数理解析研究所講究録
第 1756 巻 2011 年 41-50

Generalizations of Burnside ring and their applications

Yugen Takegahara
(竹ケ原 裕元)

Muroran Institute of Technology
(室蘭工業大学)

There are various generalizations of Burnside ring. Among others Boltje's plus construction is useful for finding explicit induction formulae. In this report, we apply it to representation theory of the quantum double of a finite group.

1 Mackey functor and restriction functor

Let G be a finite group, and let k be a commutative ring. A restriction functor for G is a triple $A=(A, \mathrm{con}, \mathrm{res})$ consisting of a family $(A(H))_{H\leq G}$ of k-modules $A(H)$, a family of k-module homomorphisms

$$\mathrm{con}_H^g : A(H) \to A({}^gH),$$

the conjugation maps, for $H \leq G$ and $g \in G$, and a family of k-module homomorphisms

$$\mathrm{res}_K^H : A(H) \to A(K),$$

the restriction maps, for $K \leq H \leq G$, satisfying the axioms

$$\begin{array}{lll}
\text{(G.1)} & \mathrm{con}_{{}^rH}^g \circ \mathrm{con}_H^r = \mathrm{con}_H^{gr}, & \mathrm{con}_H^h = \mathrm{id}_{A(H)}, \\
\text{(G.2)} & \mathrm{res}_L^K \circ \mathrm{res}_K^H = \mathrm{res}_L^H, & \mathrm{res}_H^H = \mathrm{id}_{A(H)}, \\
\text{(G.3)} & \mathrm{con}_K^g \circ \mathrm{res}_K^H = \mathrm{res}_{{}^gK}^{{}^gH} \circ \mathrm{con}_H^g &
\end{array}$$

for all $L \leq K \leq H \leq G$, g, $r \in G$, and $h \in H$ [3]. An algebra restriction functor for G is a restriction functor $A=(A, \mathrm{con}, \mathrm{res})$ for G such that $A(H)$ with $H \leq G$ are k-algebras and con and res are k-algebra homomorphisms [3]. A Mackey functor for G is a quadruple $A=(A, \mathrm{con}, \mathrm{res}, \mathrm{ind})$ consisting of a restriction functor $(A, \mathrm{con}, \mathrm{res})$ for G and a family of k-module homomorphisms

$$\mathrm{ind}_K^H : A(K) \to A(H),$$

the induction maps, for $K \leq H \leq G$, satisfying the axioms

$$\begin{array}{lll}
\text{(G.4)} & \mathrm{ind}_K^H \circ \mathrm{ind}_L^K = \mathrm{ind}_L^H, & \mathrm{ind}_H^H = \mathrm{id}_{A(H)}, \\
\text{(G.5)} & \mathrm{con}_H^g \circ \mathrm{ind}_K^H = \mathrm{ind}_{{}^gK}^{{}^gH} \circ \mathrm{con}_K^g, & \\
\text{(G.6)} & \text{(Mackey axiom)} &
\end{array}$$

$$\mathrm{res}_K^H \circ \mathrm{ind}_U^H = \sum_{KhU \in K\backslash H/U} \mathrm{ind}_{K\cap {}^hU}^K \circ \mathrm{res}_{K\cap {}^hU}^{{}^hU} \circ \mathrm{con}_U^h$$

for all $L \leq K \leq H \leq G$, $U \leq H$, and $g \in G$, where in (G.6), the sum is taken over all (K, U)-double cosets KhU, $h \in H$, of H. A Green functor for G is a Mackey functor $A = (A, \text{con}, \text{res}, \text{ind})$ for G such that $A = (A, \text{con}, \text{res})$ is an algebra restriction functor for G and

(G.7) (Frobenius axioms)

$$\sigma \cdot \text{ind}_K^H(\tau) = \text{ind}_K^H(\text{res}_K^H(\sigma) \cdot \tau), \quad \text{ind}_K^H(\tau) \cdot \sigma = \text{ind}_K^H(\tau \cdot \text{res}_K^H(\sigma))$$

for all $K \leq H \leq G$, $\sigma \in A(H)$, and $\tau \in A(K)$.

