{-1}]], \\ & v \mapsto Y(v, x) \end{aligned}$$

and a vector $\mathbf{1} \in V$, satisfying the conditions that $Y(\mathbf{1}, x) = 1$,

$$Y(v, x)\mathbf{1} \in V[[x]] \text{ and } \lim_{x \to 0} Y(v, x)\mathbf{1} = v \quad \text{for } v \in V,$$

and that for $u, v, w \in V$, there exists a nonnegative integer l such that

$$(x_0 + x_2)^l Y(u, x_0 + x_2) Y(v, x_2) w = (x_0 + x_2)^l Y(Y(u, x_0)v, x_2) w. \tag{2.2}$$

Let V be a nonlocal vertex algebra. Define a linear operator $\mathcal{D}$ on V by

$$\mathcal{D}(v) = v_{-2}\mathbf{1} \quad \text{for } v \in V. \tag{2.3}$$

Then

$$[\mathcal{D}, Y(v, x)] = Y(\mathcal{D}v, x) = \frac{d}{dx} Y(v, x) \quad \text{for } v \in V. \tag{2.4}$$

The following notion singles out an important family of nonlocal vertex algebras:

Definition 2.2. A *weak quantum vertex algebra* is a nonlocal vertex algebra V which satisfies $\mathcal{S}$*-locality* in the sense that for $u, v \in V$, there exist

$$u^{(i)},\ v^{(i)} \in V,\quad f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \dots, r)$$

(finitely many) such that

$$(x_1 - x_2)^k Y(u, x_1) Y(v, x_2) = (x_1 - x_2)^k \sum_{i=1}^{r} f_i(x_2 - x_1) Y(v^{(i)}, x_2) Y(u^{(i)}, x_1) \tag{2.5}$$

for some nonnegative integer k.

The notion of weak quantum vertex algebra naturally generalizes the notion of vertex algebra and that of vertex superalgebra.

We have the following basic facts (see [Li2]):

Proposition 2.3. *Let V be a nonlocal vertex algebra and let*

$$u, v,\ u^{(i)},\ v^{(i)} \in V,\quad f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \dots, r).$$

Then the $\mathcal{S}$-locality relation (2.5) is equivalent to

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1) Y(v, x_2) \\ & \qquad - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) \sum_{i=1}^{r} f_i(-x_0) Y(v^{(i)}, x_2) Y(u^{(i)}, x_1) \\ & = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2) \end{aligned} \tag{2.6}$$

*(the $\mathcal{S}$-*Jacobi identity*), and is also equivalent to*

$$Y(u,x)v = e^{x\mathcal{D}}\sum_{i=1}^{r} f_i(-x)Y(v^{(i)},-x)u^{(i)} \tag{2.7}$$

*(the $\mathcal{S}$-*skew symmetry*).*

Definition 2.4. Let V be a nonlocal vertex algebra. A *V-module* is a vector space W, equipped with a linear map

$$\begin{aligned} Y_W(\cdot,x): \quad & V \to \mathrm{Hom}(W,W((x))) \subset (\mathrm{End}W)[[x,x^{-1}]], \\ & v \mapsto Y_W(v,x), \end{aligned}$$

satisfying the conditions that

$$Y_W(\mathbf{1},x) = 1_W \quad \text{(the identity operator on } W)$$

and that for $u,v \in V,\ w \in W$, there exists a nonnegative integer l such that

$$(x_0+x_2)^l Y_W(u,x_0+x_2)Y_W(v,x_2)w = (x_0+x_2)^l Y_W(Y(u,x_0)v,x_2)w.$$

We also define a *quasi V-module* by replacing the last condition with that for $u,v \in V,\ w \in W$, there exists a nonzero polynomial $p(x_1,x_2)$ such that

$$p(x_0+x_2,x_2)Y_W(u,x_0+x_2)Y_W(v,x_2)w = p(x_0+x_2,x_2)Y_W(Y(u,x_0)v,x_2)w.$$

Proposition 2.5. *Let V be a weak quantum vertex algebra and let (W,Y_W) be a module for V viewed as a nonlocal vertex algebra. Assume*

$$u,v,\ u^{(i)},\ v^{(i)} \in V,\quad f_i(x) \in \mathbb{C}((x)) \quad (i=1,\dots,r)$$

such that the $\mathcal{S}$-locality relation (2.5) holds. Then

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) Y_W(u,x_1)Y_W(v,x_2) \\ & \qquad -x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\sum_{i=1}^{r} f_i(-x_0)Y_W(v^{(i)},x_2)Y_W(u^{(i)},x_1) \\ & = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) Y_W(Y(u,x_0)v,x_2). \end{aligned}$$

