数理解析研究所講究録
第 1759 巻 2011 年 63-67

On the number of slack variables used in representation of semi-algebraic sets

Issei Yoshida
IBM Research - Tokyo *

Abstract

In real algebraic geometry, slack variables play a fundamental role in transforming a semi-algebraic set into an algebraic set. Specifically, for a semi-algebraic set $S \subset \mathbb{R}^n$, there exists an algebraic set $\widetilde{S} \subset \mathbb{R}^{n+k}$ such that the natural projection of $\widetilde{S}$ onto $\mathbb{R}^n$ is S, where k slack variables are used to convert inequalities to equations.

We consider the number of slack variables necessary for the transformation, corresponding to the auxiliary dimension k in $\mathbb{R}^{n+k}$. In general such k depends on the boolean expression of a given semi-algebraic set. It is known that if a semi-algebraic set consists of only inequations, then only one slack variable is sufficient for the transformation. We show that for a general semi-algebraic set expressed by a disjunctive normal form (DNF), $\lfloor \frac{m}{2} \rfloor$ variables are sufficient for the transformation where m is the maximal number of inequalities appeared in a single conjunctive clause which comprises the DNF.

1 Introduction

Semi-algebraic sets have been actively studied in real algebraic geometry. Since a semi-algebraic set can express open conditions defined by a set of inequalities, it has a wide variety of real-world applications including robotics, automatic geometry theorem-proving [1] and algebraic statistics [2].

In order to analyze inequations within a framework of algebraic geometry that treats only equations, slack variables are typically used to "transform" inequalities into equations. For example, let $S \subset \mathbb{R}^2$ be a semi-algebraic set in the xy-plane defined by an inequality $f(x,y) := y - x^2 \geq 0$. When we define $\widetilde{S} \subset \mathbb{R}^3$ as $\widetilde{S} := \{(x,y,z) \in \mathbb{R}^3 \mid z^2 - f(x,y) = 0\}$, $\widetilde{S}$ is an algebraic set in $\mathbb{R}^3$. Then it is easy to see that $(x,y) \in S$ if and only if there exists a $z \in \mathbb{R}$ such that $(x,y,z) \in \widetilde{S}$. Hence we can study S by investigating $\widetilde{S}$ with the tools of algebraic geometry.

In this paper we consider the number of slack variables that are necessary to transform inequalities into equations (formal definition is given later). First, we present our problem definition. Next, we briefly review the result of [3] for the case when all inequalities appeared in the given semi-algebraic set are inequations. To the best of our knowledge, it is an open problem to determine the minimum number of slack variables for general semi-algebraic sets. We show an upper bound of the number which is about a half of the maximal number of inequalities appeared in a single clause of the given semi-algebraic set in disjunctive normal form. Finally, we give some remarks on the problem.

2 Problem Definition

Let k be a real closed field, for example, $k = \mathbb{R}$. Let $R := k[x_1, \cdots, x_n]$ be an n-dimensional polynomial ring over k. For $f \in R$ let $V(f) := \{(a_1, \cdots, a_n) \in k^n \mid f(a_1, \cdots, a_n) = 0\}$ be an algebraic set defined by f, and $D(f) := k^n \setminus V(f)$. Also, we define $V_>(f) := \{(a_1, \cdots, a_n) \in k^n \mid f(a_1, \cdots, a_n) > 0\}$.

We say that a subset $X \subset k^n$ is a semi-algebraic set of k^n if there exist $f_i, g_j \in R$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ such that X is represented as

*issei@jp.ibm.com

$$X = \left(\bigcap_{i=1}^{p} V(f_i)\right) \bigcap \left(\bigcap_{j=1}^{q} V_>(g_j)\right) \tag{1}$$

When $p = q = 0$ we define the right-hand side of (1) as k^n.

More generally, we define an algebraic constraint set as follows. We define a literal over R to be a symbol f or $f_>$ for $f \in R$. We define the set of algebraic constraints $\mathcal{AC}(R)$ over R to be a Boolean Algebra generated by literals over R. For example, $f \wedge (g_> \vee h) \in \mathcal{AC}(R)$ for $f, g, h \in R$.