2 Plus construction

Let A be a restriction functor for G. Let $H \leq G$, and set

$$M(H) = \prod_{U \leq H} A(U).$$

We consider $M(H)$ to be a left kH-module with the action given by

$$h.(y_U)_{U \leq H} = (\text{con}_U^h(y_U))_{{}^hU \leq H}$$

for all $h \in H$ and $(y_U)_{U \leq H} \in M(H)$. Let $I(M(H))$ be the smallest kH-submodule of $M(H)$ such that H acts trivially on $M(H)/I(M(H))$. We write $\overline{d} = d + I(M(H))$ for each $d \in M(H)$. For any $K \leq H \leq G$ and $\sigma \in A(K)$, define

$$[K, \sigma] = \overline{(x_U)_{U \leq H}} \in M(H)/I(M(H))$$

by

$$x_U = \begin{cases} \sigma & \text{if } U = K, \\ 0 & \text{otherwise.} \end{cases}$$

Following [3], we define a Mackey functor $A_+ = (A_+, \text{con}_+, \text{res}_+, \text{ind}_+)$ for G by

$$\begin{aligned}
&A_+(H) = M(H)/I(M(H)), \\
&\text{con}_{+H}^{\ g}([K, \sigma]) = [{}^gK, \text{con}_H^g(\sigma)], \\
&\text{res}_{+L}^{\ H}([K, \sigma]) = \sum_{LhK \in L \backslash H/K} [L \cap {}^hK, \text{res}_{L \cap {}^hK}^{{}^hK} \circ \text{con}_K^h(\sigma)], \\
&\text{ind}_{+L}^{\ H}([U, \tau]) = [U, \sigma]
\end{aligned}$$

for all $K \leq H \leq G$, $U \leq L \leq H$, $\sigma \in A(K)$, and $\tau \in A(U)$. If A is an algebra restriction functor, then we view A_+ as a Green functor whose multiplication on the elements $[K, \sigma]$ with $K \leq H \leq G$ and $\sigma \in A(K)$ of $A_+(H)$ is given by

$$[K, \sigma] \cdot [U, \tau] = \sum_{KhU \in K \backslash H/U} [K \cap {}^hU, \text{res}_{K \cap {}^hU}^K(\sigma) \cdot \text{res}_{K \cap {}^hU}^{{}^hU} \circ \text{con}_U^h(\tau)].$$

3 Burnside homomorphism

Let A be a restriction functor for G. Suppose that $\mathcal{B} = (\mathcal{B}(H))_{H\leq G}$ is a stable k-basis of A, *i.e.*, $\mathcal{B}(H)$ is a k-basis of $A(H)$ for each $H \leq G$ and $\mathrm{con}_H^g(\mathcal{B}(H)) = \mathcal{B}({}^gH)$ for each $H \leq G$ and for all $g \in G$ [3]. Let $H \leq G$, and set

$$\mathcal{S}_{\mathcal{B}}(H) = \{(K,\sigma) \mid K \leq H \quad \text{and} \quad \sigma \in \mathcal{B}(K)\}.$$

Given $K \leq H$ and $\chi \in A(K)$, we define elements $\langle \chi, \sigma\rangle$ with $\sigma \in \mathcal{B}(K)$ of k by

$$\chi = \sum_{\sigma\in\mathcal{B}(K)} \langle \chi,\sigma\rangle\sigma.$$

A partially order $\leq$ of $\mathcal{S}_{\mathcal{B}}(H)$ is defined by

$$(U,\tau) \leq (K,\sigma) \iff U \leq K \quad \text{and} \quad \langle \mathrm{res}_U^K(\sigma),\tau\rangle \neq 0.$$