A *rational quantum Yang-Baxter operator* on a vector space U is a linear operator

$$\mathcal{S}(x):\ U\otimes U \to U\otimes U\otimes \mathbb{C}((x))$$

satisfying the quantum Yang-Baxter equation

$$\mathcal{S}^{12}(x)\mathcal{S}^{13}(x+z)\mathcal{S}^{23}(z) = \mathcal{S}^{23}(z)\mathcal{S}^{13}(x+z)\mathcal{S}^{12}(x).$$

It is said to be *unitary* if

$$\mathcal{S}(x)\mathcal{S}^{21}(-x) = 1,$$

where $\mathcal{S}^{21}(x) = \sigma\mathcal{S}(x)\sigma$ with σ denoting the flip operator on $U\otimes U$.

Definition 2.6. A *quantum vertex algebra* is a weak quantum vertex algebra V equipped with a unitary rational quantum Yang-Baxter operator $\mathcal{S}(x)$ on V, satisfying

$$\mathcal{S}(x)(\mathbf{1}\otimes v)=\mathbf{1}\otimes v \quad \text{for } v\in V, \tag{2.8}$$

$$[\mathcal{D}\otimes 1,\mathcal{S}(x)]=-\frac{d}{dx}\mathcal{S}(x), \tag{2.9}$$

$$Y(u,x)v=e^{x\mathcal{D}}Y(-x)\mathcal{S}(-x)(v\otimes u) \quad \text{for } u,v\in V, \tag{2.10}$$

$$\mathcal{S}(x_1)(Y(x_2)\otimes 1)=(Y(x_2)\otimes 1)\mathcal{S}^{23}(x_1)\mathcal{S}^{13}(x_1+x_2). \tag{2.11}$$

We denote a quantum vertex algebra by a pair $(V,\mathcal{S})$.

In the study of quantum vertex (operator) algebras, the notion of non-degeneracy, which was introduced by Etingof-Kazhdan in [EK], has played a very important role.

Definition 2.7. A nonlocal vertex algebra V is said to be *non-degenerate* if for every positive integer n, the linear map

$$Z_n : V^{\otimes n}\otimes \mathbb{C}((x_1))\cdots((x_n))\rightarrow V((x_1))\cdots((x_n)),$$

defined by

$$Z_n(v^{(1)}\otimes\cdots\otimes v^{(n)}\otimes f)=fY(v^{(1)},x_1)\cdots Y(v^{(n)},x_n)\mathbf{1}$$

for $v^{(1)},\dots,v^{(n)}\in V$, $f\in \mathbb{C}((x_1))\cdots((x_n))$, is injective.

It was proved in [Li2] (cf. [EK]).

Proposition 2.8. *Let V be a weak quantum vertex algebra. Assume that V is non-degenerate. Then there exists a linear map $\mathcal{S}(x): V\otimes V\rightarrow V\otimes V\otimes\mathbb{C}((x))$, which is uniquely determined by*

$$Y(u,x)v=e^{x\mathcal{D}}Y(-x)\mathcal{S}(-x)(v\otimes u) \quad \textit{for } u,v\in V,$$

and $(V,\mathcal{S})$ carries the structure of a quantum vertex algebra. Moreover, the following relation holds

$$[1\otimes\mathcal{D},\mathcal{S}(x)]=\frac{d}{dx}\mathcal{S}(x). \tag{2.12}$$

The following is a general result on non-degeneracy (see [Li7], cf. [Li4]):

Proposition 2.9. *Let V be a nonlocal vertex algebra such that V as a V-module is irreducible and of countable dimension (over $\mathbb{C}$). Then V is non-degenerate.*

Next, we discuss the conceptual construction of weak quantum vertex algebras. Let W be a general vector space. Set

$$\mathcal{E}(W) = \mathrm{Hom}(W, W((x))) \subset (\mathrm{End}W)[[x, x^{-1}]]. \tag{2.13}$$

The identity operator on W, denoted by 1_W, is a special element of $\mathcal{E}(W)$.