For $C \in \mathcal{AC}(R)$, we define $V_c(C)$ to be the subset of k^n obtained by replacing f, $f_>$ with $V(f)$, $V_>(f)$ respectively, and replacing $\wedge$, $\vee$, $\neg$ with $\cap$, $\cup$, and the complement of the specified subset respectively. We call $V_c(C)$ the algebraic constraint set corresponding to C, or simply an algebraic constraint set when C is clear from the context.

It is well-known that any boolean formula can be converted into DNF, so any algebraic constraint is equivalent to a constraint C in the following form:

$$C = \bigvee_{i=1}^{r} \left((\bigwedge_{j=1}^{p_i} f_{ij}) \bigwedge (\bigwedge_{j=1}^{q_i} h_{ij>}) \right), \tag{2}$$

where f_{ij} and h_{ij} are elements of R, and one of p_i or q_i can be zero. When $r = 0$ we define $C = 0$, and when $r = 1$ we call C a minimal algebraic constraint.

In this case, $V_c(C)$ is given by

$$V_c(C) = \bigcup_{i=1}^{r} \left((\bigcap_{j=1}^{p_i} V(f_{ij})) \bigcap (\bigcap_{j=1}^{q_i} V_>(h_{ij})) \right) \tag{3}$$

It is clear from definition that an algebraic constraint set for a minimal constraint (that is, $r = 1$ in (3)) is simply a semi-algebraic set. Note that $D(f)$ is naturally expressed in a form of (3) because $D(f) = V_>(f) \cup V_>(-f)$ and hence $f \vee f_>$ is the corresponding constraint. Let $\neg f := (f \vee f_>)$. Then an inequality $(f > 0)$ does not essentially appear in

$$C = \bigvee_{i=1}^{r} \left((\bigwedge_{j=1}^{p_i} f_{ij}) \bigwedge (\bigwedge_{j=1}^{q_i} \neg h_{ij}) \right), \tag{4}$$

and the corresponding algebraic constraint set is given by

$$V_c(C) = \bigcup_{i=1}^{r} \left((\bigcap_{j=1}^{p_i} V(f_{ij})) \bigcap (\bigcap_{j=1}^{q_i} D(h_{ij})) \right) \tag{5}$$

In general, an algebraic constraint set ((3) or (5)) is not an algebraic set in k^n. We consider an algebraic set that fully reflects the property of a given algebraic constraint in the following discussion.

For a given algebraic constraint C given by (2) or (4), in most practical cases it is interesting to see whether there exists a point $x \in k^n$ satisfying C. In other words, we want to know whether $V_c(C) = \emptyset$ or not. We introduce the notion of slack dimension, which indicates the number of "auxiliary" dimensions necessary for embedding $V_c(C)$, in some sense, into a space of higher dimension.

Definition 1
Let C be an algebraic constraint over R. The slack dimension of C over R, denoted by $sdim_R(C)$, is the minimum of $t \in \mathbb{N}$ such that there exists an algebraic set $\widetilde{V}_c(C) \subset k^{n+t}$ and the natural projection $\pi : k^{n+t} \to k^n$, $(x_1, \cdots, x_n, z_1, \cdots, z_t) \mapsto (x_1, \cdots, x_n)$ satisfying $\pi(\widetilde{V}_c(C)) = V_c(C)$.

Note that $sdim_R(C)$ is always finite. For each $V_>(f)$ or $D(f)$ that appears in $V_c(C)$ we can introduce a new "slack" variable z and consider $V(1-z^2 f)$ or $V(1-zf)$ (these are algebraic sets in k^{n+1}) respectively. For example, $V_>(f) \neq \emptyset \;\Leftrightarrow\; \exists x \in k^n, f(x) > 0 \;\Leftrightarrow\; \exists x \in k^n, z \in k, 1 - z^2 f(x) = 0 \;\Leftrightarrow\; \exists (x, z) \in k^{n+1}, (x, z) \in \widetilde{V}_c(C)$. Hence $\pi(V(1 - z^2 f)) = V_>(f)$ as expected. If $k = \mathbb{Q}$ then $1 - 3z^2 = 0$ does not have a solution in k, so $V(1 - z^2 f)$ does not work. Thus k must be a real closed field.

We are interested in the minimal number of slack variables necessary for constructing $V_c(C)$.