View $\mathcal{S}_{\mathcal{B}}(H)$ as a left H-set with the action given by

$$h.(K,\sigma) = ({}^hK, \mathrm{con}_K^h(\sigma))$$

for all $h \in H$ and $(K,\sigma) \in \mathcal{S}_{\mathcal{B}}(H)$, and denote by $\mathcal{R}_{\mathcal{B}}(H)$ a complete set of representatives of H-orbits in $\mathcal{S}_{\mathcal{B}}(H)$. The elements $[K,\sigma]$, $(K,\sigma) \in \mathcal{R}_{\mathcal{B}}(H)$, form a k-basis of $A_+(H)$. Given $(U,\tau) \in \mathcal{R}_{\mathcal{B}}(H)$, we set

$$N_H(U,\tau) = \{h \in N_H(U) \mid \mathrm{con}_U^h(\tau) = \tau\},$$

$$W_H(U,\tau) = N_H(U,\tau)/U,$$

and

$$\mathcal{S}_{\mathcal{B}}(H)_{\geq(U,\tau)} = \{(K,\sigma) \in \mathcal{S}_{\mathcal{B}}(H) \mid (K,\sigma) \geq (U,\tau)\}.$$

Let $\mathrm{Cl}(H)$ be a full set of nonconjugate subgroups of H. Set

$$\mathcal{G}_A(H) = \prod_{K\in\mathrm{Cl}(H)} A(K)^{N_H(K)},$$

where

$$A(K)^{N_H(K)} = \{\sigma \in A(K) \mid \mathrm{con}_K^h(\sigma) = \sigma \text{ for all } h \in N_H(K)\}.$$

Given $(K,\sigma) \in \mathcal{R}_{\mathcal{B}}(H)$, we set

$$(K,\sigma)^+ = \sum_{hN_H(K,\sigma)\in N_H(K)/N_H(K,\sigma)} \mathrm{con}_K^h(\sigma).$$

The elements $(K,\sigma)^+$, $(K,\sigma) \in \mathcal{R}_{\mathcal{B}}(H)$, form a k-basis of $\mathcal{G}_A(H)$. A k-module homomorphism $\varphi_{A,H} : A_+(H) \to \mathcal{G}_A(H)$ defined by

$$\varphi_{A,H}([U,\tau]) = \sum_{(K,\sigma)\in\mathcal{R}_{\mathcal{B}}(H)} \sum_{hU\in H/U,\, K\leq {}^hU} \langle \mathrm{res}_K^{{}^hU} \circ \mathrm{con}_U^h(\tau),\sigma\rangle (K,\sigma)^+$$

for all $(U,\tau) \in \mathcal{R}_{\mathcal{B}}(H)$ is called a Burnside homomorphism.

Example 3.1 Define a k-algebra restriction functor $\underline{k} = (\underline{k}, \mathrm{con}, \mathrm{res})$ for G by

$$\begin{aligned}\underline{k}(H) &= k,\\ \mathrm{con}^g_H &= \mathrm{id}_k,\\ \mathrm{res}^H_K &= \mathrm{id}_k\end{aligned}$$

for all $K \leq H \leq G$ and $g \in G$. Then the Green functor $\underline{k}_+$ for G is isomorphic to the Burnside ring functor $k \otimes_{\mathbb{Z}} \Omega$ [3]. If $k = \mathbb{Z}$, then Burnside homomorphisms $\varphi_{\underline{\mathbb{Z}},H} : \Omega(H) \to \prod_{K \in \mathrm{Cl}(H)} \mathbb{Z}$ with $H \leq G$ are ring monomorphisms.

4 Canonical induction formula

Let X be a Mackey functor for G, and suppose that A is a restriction subfunctor of X. So $A(H) \subseteq X(H)$ for each $H \leq G$, and the conjugation and restriction maps of A are compatible with those of X. We define a morphism $\Theta^{X,A} : A_+ \to X$ of Mackey functors for G by

$$\Theta^{X,A}_H([K,\sigma]) = \mathrm{ind}^H_K(\sigma)$$

for all $H \leq G$ and $[K,\sigma] \in A_+(H)$. A morphism $\Psi : X \to A_+$ of restriction functors for G is called a canonical induction formula for X from A if

$$\Theta^{X,A} \circ \Psi = \mathrm{id}_X$$

[3, 3.3. Definition].