Definition 2.10. A finite sequence $a_1(x), \ldots, a_r(x)$ in $\mathcal{E}(W)$ is said to be *quasi compatible* if there exists a nonzero polynomial $p(x, y)$ such that

$$\left(\prod_{1\le i<j\le r} p(x_i, x_j)\right) a_1(x_1)\cdots a_r(x_r) \in \mathrm{Hom}(W, W((x_1, \ldots, x_r))). \tag{2.14}$$

The sequence $a_1(x), \ldots, a_r(x)$ is said to be *compatible* if (2.14) holds with $p(x_1, x_2) = (x_1 - x_2)^k$ for some nonnegative integer k. Furthermore, a subset T of $\mathcal{E}(W)$ is said to be *quasi compatible* (resp. *compatible*) if every finite sequence in T is quasi compatible (resp. compatible).

Let $(a(x), b(x))$ be a quasi compatible ordered pair in $\mathcal{E}(W)$. That is, there is a nonzero polynomial $p(x, y)$ such that

$$p(x_1, x_2)a(x_1)b(x_2) \in \mathrm{Hom}(W, W((x_1, x_2))). \tag{2.15}$$

We define $Y_{\mathcal{E}}(a(x), x_0)b(x) \in \mathcal{E}(W)((x_0))$ by

$$Y_{\mathcal{E}}(a(x), x_0)b(x) = \iota_{x,x_0}\left(\frac{1}{p(x + x_0, x)}\right)(p(x_1, x)a(x_1)b(x))\,|_{x_1=x+x_0} \tag{2.16}$$

and we then define $a(x)_n b(x) \in \mathcal{E}(W)$ for $n \in \mathbb{Z}$ by

$$Y_{\mathcal{E}}(a(x), x_0)b(x) = \sum_{n\in\mathbb{Z}} a(x)_n b(x) x_0^{-n-1}. \tag{2.17}$$

One can show that this is well defined; the expression on the right-hand side is independent of the choice of $p(x, y)$. In this way we have defined partial operations $(a(x), b(x)) \mapsto a(x)_n b(x)$ for $n \in \mathbb{Z}$ on $\mathcal{E}(W)$. We say that a quasi compatible subspace U of $\mathcal{E}(W)$ is *$Y_{\mathcal{E}}$-closed* if

$$a(x)_n b(x) \in U \quad \text{for } a(x), b(x) \in U,\ n \in \mathbb{Z}. \tag{2.18}$$

We have the following conceptual results (see [Li2], cf. [Li1]):

Theorem 2.11. *Let W be a vector space and let U be any (resp. quasi) compatible subset of $\mathcal{E}(W)$. Then there exists a (unique) smallest $Y_{\mathcal{E}}$-closed (resp. quasi) compatible subspace $\langle U\rangle$ that contains U and 1_W. Furthermore, $(\langle U\rangle, Y_{\mathcal{E}}, 1_W)$ carries the structure of a nonlocal vertex algebra with W as a (resp. quasi) module where $Y_W(\alpha(x), x_0) = \alpha(x_0)$ for $\alpha(x) \in \langle U\rangle$.*

Definition 2.12. Let W be a vector space. A subset U of $\mathcal{E}(W)$ is said to be *$\mathcal{S}$-local* if for any $a(x), b(x) \in U$, there exist

$$c^{(i)}(x), d^{(i)}(x) \in U, \; f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \ldots, r)$$

(with r finite) such that

$$(x-z)^k a(x) b(z) = (x-z)^k \sum_{i=1}^{r} f_i(-z+x) c^{(i)}(z) d^{(i)}(x) \tag{2.19}$$

for some nonnegative integer k.

Every $\mathcal{S}$-local subset was proved to be compatible. Furthermore, we have:

Theorem 2.13. *For any $\mathcal{S}$-local subset U of $\mathcal{E}(W)$, $\langle U \rangle$ is a weak quantum vertex algebra with W as a module.*

3 ϕ-coordinated modules for nonlocal vertex algebras and quantum vertex algebras

In this section, we present the theory of ϕ-coordinated quasi modules for nonlocal vertex algebras and for weak quantum vertex algebras, which was established in [Li8].