3 Slack Variables for Inequations

First we consider the case that a given constraint C is expressed by (4). In this section we describe the result given in [3].

Definition 2
Let $(x_1, \cdots, x_n)$ be the coordinate system of k^n and $\pi : k^{n+1} \to k^n$ be a natural projection defined by $(x_1, \cdots, x_n, z) \mapsto (x_1, \cdots, x_n)$. For an algebraic constraint C defined by (4), we define an algebraic set $\widetilde{V}_c(C)$ on k^{n+1} as

$$\widetilde{V}_c(C) := \bigcup_{i=1}^{r}\Big(\big(\bigcap_{j=1}^{p_i} V(f_{ij})\big)\bigcap V(1 - zh_{i1}\cdots h_{iq_i})\Big) \tag{6}$$

When $C = 0$ we define $\widetilde{V}_c(C) = k^{n+1}$.

Note that f_{ij} and h_{ij} can be seen as polynomials of $R[z]$ even though they do not actually contain the variable z. Then the following holds.

Proposition 3 (see [3])
$\pi(\widetilde{V}_c(C)) = V_c(C)$ for any algebraic constraint C given by (4).

Proposition 3 implies that there exists a point satisfying the given algebraic constraint if and only if $\widetilde{V}_c(C)$ is non-empty. The variable z newly introduced here is sometimes called a *slack* variable. The proposition says that only one slack variable is sufficient for expressing the given constraint C by an algebraic set, in particular the number of slack variables is independent of C or the dimension n. Refer to [3] for a constructive proof of Proposition 3 in which a set of algebraic equations defining $\widetilde{V}_c(C)$ is explicitly calculated from the given C. The next proposition just restates Proposition 3.

Proposition 4
$sdim_R(C) = 1$ for any algebraic constraint C given by (4).

4 Slack Variables for Semi-algebraic Sets

The following two facts are essential to show that only one slack variable is sufficient for any case when C is given by (4).

Lemma 5 (see [3])
(1) Let C_1, C_2 be algebraic constraints in a form of (4). If $X, Y \subset k^{n+t}$ such that $\pi(X) = V_c(C_1)$ and $\pi(Y) = V_c(C_2)$, then $\pi(X \cup Y) = V_c(C_1 \vee C_2)$

(2) $f_1(x) \neq 0 \ \wedge \ \cdots \ \wedge \ f_p(x) \neq 0 \ \Leftrightarrow \ \exists z \in k, \ 1 - zf_1(x)\cdots f_p(x) = 0$

(1) says that when we consider the number of slack variables it is sufficient to concentrate on a single conjunction that appear in a DNF of (4). This shall be understood by the following example: $f(x) > 0 \vee g(x) > 0 \ \Leftrightarrow \ \exists z, 1 - zf(x) = 0 \vee \exists w, 1 - wg(x) = 0 \ \Leftrightarrow \ \exists z,$ "$1 - zf(x) = 0 \vee 1 - zg(x) = 0$". This technique can also be applied to the case when C is in a form of (2). Hence it is sufficient to consider the case when C is represented as

$$C = (\bigwedge_{i=1}^{p} f_i) \bigwedge (\bigwedge_{j=1}^{q} h_{j>})$$

Moreover, since we do not have use slack variables for each f_i, the problem is reduced to the case when C is represented as

$$C = \bigwedge_{i=1}^{p} h_{i>}$$

and the corresponding semi-algebraic set is

$$V_c(C) = \bigcap_{i=1}^{p} V_>(h_i) = \{a = (a_1, \cdots, a_n) \in k^n \mid h_1(a) > 0, \cdots, h_p(a) > 0 \}$$

On the other hand, (2) cannot applied to this case. For $a \in k^n$, apparently $h_1(a) > 0 \wedge h_2(a) > 0$ is not equivalent to that $1 - z^2 h_1(a) h_2(a) > 0$ for some $z \in k$.

Proposition 6 is the main result in this paper.

Proposition 6
Let $h_1, \cdots, h_p \in R = k[x_1, \cdots, x_n]$. Then there exists an algebraic set $V \subset k^{n+\lceil \frac{p}{2} \rceil}$ such that

$$\pi(V) = \bigcap_{i=1}^{p} V_>(h_i)$$

where $\pi : k^{n+\lceil \frac{p}{2} \rceil} \to k^n$ is the natural projection.