Example 4.1 Assume that $k = \mathbb{Z}$. For each $H \leq G$, let $R(H)$ be the character ring of $\mathbb{C}H$ (see, *e.g.*, [5, §9C]). The character ring functor for G is the Green functor $R = (R, \mathrm{con}, \mathrm{res}, \mathrm{ind})$ for G with usual conjugation, restriction, and induction. Given $H \leq G$, let $\mathrm{Lin}(H)$ be the set of linear $\mathbb{C}$-characters of H. We denote by R^{ab} the restriction subfunctor of the character ring functor for G such that $R^{\mathrm{ab}}(H)$, where $H \leq G$, is the $\mathbb{Z}$-span of $\mathrm{Lin}(H)$. Observe that $\mathcal{B}^{\mathrm{Lin}} = (\mathrm{Lin}(H))_{H \leq G}$ is a stable $\mathbb{Z}$-basis of R^{ab}. For each $H \leq G$, define a $\mathbb{Z}$-module homomorphism $\alpha^{\mathrm{Lin}}_H : R(H) \to R^{\mathrm{ab}}(H)$ by

$$\alpha^{\mathrm{Lin}}_H(\chi) = \begin{cases} \chi & \text{if } \chi \in \mathrm{Lin}(H),\\ 0 & \text{otherwise}\end{cases}$$

for all irreducible $\mathbb{C}$-characters χ of H. Let μ be the Möbius function of the poset $(\mathcal{S}(G), \leq)$, $\mathcal{S}(G)$ the set of subgroups of G.

According to [3, 1.8.(a), 6.13.(a), 9.7. Example], there exists a canonical induction formula $\Psi^{R,R^{\mathrm{ab}}}$ for R from R^{ab} defined by

$$\Psi^{R,R^{\mathrm{ab}}}_H(\chi) = \sum_{(U,\tau) \in \mathcal{R}_{\mathcal{B}^{\mathrm{Lin}}}(H)} m_\tau(\chi)[U,\tau],$$

where

$$m_\tau(\chi) = \frac{1}{|W_H(U,\tau)|} \sum_{(K,\sigma)\in\mathcal{S}_{\mathcal{B}\mathrm{Lin}}(H)_{\geq(U,\tau)}} \mu(U,K)\langle\alpha_K^{\mathrm{Lin}} \circ \mathrm{res}_K^H(\chi), \sigma\rangle,$$

for all $H \leq G$ and $\chi \in R(H)$. In particular, $m_\tau(\chi) \in \mathbb{Z}$ for all $H \leq G$, $\chi \in R(H)$, and $(U,\tau) \in \mathcal{R}_{\mathcal{B}\mathrm{Lin}}(H)$. Consequently,

$$\chi = \sum_{(U,\tau)\in\mathcal{R}_{\mathcal{B}\mathrm{Lin}}(G)} m_\tau(\chi)\mathrm{ind}_U^G(\tau)$$

for all $\chi \in R(G)$. This explicit Brauer induction formula is due to Boltje [2].

5 An induction formula for Mackey functors

Let X be a Mackey functor for G. Given $H \leq G$, we set

$$\mathcal{T}^X(H) = \sum_{K<H} \{\mathrm{ind}_K^H(y) \mid y \in X(K)\}$$

and

$$\mathcal{K}^X(H) = \bigcap_{K<H} \{x \in X(H) \mid \mathrm{res}_K^H(x) = 0\}.$$

A subgroup H of G is called primordial for X if $\mathcal{T}^X(H) \neq X(H)$ [10], and is called coprimordial for X if $\mathcal{K}^X(H) \neq \{0\}$ [3]. Let $\mathcal{P}(X)$ be the set of primordial subgroups of G, and let $\mathcal{C}(X)$ be the set of coprimordial subgroups of G. If $|G|$ is invertible in k, then

$$X(H) = \mathcal{T}^X(H) \oplus \mathcal{K}^X(H)$$

for all $H \leq G$, and thereby, $\mathcal{P}(X) = \mathcal{C}(X)$ (see *e.g.*, [3, 6.2 Proposition]).