Set

$$F_{\mathrm{a}}(x, y) = x + y \in \mathbb{C}[x, y], \tag{3.1}$$

which is known as the one-dimensional additive formal group. The following notion, introduced in [Li8], is an analog of the notion of G-set for a group G:

Definition 3.1. An *associate* of $F_{\mathrm{a}}(x, y)$ is a formal series $\phi(x, z) \in \mathbb{C}((x))[[z]]$, satisfying

$$\phi(x, 0) = x, \qquad \phi(\phi(x, x_0), x_2) = \phi(x, x_0 + x_2). \tag{3.2}$$

We have the following explicit construction of associates (see [Li8]):

Proposition 3.2. *For $p(x) \in \mathbb{C}((x))$, set*

$$\phi_{p(x)}(x, z) = e^{zp(x)\frac{d}{dx}} x = \sum_{n \geq 0} \frac{z^n}{n!} \left(p(x) \frac{d}{dx} \right)^n x \in \mathbb{C}((x))[[z]].$$

Then $\phi_{p(x)}(x, z)$ is an associate of F_{a}. Furthermore, every associate of F_{a} is of this form with $p(x)$ uniquely determined.

Using Proposition 3.2, we obtain particular associates of F_a: $\phi_{p(x)}(x,z)=x$ with $p(x)=0$; $\phi_{p(x)}(x,z)=x+z$ with $p(x)=1$; $\phi_{p(x)}(x,z)=xe^z$ with $p(x)=x$; $\phi_{p(x)}(x,z)=x(1-zx)^{-1}$ with $p(x)=x^2$.

Definition 3.3. Let V be a nonlocal vertex algebra and let ϕ be an associate of F_a. A *ϕ-coordinated quasi V-module* is defined as in Definition 2.4 except replacing the weak associativity axiom with the condition that for $u,v\in V$, there exists a (nonzero) polynomial $p(x,y)$ such that $p(\phi(x,z),x)\neq 0$,

$$p(x_1,x_2)Y_W(u,x_1)Y_W(v,x_2)\in \mathrm{Hom}(W,W((x_1,x_2))), \tag{3.3}$$

and

$$p(\phi(x_2,x_0),x_2)Y_W(Y(u,x_0)v,x_2)=(p(x_1,x_2)Y_W(u,x_1)Y_W(v,x_2))\,|_{x_1=\phi(x_2,x_0)} \tag{3.4}$$

A *ϕ-coordinated V-module* is defined as above except that $p(x_1,x_2)$ is assumed to be a polynomial of the form $(x_1-x_2)^k$ with $k\in\mathbb{N}$.

Let W be a vector space and let $\phi(x,z)$ be an associate of $F_a(x,y)$, which are both fixed for the moment. We define a notion of *ϕ-quasi compatible subset* of $\mathcal{E}(W)$ as in Definition 2.10 but in addition assuming $p(\phi(x,z),x)\neq 0$. For a ϕ-quasi compatible pair $(a(x),b(x))$ in $\mathcal{E}(W)$, by definition there exists a polynomial $p(x,y)$ such that $p(\phi(x,z),x)\neq 0$ and

$$p(x_1,x_2)a(x_1)b(x_2)\in \mathrm{Hom}(W,W((x_1,x_2))). \tag{3.5}$$

Definition 3.4. Let $a(x),b(x)\in\mathcal{E}(W)$ be such that $(a(x),b(x))$ is ϕ-quasi compatible. We define

$$a(x)_n^{\phi}b(x)\in\mathcal{E}(W)\quad \text{for } n\in\mathbb{Z}$$

in terms of the generating function

$$Y_{\mathcal{E}}^{\phi}(a(x),z)b(x)=\sum_{n\in\mathbb{Z}}a(x)_n^{\phi}b(x)z^{-n-1} \tag{3.6}$$

by

$$Y_{\mathcal{E}}^{\phi}(a(x),z)b(x)=p(\phi(x,z),x)^{-1}\left(p(x_1,x)a(x_1)b(x)\right)|_{x_1=\phi(x,z)}, \tag{3.7}$$

which lies in $(\mathrm{Hom}(W,W((x))))((z))=\mathcal{E}(W)((z))$, where $p(x_1,x_2)$ is any polynomial with $p(\phi(x,z),x)\neq 0$ such that (3.5) holds and where $p(\phi(x,z),x)^{-1}$ stands for the inverse of $p(\phi(x,z),x)$ in $\mathbb{C}((x))((z))$.