This proposition says that the number of slack variables necessary for construction of an algebraic set is about half the number of inequalities in the constraint.

Proof We prove that for $h_1, h_2 \in R$ there exists $V \subset k^{n+1}$ such that $\pi(V) = V_>(h_1) \cap V_>(h_2)$. The proposition follows from this by introducing a new slack variable for each pair of (h_{2i-1}, h_{2i}) $(i = 1, \cdots, \lceil \frac{p}{2} \rceil - 1)$.

For given h_1 and h_2, define the hypersurface $Z(h_1, h_2) \subset k^{n+1}$ by

$$Z(h_1, h_2) := V((1 - z^2 h_1(x))^2 + (1 - z^2 h_2(x))^2 - z^4 h_1(x)^2 h_2(x)^2) \tag{7}$$

where z is a new variable corresponding to the $(n+1)$-th coordinate of k^{n+1}. We show that $\pi(Z(h_1, h_2)) = V_>(h_1) \cap V_>(h_2)$. Let $(x, z) = (x_1, \cdots, x_n, z) \in k^{n+1}$.

Suppose $x \in \pi(Z(h_1, h_2))$. There exists $z \in k$ such that $(x, z) \in Z(h_1, h_2)$. By definition of $Z(h_1, h_2)$,

$$(1 - z^2 h_1(x))^2 + (1 - z^2 h_2(x))^2 - z^4 h_1(x)^2 h_2(x)^2 = 0$$

By substituting $h_1(x) = 0$ or $h_2(x) = 0$ into this equation, neither $h_1(x)$ nor $h_2(x)$ can be zero. Hence by dividing both sides by $(h_1(x) h_2(x))^2$, we obtain

$$\left(\frac{1}{h_1(x)} - z^2\right)^2 + \left(\frac{1}{h_2(x)} - z^2\right)^2 = z^4 \tag{8}$$

(8) means that in the ab-plane k^2, the point $(h_1(x), h_2(x))$ is on the circle $(a - z^2)^2 + (b - z^2)^2 = z^4$. Since z cannot be zero, the circle is contained in the first quadrant of ab-plane $\{(a, b) \mid a > 0, b > 0\}$. This shows $h_1(x) > 0$ and $h_2(x) > 0$, hence $x \in V_>(h_1) \cap V_>(h_2)$.

Conversely, suppose $x \in V_>(h_1) \cap V_>(h_2)$, that is, $h_1(x) > 0$ and $h_2(x) > 0$. It is easily seen that $\bigcup_{z>0} \{(a, b) \mid (a - z^2)^2 + (b - z^2)^2 = z^4\} = \{(a, b) \mid a > 0, b > 0\}$. Hence there exists $z \in k$ satisfying the equation (8), from which $x \in \pi(Z(h_1, h_2))$ follows. ∎

5 Concluding Remarks

The technique we used in the proof of Proposition 6 cannot be extended straightforward to three or more h_i, because for $p \geq 3$ the family of p-dimensional spheres parametrized by $z \in k$, as we constructed in the proof, is not a covering of $a_1 > 0, \cdots, a_p > 0$ anymore. It is an open problem to determine $sdim_R(C)$ for a general algebraic constraint C. Proposition 6 gives a non-trivial upperbound of $sdim_R(C)$ for an individual C.

Related to our problem, minimization of the number of inequalities expressing a semi-algebraic set having certain properties (for example, convexity) is actively studied [4]. To begin with such special semi-algebraic sets is a possible direction to give a tighter lower-bound of $sdim_R(C)$.

References

[1] D. Cox, J. Little and D. O'Shea, "Ideals, Varieties, and Algorithms", Springer, 2006.

[2] M. Drton, B. Sturmfels and S. Sullivant, "Lectures on Algebraic Statistics", Birkhaeuser Basel, 2008.

[3] Issei Yoshida, "A new model of algebraic constraints for representing multiple negations with one slack variable", Bulletin of JSSAC, Vol.15, No.2, 2008.

[4] Hartwig Bosse, Martin Grotschel and Martin Henk, "Polynomial inequalities representing polyhedra", Mathematical Programming, Volume 103, Number 1, 2005.