We define a restriction functor $\overline{X} = (\overline{X}, \overline{\mathrm{con}}, \overline{\mathrm{res}})$ for G by

$$\begin{aligned} \overline{X}(H) &= X(H)/\mathcal{T}^X(H), \\ \overline{\mathrm{con}}_H^g(\overline{x}) &= \overline{\mathrm{con}_H^g(x)}, \\ \overline{\mathrm{res}}_K^H(\overline{x}) &= \overline{\mathrm{res}_K^H(x)} \end{aligned}$$

for all $K \leq H \leq G$, $x \in X(H)$, and $g \in G$. Here $\overline{y} = y + \mathcal{T}^X(L)$ for all $L \leq G$ and for each $y \in X(L)$.

The first assertion of the following proposition is due to Puig [9, Proposition 3.4(iii)] (see also [10]), and the second one is part of [3, 6.9 Example] (see also [10, Proposition 7.7]), which is a generalization of Brauer's explicit formula for Artin's induction theorem or Witherspoon's explicit formula for Conlon's induction theorem (cf. [10, Section 7], [11, Proposition 3.7]).

Proposition 5.1 *Suppose that $|G|$ is invertible in k. Let X be a Mackey functor for G, and let $H \leq G$. Then there exists a k-module isomorphism*

$$X(H) \xrightarrow{\sim} \prod_{K \in \mathrm{Cl}(H) \cap \mathcal{P}(X)} \overline{X}(K)^{N_H(K)}, \quad x \mapsto (\overline{\mathrm{res}_K^H(x)})_{K \in \mathrm{Cl}(H) \cap \mathcal{P}(X)}.$$

Moreover,

$$x = \sum_{K \in \mathrm{Cl}(H) \cap \mathcal{P}(X)} \frac{1}{|N_H(K)|} \sum_{U \leq K} |U| \mu(U, K) \mathrm{ind}_U^H \circ \mathrm{res}_U^H(x)$$

for all $x \in X(H)$.

6 Crossed Mackey functor

Let S be a finite G-monoid, *i.e.*, a finite semigroup with identity on which G acts as monoid homomorphisms. Given a restriction functor A for G, we define a restriction functor $A_{\otimes S} = (A_{\otimes S}, \mathrm{con}_{\otimes S}, \mathrm{res}_{\otimes S})$ for G by

$$\begin{aligned}
A_{\otimes S}(H) &= A(H) \otimes_k kC_S(H), \\
\mathrm{con}_{\otimes S}{}_H^g(x \otimes s) &= \mathrm{con}_H^g(x) \otimes {}^g s, \\
\mathrm{res}_{\otimes S}{}_K^H(x \otimes s) &= \mathrm{res}_K^H(x) \otimes s
\end{aligned}$$

for all $K \leq H \leq G$, $x \in A(H)$, $s \in C_S(H)$, and $g \in G$. If A is an algebra restriction functor, then we define multiplication on $A_{\otimes S}(H)$ by

$$\left(\sum_{s \in C_S(H)} x(s) \otimes s \right) \left(\sum_{t \in C_S(H)} y(t) \otimes t \right) = \sum_{r \in C_S(H)} \sum_{(s,t) \in S \times S,\, st = r} x(s) y(t) \otimes r.$$