Let U be a subspace of $\mathcal{E}(W)$ such that every ordered pair in U is ϕ-quasi compatible. We say that U is *$Y_{\mathcal{E}}^{\phi}$-closed* if

$$a(x)_n^{\phi}b(x)\in U\quad \text{for } a(x),b(x)\in U,\ n\in\mathbb{Z}. \tag{3.8}$$

We have (see [Li8]):

Theorem 3.5. *Let W be a vector space, $\phi(x,z)$ an associate of $F_{\mathrm{a}}(x,y)$, and U a ϕ-quasi compatible subset of $\mathcal{E}(W)$. There exists a $Y_{\mathcal{E}}^{\phi}$-closed ϕ-quasi compatible subspace of $\mathcal{E}(W)$, that contains U and 1_W. Denote by $\langle U\rangle_\phi$ the smallest such subspace. Then $(\langle U\rangle_\phi, Y_{\mathcal{E}}^{\phi}, 1_W)$ carries the structure of a nonlocal vertex algebra and W is a ϕ-coordinated quasi $\langle U\rangle_\phi$-module with $Y_W(\alpha(x),z) = \alpha(z)$ for $\alpha(x) \in \langle U\rangle_\phi$.*

Definition 3.6. Let W be a vector space. A subset U of $\mathcal{E}(W)$ is said to be *quasi $\mathcal{S}_{trig}$-local* if for any $a(x), b(x) \in U$, there exist finitely many

$$u^{(i)}(x),\ v^{(i)}(x) \in U,\ \ f_i(x) \in \mathbb{C}(x)\ \ (i = 1,\dots,r)$$

such that

$$p(x_1,x_2)a(x_1)b(x_2) = \sum_{i=1}^{r} p(x_1,x_2)\iota_{x_2,x_1}(f_i(x_1/x_2))u^{(i)}(x_2)v^{(i)}(x_1) \tag{3.9}$$

for some nonzero polynomial $p(x_1,x_2)$, depending on $a(x)$ and $b(x)$. We define *$\mathcal{S}_{trig}$-locality* by strengthening (3.9) as

$$(x_1-x_2)^k a(x_1)b(x_2) = (x_1-x_2)^k \sum_{i=1}^{r} \iota_{x_2,x_1}(f_i(x_1/x_2))u^{(i)}(x_2)v^{(i)}(x_1) \tag{3.10}$$

for some nonnegative integer k.

The fact is that quasi $\mathcal{S}_{trig}$-local subsets of $\mathcal{E}(W)$ are quasi compatible whereas $\mathcal{S}_{trig}$-local subsets are compatible. Furthermore, we have (see [Li8]):

Theorem 3.7. *Let W be a vector space and let U be any (resp. quasi) $\mathcal{S}_{trig}$-local subset of $\mathcal{E}(W)$. Set $\phi(x,z) = xe^z$. Then the nonlocal vertex algebra $\langle U\rangle_\phi$ generated by U is a weak quantum vertex algebra and (W, Y_W) is a (resp. quasi) ϕ-coordinated module for $\langle U\rangle_\phi$, where*

$$Y_W(\alpha(x), x_0) = \alpha(x_0) \quad \textit{for } \alpha(x) \in \langle U\rangle_\phi.$$

4 Quantum affine algebras and weak quantum vertex algebras

In this section we show how to associate weak quantum vertex algebras to quantum affine algebras by using the conceptual construction of weak quantum vertex algebras and their ϕ-coordinated quasi modules.

First, we follow [FJ] (cf. [Dr]) to present the quantum affine algebras. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank l of type A, D, or E and let $A = (a_{ij})$ be the Cartan matrix. Let q be a nonzero complex number. For $1 \le i, j \le l$, set

$$f_{ij}(x) = (q^{a_{ij}}x - 1)/(x - q^{a_{ij}}) \in \mathbb{C}(x). \tag{4.1}$$

Then we set

$$g_{ij}(x)^{\pm 1} = \iota_{x,0} f_{ij}(x)^{\pm 1} \in \mathbb{C}[[x]], \tag{4.2}$$

where $\iota_{x,0} f_{ij}(x)^{\pm 1}$ are the formal Taylor series expansions of $f_{ij}(x)^{\pm 1}$ at 0. Let $\mathbb{Z}_+$ denote the set of positive integers. The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ is (isomorphic to) the associative algebra with identity 1 with generators

$$X_{ik}^{\pm}, \quad \phi_{im}, \quad \psi_{in}, \quad \gamma^{1/2}, \quad \gamma^{-1/2} \tag{4.3}$$

for $1 \le i \le l$, $k \in \mathbb{Z}$, $m \in -\mathbb{Z}_+$, $n \in \mathbb{Z}_+$, where $\gamma^{\pm 1/2}$ are central, satisfying the relations below, written in terms of the following generating functions in a formal variable z:

$$X_i^{\pm}(z) = \sum_{k \in \mathbb{Z}} X_{ik}^{\pm} z^{-k}, \quad \phi_i(z) = \sum_{m \in -\mathbb{Z}_+} \phi_{im} z^{-m}, \quad \psi_i(z) = \sum_{n \in \mathbb{Z}_+} \psi_{in} z^{-n}. \tag{4.4}$$

The relations are

$$\begin{aligned}
&\gamma^{1/2}\gamma^{-1/2} = \gamma^{-1/2}\gamma^{1/2} = 1, \\
&\phi_{i0}\psi_{i0} = \psi_{i0}\phi_{i0} = 1, \\
&[\phi_i(z), \phi_j(w)] = 0, \quad [\psi_i(z), \psi_j(w)] = 0, \\
&\phi_i(z)\psi_j(w)\phi_i(z)^{-1}\psi_j(w)^{-1} = g_{ij}(z/w\gamma)/g_{ij}(z\gamma/w), \\
&\phi_i(z)X_j^{\pm}(w)\phi_i(z)^{-1} = g_{ij}(z/w\gamma^{\pm 1/2})^{\pm 1} X_j^{\pm}(w), \\
&\psi_i(z)X_j^{\pm}(w)\psi_i(z)^{-1} = g_{ij}(w/z\gamma^{\pm 1/2})^{\mp 1} X_j^{\pm}(w), \\
&(z - q^{\pm 4a_{ij}}w)X_i^{\pm}(z)X_j^{\pm}(w) = (q^{\pm 4a_{ij}}z - w)X_j^{\pm}(w)X_i^{\pm}(z), \\
&[X_i^{+}(z), X_j^{-}(w)] = \frac{\delta_{ij}}{q - q^{-1}}\left(\delta\left(\frac{z}{w\gamma}\right)\psi_i(w\gamma^{1/2}) - \delta\left(\frac{z\gamma}{w}\right)\phi_i(z\gamma^{1/2})\right),
\end{aligned}$$

and there is one more set of relations of Serre type.

A $U_q(\hat{\mathfrak{g}})$-module W is said to be *restricted* if for any $w \in W$, $X_{ik}^{\pm}w = 0$ and $\psi_{ik}w = 0$ for $1 \le i \le l$ and for k sufficiently large. We say W is of *level* $\ell \in \mathbb{C}$ if $\gamma^{\pm 1/2}$ act on W as scalars $q^{\pm \ell/4}$. (Rigorously speaking, one needs to choose a branch of $\log q$.) We have (see [Li8]; cf. [Li2], Proposition 4.9):

Proposition 4.1. *Let q and ℓ be complex numbers with $q \ne 0$ and let W be a restricted $U_q(\hat{\mathfrak{g}})$-module of level ℓ. Set*

$$U_W = \{\phi_i(x), \psi_i(x), X_i^{\pm}(x) \mid 1 \le i \le l\}.$$

Then U_W is a quasi $\mathcal{S}_{trig}$-local subset of $\mathcal{E}(W)$ and $\langle U_W \rangle_\phi$ is a weak quantum vertex algebra with W as a ϕ-coordinated quasi module, where $\phi(x, z) = xe^z$.

With Proposition 4.1 on hand, the remaining problem is to determine the weak quantum vertex algebras $\langle U_W \rangle_\phi$ explicitly and to show that they are quantum vertex algebras, sufficiently by establishing the non-degeneracy. We expect that these weak quantum vertex algebras are vacuum modules for certain associative algebras derived from quantum affine algebras.

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