Let X be a Mackey functor for G. Following [8], we define a crossed Mackey functor $X_S = (X_S, \mathrm{con}_S, \mathrm{res}_S, \mathrm{ind}_S)$ for G by

$$\begin{aligned}
X_S(H) &= \left\{ (x(s))_{s \in S} \in \prod_{s \in S} X(H_s) \,\middle|\, \mathrm{con}_{H_s}^h(x(s)) = x({}^h s) \text{ for all } h \in H \right\}, \\
\mathrm{con}_S{}_H^g((x(s))_{s \in S}) &= (\mathrm{con}_{H_s}^g(x(s)))_{{}^g s \in S}, \\
\mathrm{res}_S{}_K^H((x(s))_{s \in S}) &= (\mathrm{res}_{K_s}^{H_s}(x(s)))_{s \in S}, \\
\mathrm{ind}_S{}_K^H((y(s))_{s \in S}) &= \left(\sum_{H_s h K \in H_s \backslash H / K} \mathrm{ind}_{({}^h K)_s}^{H_s} \circ \mathrm{con}_{K_{h^{-1}s}}^h (y({}^{h^{-1}} s)) \right)_{s \in S},
\end{aligned}$$

where H_s is the stabilizer of s in H, for all $K \leq H \leq G$ and $g \in G$. If X is a Green functor, then we define multiplication on $X_S(H)$ by

$$(x(s))_{s\in S}(y(t))_{t\in S} = \left(\sum_{(s,t)\in\overline{H_r\backslash S\times S},\, st=r} \mathrm{ind}_{H_{s,t}}^{H_r}(\mathrm{res}_{H_{s,t}}^{H_s}(x(s)) \cdot \mathrm{res}_{H_{s,t}}^{H_t}(y(t))) \right)_{r\in S},$$

where $\overline{H_r\backslash S \times S}$ is a complete set of H_r-orbits of the diagonal action on $S \times S$.

The second assertion of the following theorem is a consequence of Proposition 5.1, and is related to [12, Theorem 5.5].

Theorem 6.1 *Suppose that $|G|$ is invertible in k. Let S be a finite G-monoid, and let X be a Mackey functor for G. Then*

$$\mathcal{P}(X_S) = \mathcal{P}(X),$$

and there exists a k-module isomorphism

$$X_S(H) \xrightarrow{\sim} \prod_{K\in \mathrm{Cl}(H)\cap\mathcal{P}(X)} \overline{X}_{\otimes S}(K)^{N_H(K)},$$

$$(x_s)_{s\in S} \mapsto \left(\sum_{s\in C_S(K)} \overline{x_s} \otimes s \right)_{K\in \mathrm{Cl}(H)\cap\mathcal{P}(X)}$$

for all $H \leq G$. If X is a Green functor, then this is a k-algebra isomorphism.

7 Green ring functor

Hereafter we assume that $k = \mathbb{Z}$. For each $H \leq G$, let $a(H)$ be the Green ring of $\mathbb{C}H$ (see, *e.g.*, [5, §80D]). The Green ring functor for G is the Green functor $a = (a, \mathrm{con}, \mathrm{res}, \mathrm{ind})$ for G with usual conjugation, restriction, and induction. The character ring functor for G and the Green ring functor for G are isomorphic.

We view G as a G-monoid with the action given by conjugation ${}^r g$ where r, $g \in G$. Let $\mathrm{conj}(G)$ be a full set of nonconjugate elements in G.

The next proposition is a consequence of Theorem 6.1, and is a special case of Lusztig [6, 2.2(g)] (cf. [12, p. 316], Theorem 8.1).

Proposition 7.1 *We have*

$$\mathbb{C}a_G(G) \cong \prod_{g\in\mathrm{conj}(G)} \mathbb{C}Z(C_G(g))$$

as $\mathbb{C}$-algebras.

8 The quantum double of a finite group

We denote by $(\mathbb{C}G)^*$ the set of all linear mappings from $\mathbb{C}G$ to $\mathbb{C}$, and view it as the $\mathbb{C}$-algebra with pointwise addition and multiplication. Suppose that $(\mathbb{C}G)^*$ is the left G-set with the action given by

$$^g f(x) = f(g^{-1}xg)$$

for all $f \in (\mathbb{C}G)^*$, $g \in G$, and $x \in \mathbb{C}G$. We set $D(G) = \mathbb{C}G \otimes_{\mathbb{C}} (\mathbb{C}G)^*$. The quantum double of G is the $\mathbb{C}$-algebra $D(G)$ with multiplication given by

$$(g_1 \otimes f_1)(g_2 \otimes f_2) = g_1 g_2 \otimes f_1 \, {}^{g_1}f_2$$

for all g_1, $g_2 \in G$ and f_1, $f_2 \in (\mathbb{C}G)^*$ (see, *e.g.*, [7, 12]). There is a basis $\{\phi_g \mid g \in G\}$ of $(\mathbb{C}G)^*$ given by

$$\phi_g(s) = \begin{cases} 1 & \text{if } s = g, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $H \leq G$, and define a subalgebra $D_G(H)$ of $D(G)$ by

$$D_G(H) = \sum_{s \in G,\, h \in H} \mathbb{C}h \otimes \phi_s.$$

Let $D_G(H)$-**mod** be the set of finitely generated $D_G(H)$-modules. We define a tensor product $M_1 \otimes M_2$ with M_1, $M_2 \in D_G(H)$-**mod** by the left $D_G(H)$-module $M_1 \otimes_{\mathbb{C}} M_2$ with the action of $D_G(H)$ given by

$$(h \otimes \phi_s)(u \otimes v) = \sum_{g \in G} (h \otimes \phi_{g^{-1}})u \otimes (h \otimes \phi_{gs})v$$

for all $h \otimes \phi_s \in D_G(H)$ and $u \otimes v \in M_1 \otimes_{\mathbb{C}} M_2$.

For each $M \in D_G(H)$-**mod**, we denote by $[M]$ the isomorphism class containing M. Let $D_G(H)$-$\overline{\textbf{mod}}$ be the set of isomorphism classes $[M]$ with $M \in D_G(H)$-**mod**, and denote by $R(D_G(H))$ the ring consisting of $\mathbb{Z}$-linear combinations of isomorphism classes $[M] \in D_G(H)$-$\overline{\textbf{mod}}$ with direct sum for addition and tensor product for multiplication.

We now define a Green functor $R_{D_G} = (R_{D_G}, \mathrm{Dcon}, \mathrm{Dres}, \mathrm{Dind})$ for G by

$$\begin{aligned}
&R_{D_G}(H) = R(D_G(H)),\\
&\mathrm{Dcon}_H^g([M]) = [g \otimes M],\\
&\mathrm{Dres}_K^H([M]) = [M|_{D_G(K)}],\\
&\mathrm{Dind}_K^H([N]) = [D_G(H) \otimes_{D_G(K)} N]
\end{aligned}$$

for all $K \leq H \leq G$, $g \in G$, $M \in D_G(H)$-**mod**, and $N \in D_G(K)$-**mod**, as usual ([1, 12]).

The following theorem is due to Oda and Yoshida [8, 5.5. Theorem] (see also [4]):

Theorem 8.1 *The Green functors R_{D_G} and a_G are isomorphic.*

9 An induction formula for the quantum double of a finite group

Let $H \leq G$, and let $\overline{H\backslash G}$ be a complete set of representatives of H-orbits in G. From Theorem 8.1, we know that $D_G(H)$ is a semisimple algebra. The set of isomorphism classes of irreducible $D_G(H)$-modules is

$$\mathrm{Irr}(D_G(H)) := \left\{ [M] \in D_G(H)\text{-}\overline{\mathbf{mod}} \;\middle|\; \begin{array}{l} (1 \otimes \phi_s)M \text{ is an irreducible } \mathbb{C}H_s\text{-module} \\ \text{for some } s \in \overline{H\backslash G}, \text{ and } (1 \otimes \phi_t)M = 0 \\ \text{for any } t \in \overline{H\backslash G} \text{ with } t \neq s \end{array} \right\}.$$

Let $R^{\mathrm{ab}}(D_G(H))$ be the ring consisting of $\mathbb{Z}$-linear combinations of isomorphism classes $[N] \in \mathrm{Lin}(D_G(H))$, where

$$\mathrm{Lin}(D_G(H)) = \left\{ [N] \in D_G(H)\text{-}\overline{\mathbf{mod}} \;\middle|\; \begin{array}{l} \dim_{\mathbb{C}}((1 \otimes \phi_s)N) = 1 \text{ for some } s \in \overline{H\backslash G}, \\ \text{and } (1 \otimes \phi_t)N = 0 \text{ for any } t \in \overline{H\backslash G} \\ \text{with } t \neq s \end{array} \right\}.$$

Define a $\mathbb{Z}$-module homomorphism $D\alpha_H : R(D_G(H)) \to R^{\mathrm{ab}}(D_G(H))$ by

$$D\alpha_H([M]) = \begin{cases} [M] & \text{if } [M] \in \mathrm{Lin}(D_G(H)), \\ 0 & \text{otherwise} \end{cases}$$

for all $[M] \in \mathrm{Irr}(D_G(H))$.

We define a restriction subfunctor $R^{\mathrm{ab}}_{D_G}$ of $R_{D_G} = (R_{D_G}, \mathrm{Dcon}, \mathrm{Dres}, \mathrm{Dind})$ by $R^{\mathrm{ab}}_{D_G}(H) = R^{\mathrm{ab}}(D_G(H))$ for all $H \leq G$. Observe that $\mathcal{B}^{D_G} := (\mathrm{Lin}(D_G(H)))_{H \leq G}$ is a stable $\mathbb{Z}$-basis of $R^{\mathrm{ab}}_{D_G}$.

We can now state an analogy of an explicit Brauer induction formula given in Example 4.1.

Theorem 9.1 *Let $H \leq G$, and let $M \in D_G(H)$-**mod**. Then*

$$M \simeq \sum_{(U,[N]) \in \mathcal{R}_{\mathcal{B}^{D_G}}(H)} m_N(M) \cdot D_G(H) \otimes_{D_G(U)} N,$$

where

$$m_N(M) = \frac{1}{|W_H(U,N))|} \sum_{(K,[\tilde{N}]) \in \mathcal{S}_{\mathcal{B}^{D_G}}(H)_{\geq (U,[N])}} \mu(U,K) \langle D\alpha_K \circ \mathrm{Dres}^H_K([M]), [\tilde{N}] \rangle.$$

References

[1] P. R. Boisen, The representation theory of fully group-graded algebras, J. Algebra **151** (1992), 160–179.

[2] R. Boltje, A canonical Brauer induction formula, Astérisque, **181–182** (1990), 31–59.

[3] R. Boltje, A general theory of canonical induction formulae, J. Algebra **206** (1998), 293–343.

[4] S. Bouc, Hochschild constructions for Green functors, Comm. Algebra **31** (2003), 403–436.

[5] C. W. Curtis and I. Reiner, Methods of Representation Theory, Vol. I, II, Wiley-Interscience, New York, 1981, 1987.

[6] G. Lusztig, Leading coefficients of character values of Hecke algebras, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), 235–262, Proc. Sympos. Pure Math., 47, Part 2, Amer. Math. Soc., Providence, RI, 1987.

[7] G. Mason, The quantum double of a finite group and its role in conformal field theory, *in* "Groups '93 Galway/St. Andrews, Vol. 2", London Math. Soc. Lecture Note Ser., 212, 405–417, Cambridge Univ. Press, Cambridge, 1995.

[8] F. Oda and T. Yoshida, Crossed Burnside rings II: The Dress construction of a Green functor, J. Algebra **282** (2004), 58–82.

[9] L. Puig, Pointed groups and construction of modules, J. Algebra **116** (1988), 7–129.

[10] J. Thévenaz, Some remarks on G-functors and the Brauer morphism, J. Reine Angew. Math. **384** (1988), 24–56.

[11] S. J. Witherspoon, The ring of equivariant vector bundles on finite sets, J. Algebra **175** (1995), 274–286.

[12] S. J. Witherspoon, The representation ring of the quantum double of a finite group, J. Algebra **179** (1996), 305–329.